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Global existence of solutions of the Keller-Segel model with a nonlinear chemotactical sensitivity function

退化放物型 Keller-Segel 系の時間大域解の存在と漸近挙動

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1 Introduction

We consider the following degenerate quasi-linear parabolic system:

$$
\text{(KS)} \qquad \begin{cases} u_t = \nabla \cdot \left(\nabla u^m - u^{q-1} \cdot \nabla v \right), & x \in \mathbb{R}^N,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), \quad \tau v(x,0) = \tau v_0(x), & x \in \mathbb{R}^N, \end{cases}
$$

where $m > 1, q \geq 2, \tau = 0$ or 1, and $N \geq 1$. The initial data (u_0, v_0) is a non-negative function and in $L^1 \cap L^\infty(\mathbb{R}^N) \times L^1 \cap H^1 \cap W^{1,\infty}(\mathbb{R}^N)$, $u_0^m \in H^1\mathbb{R}^N)$. This equation is often called as the Keller-Segel model describing the motion of the chemotaxis molds.

Our aim of this paper is to prove the existence of a global weak solution of (KS) under some appropriate conditions without any restriction on the size of the initial data. Specifically, we show that a solution (u, v) of (KS) exists globally in time either

(i) $q < m$ for a large initial data or (ii) $1 < m \leq q - \frac{2}{N}$ for a small initial data.

Our results are the expansions of our previous work [9], which deals with the case of $q = 2$.

Definition 1 *For $m > 1$, non-negative functions (u, v) defined in $[0, \infty) \times \mathbb{R}^N$ are said to be a weak solution of* (KS) *for $u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$, $u_0^m \in H^1(\mathbb{R}^N)$ and $v_0 \in L^1 \cap H^1 \cap W^{1,\infty}(\mathbb{R}^N)$ if*

i) $u \in L^\infty(0, \infty; L^2(\mathbb{R}^N))$, $u^m \in L^2(0, \infty; H^1(\mathbb{R}^N))$,

ii) $v \in L^\infty(0, \infty; H^1(\mathbb{R}^N))$,

iii) (u, v) *satisfies the equations in the sense of distribution: i.e.*

$$
\int_0^\infty \int_{\mathbb{R}^N} \left(\nabla u^m \cdot \nabla \varphi - u^{q-1} \nabla v \cdot \nabla \varphi - u \cdot \varphi_t \right) \, dx dt = \int_{\mathbb{R}^N} u_0(x) \cdot \varphi(x, 0) \, dx,
$$
$$
\int_0^\infty \int_{\mathbb{R}^N} \left(\nabla v \cdot \nabla \varphi + v \cdot \varphi - u \cdot \varphi - \tau v \cdot \varphi_t \right) \, dx dt = \int_{\mathbb{R}^N} v_0(x) \cdot \varphi(x, 0) \, dx,
$$

for every smooth test function φ which vanishes for all $|x|$ and t large enough.

The first theorem gives the existence of a time global weak solution to (KS) with $\tau = 1$ and the uniform bound of the solution when $u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$ and $v_0 \in L^1 \cap H^1 \cap W^{1,\infty}(\mathbb{R}^N)$. The first theorem also ensures the weak solution obtained here neither blows up nor grows up. We note that the initial data is not assumed to be small.

Theorem 1.1 (time global existence of $\tau = 1$ case) *Let $\tau = 1$, $q \geq 2, m > q$ and suppose that u_0 and v_0 are non-negative everywhere. Then* (KS) *has a global weak solution (u, v). Moreover, $u^m \in C((0, \infty); L^2(\mathbb{R}^N))$ and (u, v) satisfies a uniform estimate, i.e., that there exists a constant $K_1 = K_1(\|u_0\|_{L^1(\mathbb{R}^N)}, \|u_0\|_{L^\infty(\mathbb{R}^N)}, \|v_0\|_{L^1(\mathbb{R}^N)}, \|v_0\|_{H^1(\mathbb{R}^N)}, \|v_0\|_{W^{1,\infty}(\mathbb{R}^N)}, m, q, N) > 0$ such that*

$$
(1.1) \qquad \sup_{t>0} \left(\|u(t)\|_{L^r(\mathbb{R}^N)} + \|v(t)\|_{L^r(\mathbb{R}^N)} \right) \leq K_1 \qquad \text{for all } r \in [1, \infty].
$$

In addition, there exists a positive constant $K_2 = K_2(\|u_0\|_{L^1(\mathbf{R}^N)}, \|u_0\|_{L^m(\mathbf{R}^N)}, \|v_0\|_{H^1(\mathbf{R}^N)}, m, q, N)$,

$$\|v_t\|_{L^2(0,\infty;L^2(\mathbf{R}^N))} + \sup_{t>0}\|v(t)\|_{H^2(\mathbf{R}^N)} \leq K_2. \tag{1.2}$$

We next consider the case when $\tau = 0$ and $m > 1$, which corresponds to a degenerate version of "the Nagai model" for the semi-linear Keller-Segel system [1], [3] –[6].

Theorem 1.2 (time global existence of $\tau = 0$ case) *Let $\tau = 0$, $q \geq 2$ and suppose that u_0 is non-negative. Then*
(i) *when $m > q$,* (KS) *has a global weak solution (u, v).*
(ii) *When $1 < m \leq q - \frac{2}{N}$, we also assume that the initial data is sufficiently small, i.e.,* $\|u_0\|_{L^{\frac{N(q-m)}{2}}(\mathbf{R}^N)} << 1$, *then* (KS) *has a global weak solution (u, v).*

Moreover it satisfies a uniform estimate, i.e., that in both cases (i) *and* (ii), *there exists* $K_1 = K_1(\|u_0\|_{L^r(\mathbf{R}^N)}, m, q, N)$ *such that*

$$\sup_{t>0}\Big(\|u(t)\|_{L^r(\mathbf{R}^N)} + \|v(t)\|_{L^r(\mathbf{R}^N)}\Big) \leq K_1 \qquad \textit{for all } r \in [1, \infty]. \tag{1.3}$$

In addition, in both cases (i) *and* (ii), *there exists a positive constant* $K_2 = K_2(\|u_0\|_{L^2(\mathbf{R}^N)}, m, q, N)$,

$$\sup_{t>0}\|v(t)\|_{H^2(\mathbf{R}^N)} \leq K_2. \tag{1.4}$$

Finally we present the decay for the solution of (KS) in the $\tau = 0$ case under the smallness assumption on $\|u_0\|_{L^{\frac{N(q-m)}{2}}(\mathbf{R}^N)}$.

Theorem 1.3 *Let $\tau = 0$, $q \geq 2$ and $1 < m \leq q - \dfrac{2}{N}$ and suppose that the initial data u_0 is non-negative everywhere. We also assume that* $\|u_0\|_{L^{\frac{N(q-m)}{2}}(\mathbf{R}^N)} << 1$, *then the weak solution (u, v) obtained in Theorem 1.2, satisfies*

$$\sup_{t>0}(1+t)^d \cdot (\|u(t)\|_{L^r(\mathbf{R}^N)} + \|v(t)\|_{L^r(\mathbf{R}^N)}) < \infty \qquad \textit{for } r \in \Big[\frac{N(q-m)}{2}, \infty\Big). \tag{1.5}$$

where

$$d = \frac{N}{\sigma}\left(1 - \frac{1}{r}\right), \quad \sigma = N(m-1) + 2.$$

We will use the simplified notations:

1) $Q_T := (0,T) \times \mathbb{R}^N$,

2) When the weak derivatives $\nabla u, D^2u$ and u_t are in $L^p(Q_T)$ for some $p \geq 1$, we say that $u \in W_p^{2,1}(Q_T)$, *i.e.*,

$$\begin{aligned} W_p^{2,1}(Q_T) \quad := \quad & \Big\{u \in L^p(0,T; W^{2,p}(\mathbb{R}^N)) \cap W^{1,p}(0,T; L^p(\mathbb{R}^N)); \\ & \|u\|_{W_p^{2,1}(Q_T)} := \|u\|_{L^p(Q_T)} + \|\nabla u\|_{L^p(Q_T)} + \|D^2u\|_{L^p(Q_T)} + \|u_t\|_{L^p(Q_T)} < \infty\Big\}. \end{aligned}$$

2 Approximated Problem

The first equation of (KS) is a quasi-linear parabolic equation of degenerate type. Therefore we can not expect the system (KS) to have a classical solution at the point where the first solution u vanishes. In order to justify all the formal arguments, we need to introduce the following approximated equation of (KS):

$$(\mathrm{KS})_\varepsilon \begin{cases} u_{\varepsilon t}(x,t) = \nabla \cdot \Big(\nabla(u_\varepsilon + \varepsilon)^m - (u_\varepsilon + \varepsilon)^{q-2}u_\varepsilon \cdot \nabla v_\varepsilon\Big), & (x,t) \in \mathbb{R}^N \times (0,T), \quad \cdots(1), \\ \tau v_{\varepsilon t}(x,t) = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, & (x,t) \in \mathbb{R}^N \times (0,T), \quad \cdots(2), \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), \ \tau v_\varepsilon(x,0) = \tau v_{0\varepsilon}(x), & x \in \mathbb{R}^N, \end{cases}$$

where ε is a positive parameter and $(u_{0\varepsilon}, v_{0\varepsilon})$ is an approximation for the initial data (u_0, v_0) such that

(A.1) $0 \le u_{0\varepsilon} \in W^{2,p}(\mathbb{R}^N),\ 0 \le \tau v_{0\varepsilon} \in W^{3,p}(\mathbb{R}^N)$ for all $p \in [1,\infty]$, for all $\varepsilon \in (0,1]$,

(A.2) $\|u_{0\varepsilon}\|_{L^p} \le \|u_0\|_{L^p},\quad \tau\|v_{0\varepsilon}\|_{W^{1,p}} \le \tau\|v_0\|_{W^{1,p}}$ for all $p \in [1,\infty]$, for all $\varepsilon \in (0,1]$,

(A.3) $\|\nabla u_{0\varepsilon}\|_{L^2} \le \|\nabla u_0\|_{L^2}$, for all $\varepsilon \in (0,1]$,

(A.4) $u_{0\varepsilon} \to u_0,\quad \tau v_{0\varepsilon} \to \tau v_0$ strongly in $L^p(\mathbb{R}^N)$ as $\varepsilon \to 0$, for some $p > \max\{2, N\}$.

We call $(u_\varepsilon, v_\varepsilon)$ a strong solution of $(\mathrm{KS})_\varepsilon$ if it belongs to $W_p^{2,1} \times W_p^{2,1}(Q_T)$ for some $p \ge 1$ and the equations (1),(2) in $(\mathrm{KS})_\varepsilon$ are satisfied almost everywhere.

The strong solution u_ε coincides with the mild solution defined in *Definition 2* if $u_\varepsilon \in L^1(0,T;L^p(\mathbb{R}^N))$ with $p \ge 1$.

Firstly, we construct the strong solution of $(\mathrm{KS})_\varepsilon$, To do this, we prepare the following two propositions:

Proposition 2.1 *Let $(u_\varepsilon, v_\varepsilon)$ be a non-negative strong solution of $(\mathrm{KS})_\varepsilon$ in $W_p^{2,1}(Q_T)$ with $\max\{2,N\} < p < \infty$ and suppose that* (A.1) and (A.2) *are satisfied. Then, u_ε and v_ε become non-negative and*

$$\sup_{t>0} \|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)} \le M_{u,r} \qquad \text{for all } r \in [1,\infty] \tag{2.1}$$

$$\begin{cases} \text{(i)} & \text{when } \tau = 1,\ q > 1,\ m > 2q-1, \\ \text{(ii)} & \text{when } \tau = 0,\ q > 1,\ m > \max\{1, q - \frac{2}{N}\}, \\ \text{(iii)} & \text{when } \tau = 0,\ q > 1,\ 1 < m \le q - \frac{2}{N},\ \text{ and } \|u_0\|_{L^{\frac{N(q-m)}{2}}} \text{ is small.} \end{cases}$$

Proposition 2.2 *Let $q > 1$, $m > 1$, $\max\{2,N\} < p < \infty$ and suppose that* (A.1) *is satisfied and assume that u_ε in the first equation of $(\mathrm{KS})_\varepsilon$ satisfies the estimate*

$$\sup_{0<t<T} \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} \le M_{u,\infty}, \tag{2.2}$$

for some constant $M_{u,\infty}$. Then, $(\mathrm{KS})_\varepsilon$ has a non-negative strong solution $(u_\varepsilon, v_\varepsilon)$ uniquely belonging to $W_p^{2,1} \times W_p^{2,1}(Q_T)$.

By combining Proposition 2.1 with 2.2, the time global strong solution $(u_\varepsilon, v_\varepsilon)$ is obtained. As for the proof of Proposition 2.2 and 2.1, we refer to [9].

3 Proof of Theorem 1.1 and 1.2

In this section, we give a proof of Theorem 1.1 and 1.2.

Let us recall (2.1) in Proposition 2.1.

We can extract a subsequence $\{u_{\varepsilon_n}\}$ such that

$$u_{\varepsilon_n} \rightharpoonup u \quad \text{weakly} \quad \text{in } L^2(0,T;L^2(\mathbb{R}^N)). \tag{3.1}$$

Moreover, we obtain a subsequence, still denoted by $\{u_{\varepsilon_n}\}$ such that

$$u_{\varepsilon_n}^m \to u^m \quad \text{strongly} \quad \text{in } C((0,T);L^2(\mathbb{R}^N)), \tag{3.2}$$

$$\nabla u_{\varepsilon_n}^m \rightharpoonup \nabla u^m \quad \text{weakly} \quad \text{in } L^2(0,T;L^2(\mathbb{R}^N)). \tag{3.3}$$

The above (3.2) and (3.3) are shown as follows.

We multiply (1) in $(\mathrm{KS})_\varepsilon$ by $\dfrac{\partial(u_\varepsilon+\varepsilon)^m}{\partial t}$ and integrate with respect to the space variable over $\mathbb{R}^N$. Then we get

$$
\begin{aligned}
&\frac{4m}{(m+1)^2}\cdot\int\left|\left((u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right)_t\right|^2 dx\\
&= -\frac{1}{2}\cdot\frac{d}{dt}\int|\nabla(u_\varepsilon+\varepsilon)^m|^2\,dx+\frac{2m}{(m+1)^2}\int\left|\left((u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right)_t\right|^2 dx\\
&\quad+\frac{4m(q-1)^2}{(m+1)^2}\cdot\|\nabla v_\varepsilon\|^2_{L^\infty}\cdot(M_{u,\infty}+\varepsilon)^{2q-4}\int|\nabla(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}|^2\,dx\\
&\quad+m\int(u_\varepsilon+\varepsilon)^{m+2q-3}\cdot|\Delta v_\varepsilon|^2\,dx.
\end{aligned}
\tag{3.4}
$$

By integrating with respect to time variable,

$$
\begin{aligned}
&\frac{2m}{(m+1)^2}\cdot\int_0^T\int\left|\left((u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right)_t\right|^2 dxdt+\frac{1}{2}\cdot\sup_{0<t<T}\int|\nabla(u_\varepsilon+\varepsilon)^m|^2\,dx\\
&= \frac{1}{2}\int|\nabla(u_{0\varepsilon}+\varepsilon)^m|^2\,dx\\
&\quad+\frac{4m(q-1)^2}{(m+1)^2}\cdot\|\nabla v_\varepsilon\|^2_{L^\infty(0,T;L^\infty)}\cdot(M_{u,\infty}+\varepsilon)^{2q-4}\int_0^T\int|\nabla(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}|^2\,dxdt\\
&\quad+m(M_{u,\infty}+\varepsilon)^{m+2q-3}\int_0^T\int|\Delta v_\varepsilon|^2\,dxdt.
\end{aligned}
\tag{3.5}
$$

On the other hand, by the multiplication (1) in $(\mathrm{KS})_\varepsilon$ by u_ε and the integration with respect to x and t, we have

$$
\begin{aligned}
&\int_0^T\int|\nabla(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}|^2\,dxdt\\
&\leq \frac{(m+1)^2}{8m}\Big(\frac{1}{q^2}\int_0^T\int u_\varepsilon^{2q}\,dxdt+\frac{\varepsilon^2}{(q-1)^2}\int_0^T\int u_\varepsilon^{2q-2}\,dxdt+2\int_0^T\int|\Delta v_\varepsilon|^2\,dxdt\Big)\\
&\quad+\frac{(m+1)^2}{8m}\|u_{0\varepsilon}\|^2_{L^2}.
\end{aligned}
\tag{3.6}
$$

From (3.5) and (3.6), we see that for $q\geq 2$ there exists a positive constant C (which is independent of ε),

$$
\begin{aligned}
&\int_0^T\int|(u_\varepsilon^m)_t|^2\,dxdt\;+\;\sup_{0<t<T}\int|\nabla u_\varepsilon^m|^2\,dx\\
&\leq \int_0^T\int\left|\left((u_\varepsilon+\varepsilon)^m\right)_t\right|^2 dxdt\;+\;\sup_{0<t<T}\int|\nabla(u_\varepsilon+\varepsilon)^m|^2\,dx\\
&\leq \frac{4m^2}{(m+1)^2}\cdot(M_u+\varepsilon)^{m-1}\int_0^T\int\left|\left(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right)_t\right|^2 dxdt\;+\;\sup_{0<t<T}\int|\nabla(u_\varepsilon+\varepsilon)^m|^2\,dx\\
&\leq C.
\end{aligned}
\tag{3.7}
$$

Thus we find that $u_\varepsilon^m\in L^\infty(0,T;H^1(\mathbb{R}^N))\cap H^1(0,T;L^2(\mathbb{R}^N))$. Hence, we can extract a subsequence such that

$$
u_{\varepsilon_n}^m\to\xi\quad\text{strongly}\qquad\text{in } C((0,T);L^2(\mathbb{R}^N)).
\tag{3.8}
$$

This gives

$$
u_{\varepsilon_n}^m(x,t)\to\xi(x,t)\qquad\text{a.a}\quad x\in\mathbb{R}^N,\ t\in(0,T).
$$

A function $g(u)=u^{\frac{1}{m}}$ is continuous with respect to u.

Thus, we see that

$$
u_{\varepsilon_n}(x,t)\;\to\;\xi^{\frac{1}{m}}(x,t)\qquad\text{a.a}\quad x\in\mathbb{R}^N,\ t\in(0,T),
\tag{3.9}
$$

Since the sequence $\{u_{\varepsilon_n}\}$ is bounded in $L^2(0,T;L^2(\mathbb{R}^N))$, we conclude by Lions's Lemma that

$$u_{\varepsilon_n} \rightharpoonup \xi^{\frac{1}{m}} \quad \text{weakly} \quad \text{in } L^2(0,T;L^2(\mathbb{R}^N)). \tag{3.10}$$

By (3.1), (3.8) and (3.10),

$$u_{\varepsilon_n}^m \to u^m \quad \text{strongly} \quad \text{in } C((0,T);L^2(\mathbb{R}^N)), \tag{3.11}$$

which prove (3.2).
Next, we multiply (1) in $(\mathrm{KS})_\varepsilon$ by u_ε^m and integrate with respect to the space variable over $\mathbb{R}^N$. Then we get

$$\frac{1}{m+1}\cdot\frac{d}{dt}\int u_\varepsilon^{m+1}\,dx \;\le\; -\frac{1}{2}\int |\nabla(u_\varepsilon+\varepsilon)^m|^2\,dx+\frac{1}{2}\cdot\|u_\varepsilon+\varepsilon\|_{L^\infty}^{2(q-1)}\cdot\|\nabla v_\varepsilon\|_{L^2}^2. \tag{3.12}$$

Integrating (3.12) with respect to t, by (2.1) in Proposition 2.1 and (A.3), we have

$$\begin{aligned}&\frac{1}{m+1}\int u_\varepsilon^{m+1}\,dx+\frac{1}{2}\cdot\int_0^T\int|\nabla u_\varepsilon^m|^2\,dxdt\\ &\le\; \frac{1}{m+1}\int u_{0\varepsilon}^{m+1}\,dx+\frac{1}{2}\|u_\varepsilon+\varepsilon\|_{L^\infty(Q_T)}^{2(q-1)}\cdot\|\nabla v_\varepsilon\|_{L^2(Q_T)}^2 \;\le\; C.\end{aligned} \tag{3.13}$$

From (3.2) and (3.13), we obtain (3.3).
By the standard argument, in both cases $\tau=0$ and $\tau=1$, we see that there exists a positive constant C which is independent of ε,

$$\int_0^T\int|(v_\varepsilon)_t|^2\,dxdt \;+\; \sup_{0<t<T}\int|\nabla v_\varepsilon|^2\,dx \;\le\; C. \tag{3.14}$$

Hence, we can extract a subsequence $\{v_{\varepsilon_n}\}$ such that

$$v_{\varepsilon_n} \to v \quad \text{strongly} \quad \text{in } C((0,T);L^2(\mathbb{R}^N)), \tag{3.15}$$
$$\nabla v_{\varepsilon_n} \rightharpoonup \chi=\nabla v \quad \text{weakly} \quad \text{in } L^2(0,T;L^2(\mathbb{R}^N)). \tag{3.16}$$

By the standard argument, we complete the proof of Theorem 1.1 and 1.2.

4 Proof of Theorem 1.3

As for the proof of Theorem 1.3, we refer to [9].

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