Global existence of solutions of the Keller-Segel model with a nonlinear chemotactical sensitivity function (Theory of Bio-Mathematics and It's Applications)

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1 Introduction

We consider the following degenerate quasi-linear parabolic system:

\[(KS) \begin{array}{l}
  u_t = \nabla \cdot (\nabla u^m - u^{q-1} \cdot \nabla v), \\
  \tau v_t = \Delta v - v + u, \\
  u(x,0) = u_0(x), \quad \tau v(x,0) = \tau v_0(x),
\end{array} \quad x \in \mathbb{R}^N, \quad t > 0,
\]

where \(m > 1, q \geq 2, \tau = 0 \) or \(1, \) and \(N \geq 1.\) The initial data \((u_0, v_0)\) is a non-negative function and in \(L^1 \cap L^\infty(\mathbb{R}^N) \times L^1 \cap H^1 \cap W^{1,\infty}(\mathbb{R}^N), u_0^m \in H^1(\mathbb{R}^N).\) This equation is often called as the Keller-Segel model describing the motion of the chemotaxis molds.

Our aim of this paper is to prove the existence of a global weak solution of \((KS)\) under some appropriate conditions without any restriction on the size of the initial data. Specifically, we show that a solution \((u, v)\) of \((KS)\) exists globally in time either

(i) \( q < m \) for a large initial data or (ii) \( 1 < m \leq q - \frac{2}{N} \) for a small initial data.

Our results are the expansions of our previous work [9], which deals with the case of \(q = 2.\)

**Definition 1.** For \(m > 1,\) non-negative functions \((u, v)\) defined in \([0, \infty) \times \mathbb{R}^N\) are said to be a weak solution of \((KS)\) for \(u_0 \in L^1 \cap L^\infty(\mathbb{R}^N), u_0^m \in H^1(\mathbb{R}^N)\) and \(v_0 \in L^1 \cap H^1 \cap W^{1,\infty}(\mathbb{R}^N)\) if

i) \( u \in L^\infty(0, \infty; L^2(\mathbb{R}^N)), u^m \in L^2(0, \infty; H^1(\mathbb{R}^N)), \)

ii) \( v \in L^\infty(0, \infty; H^1(\mathbb{R}^N)), \)

iii) \((u, v)\) satisfies the equations in the sense of distribution: i.e.

\[
\int_{0}^{\infty} \int_{\mathbb{R}^N} (\nabla u^m \cdot \nabla \varphi - u^{q-1} \nabla v \cdot \nabla \varphi - u \cdot \varphi_t) \, dx \, dt = \int_{\mathbb{R}^N} u_0(x) \cdot \varphi(x, 0) \, dx,
\]

\[
\int_{0}^{\infty} \int_{\mathbb{R}^N} (\nabla v \cdot \nabla \varphi + u \cdot \varphi - u \cdot \varphi_t - \tau v \cdot \varphi_t) \, dx \, dt = \int_{\mathbb{R}^N} v_0(x) \cdot \varphi(x, 0) \, dx,
\]

for every smooth test function \(\varphi\) which vanishes for all \(|x|\) and \(t\) large enough.

The first theorem gives the existence of a time global weak solution to \((KS)\) with \(\tau = 1\) and the uniform bound of the solution when \(u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)\) and \(v_0 \in L^1 \cap H^1 \cap W^{1,\infty}(\mathbb{R}^N).\) The first theorem also ensures the weak solution obtained here neither blows up nor grows up. We note that the initial data is not assumed to be small.

**Theorem 1.1 (time global existence of \(\tau = 1\) case)** Let \(\tau = 1,\) \(q \geq 2, m > q\) and suppose that \(u_0\) and \(v_0\) are non-negative everywhere. Then \((KS)\) has a global weak solution \((u, v).\) Moreover, \(u^m \in C((0, \infty); L^2(\mathbb{R}^N))\) and \((u, v)\) satisfies a uniform estimate, i.e., that there exists a constant \(K_1 = K_1(||u_0||_{L^1(\mathbb{R}^N)}, ||v_0||_{L^1(\mathbb{R}^N)}, ||u_0||_{H^1(\mathbb{R}^N)}, ||v_0||_{W^{1,\infty}(\mathbb{R}^N)}, m, q, N) > 0\) such that

\[
\sup_{t > 0} (||u(t)||_{L^r(\mathbb{R}^N)} + ||v(t)||_{L^r(\mathbb{R}^N)}) \leq K_1 \quad \text{for all } r \in [1, \infty].
\]
In addition, there exists a positive constant $K_2 = K_2(||u_0||_{L^1(\mathbb{R}^N)}, ||u_0||_{L^\infty(\mathbb{R}^N)}, ||v_0||_{H^1(\mathbb{R}^N)}, m, q, N),$
\begin{align}
||u(t)||_{L_r^r(\mathbb{R}^N)} + \sup_{t>0}||v(t)||_{H^1(\mathbb{R}^N)} &\leq K_2. 
\end{align}

We next consider the case when \( \tau = 0 \) and \( m > 1 \), which corresponds to a degenerate version of "the Nagai model" for the semi-linear Keller-Segel system \cite{1, 3 - 6}.

**Theorem 1.2 (time global existence of \( \tau = 0 \) case)** Let \( \tau = 0, \) \( q \geq 2 \) and suppose that \( u_0 \) is non-negative. Then
(i) when \( m > q, \) (KS) has a global weak solution \((u,v)\).
(ii) When \( 1 < m \leq q - \frac{2}{N} \), we also assume that the initial data is sufficiently small, i.e., \( ||u_0||_{L^\frac{N(q-m)}{2-}}(\mathbb{R}^N) << 1 \), then (KS) has a global weak solution \((u,v)\).

Moreover it satisfies a uniform estimate, i.e., that in both cases (i) and (ii), there exists \( K_1 = K_1(||u_0||_{L^r(\mathbb{R}^N)}, m, q, N) \) such that
\begin{align}
\sup_{t>0} \left(||u(t)||_{L_r^r(\mathbb{R}^N)} + ||v(t)||_{L_r^r(\mathbb{R}^N)}\right) &\leq K_1 \quad \text{for all } r \in [1, \infty].
\end{align}

In addition, in both cases (i) and (ii), there exists a positive constant \( K_2 = K_2(||u_0||_{L^2(\mathbb{R}^N)}, m, q, N), \)
\begin{align}
\sup_{t>0} ||v(t)||_{H^1(\mathbb{R}^N)} &\leq K_2.
\end{align}

Finally we present the decay for the solution of (KS) in the \( \tau = 0 \) case under the smallness assumption on \( ||u_0||_{L^\frac{N(q-m)}{2-}}(\mathbb{R}^N) \).

**Theorem 1.3** Let \( \tau = 0, \) \( q \geq 2 \) and \( 1 < m \leq q - \frac{2}{N} \) and suppose that the initial data \( u_0 \) is non-negative everywhere. We also assume that \( ||u_0||_{L^\frac{N(q-m)}{2-}}(\mathbb{R}^N) \leq 1 \), then the weak solution \((u,v)\) obtained in Theorem 1.2, satisfies
\begin{align}
\sup_{t>0}(1+t)^d \cdot (||u(t)||_{L^r(\mathbb{R}^N)} + ||v(t)||_{L^r(\mathbb{R}^N)}) &< \infty \quad \text{for } r \in \left[ \frac{N(q-m)}{2-}, \infty \right).
\end{align}

where
\begin{align}
d &= \frac{N}{\sigma} \left( 1 - \frac{1}{r} \right), \quad \sigma = N(m - 1) + 2.
\end{align}

We will use the simplified notations:
1) \( Q_T := (0, T) \times \mathbb{R}^N, \)
2) When the weak derivatives \( \nabla u, D^2 u \) and \( u_t \) are in \( L^p(Q_T) \) for some \( p \geq 1 \), we say that \( u \in W^{2,1}_p(Q_T), \) i.e.,
\begin{align}
W^{2,1}_p(Q_T) := \left\{ u \in L^p(0, T; W^{2,p}(\mathbb{R}^N)) \cap W^{1,p}(0, T; L^p(\mathbb{R}^N)); \right. \\
|\nabla u||_{L^p(Q_T)} + ||\nabla^2 u||_{L^p(Q_T)} + ||u_t||_{L^p(Q_T)} + ||u_t||_{L^p(Q_T)} < \infty \left. \right\}.
\end{align}

2 Approximated Problem

The first equation of (KS) is a quasi-linear parabolic equation of degenerate type. Therefore we cannot expect the system (KS) to have a classical solution at the point where the first solution \( u \) vanishes. In order to justify all the formal arguments, we need to introduce the following approximated equation of (KS):
\begin{align}
\begin{cases}
(\text{KS})_\epsilon \quad &u_{\epsilon t}(x,t) = \nabla \cdot (\nabla (u_\epsilon + \epsilon)^m - (u_\epsilon + \epsilon)^{q-2}u_\epsilon \cdot \nabla v_\epsilon), \quad (z, t) \in \mathbb{R}^N \times (0, T), \\
&\tau v_{\epsilon t}(x, t) = \Delta v_\epsilon - v_\epsilon + u_\epsilon, \quad (z, t) \in \mathbb{R}^N \times (0, T), \\
&u_\epsilon(x, 0) = u_0(\epsilon)(x), \quad \tau v_\epsilon(x, 0) = \tau v_0(\epsilon)(x), \quad x \in \mathbb{R}^N,
\end{cases}
\end{align}
where $\varepsilon$ is a positive parameter and $(u_{0\varepsilon}, v_{0\varepsilon})$ is an approximation for the initial data $(u_0, v_0)$ such that

(A.1) $0 \leq u_{0\varepsilon} \in W^{2,p}(\mathbb{R}^N)$, $0 \leq v_{0\varepsilon} \in W^{3,p}(\mathbb{R}^N)$ for all $p \in [1, \infty)$, for all $\varepsilon \in (0, 1]$,  

(A.2) $||u_{0\varepsilon}||_{L^p} \leq ||u_0||_{L^p}$, $\tau||v_{0\varepsilon}||_{W^{1,p}} \leq \tau||v_0||_{W^{1,p}}$ for all $p \in [1, \infty]$, for all $\varepsilon \in (0, 1]$,  

(A.3) $||\nabla u_{0\varepsilon}||_{L^2} \leq ||\nabla u_0||_{L^2}$, for all $\varepsilon \in (0, 1]$,  

(A.4) $u_{0\varepsilon} \to u_0$, $\tau v_{0\varepsilon} \to \tau v_0$ strongly in $L^p(\mathbb{R}^N)$ as $\varepsilon \to 0$, for some $p > \max\{2, N\}$.

We call $(u_\varepsilon, v_\varepsilon)$ a strong solution of $(KS)_\varepsilon$ if it belongs to $W^{2,1}_p \times W^{2,1}_p(Q_T)$ for some $p \geq 1$ and the equations (1),(2) in $(KS)_\varepsilon$ are satisfied almost everywhere.

The strong solution $u_\varepsilon$ coincides with the mild solution defined in Definition 2 if $u_\varepsilon \in L^1(0, T; L^p(\mathbb{R}^N))$ with $p \geq 1$.

Firstly, we construct the strong solution of $(KS)_\varepsilon$, To do this, we prepare the following two propositions:

**Proposition 2.1** Let $(u_\varepsilon, v_\varepsilon)$ be a non-negative strong solution of $(KS)_\varepsilon$ in $W^{2,1}_p(Q_T)$ with $\max\{2, N\} < p < \infty$ and suppose that (A.1) and (A.2) are satisfied. Then, $u_\varepsilon$ and $v_\varepsilon$ become non-negative and

\[
\sup_{t > 0} ||u_\varepsilon(t)||_{L^r(\mathbb{R}^N)} \leq M_{u,r} \quad \text{for all } r \in [1, \infty],
\]

\[
\begin{aligned}
(i) & \quad \text{when } \tau = 1, \quad q > 1, \quad m > 2q - 1, \\
(ii) & \quad \text{when } \tau = 0, \quad q > 1, \quad m > \max\{1, q - \frac{2}{N}\}, \\
(iii) & \quad \text{when } \tau = 0, \quad q > 1, \quad 1 < m \leq q - \frac{2}{N}, \quad \text{and } ||u_0||_{L^{\frac{N}{q-m}}} \text{ is small}.
\end{aligned}
\]

**Proposition 2.2** Let $q > 1$, $m > 1$, $\max\{2, N\} < p < \infty$ and suppose that (A.1) is satisfied and assume that $u_\varepsilon$ in the first equation of $(KS)_\varepsilon$ satisfies the estimate

\[
\sup_{0 < t < T} ||u_\varepsilon(t)||_{L^{\infty}(\mathbb{R}^N)} \leq M_{u,\infty},
\]

for some constant $M_{u,\infty}$. Then, $(KS)_\varepsilon$ has a non-negative strong solution $(u_\varepsilon, v_\varepsilon)$ uniquely belonging to $W^{2,1}_p \times W^{2,1}_p(Q_T)$.

By combining Proposition 2.1 with 2.2, the time global strong solution $(u_\varepsilon, v_\varepsilon)$ is obtained. As for the proof of Proposition 2.2 and 2.1, we refer to [9].

### 3 Proof of Theorem 1.1 and 1.2

In this section, we give a proof of Theorem 1.1 and 1.2.

Let us recall (2.1) in Proposition 2.1.

We can extract a subsequence $\{u_{n\varepsilon}\}$ such that

\[
(3.1) \quad u_{n\varepsilon} \rightharpoonup u \quad \text{weakly in } L^2(0, T; L^2(\mathbb{R}^N)).
\]

Moreover, we obtain a subsequence, still denoted by $\{u_{n\varepsilon}\}$ such that

\[
(3.2) \quad u_{n\varepsilon}^m \to u^m \quad \text{strongly in } C((0, T); L^2(\mathbb{R}^N)),
\]

\[
(3.3) \quad \nabla u_{n\varepsilon}^m \rightharpoonup \nabla u^m \quad \text{weakly in } L^2(0, T; L^2(\mathbb{R}^N)).
\]

The above (3.2) and (3.3) are shown as follows.

We multiply (1) in $(KS)_\varepsilon$ by $\frac{\partial(u_\varepsilon + \varepsilon)^m}{\partial t}$ and integrate with respect to the space variable over $\mathbb{R}^N$. Then
we get

$$\frac{4m}{(m+1)^2} \cdot \int \left| \left( \frac{\partial}{\partial t} \left( u_\epsilon + \epsilon \right)^{m+1} \right) \right|^2 \, dx$$

\[= - \frac{1}{2} \frac{d}{dt} \int |\nabla (u_\epsilon + \epsilon)|^2 \, dx + \frac{2m}{(m+1)^2} \int \left| \frac{\partial}{\partial t} \left( u_\epsilon + \epsilon \right)^{m+1} \right|^2 \, dx \]

\[+ \frac{4m(q-1)^2}{(m+1)^2} \cdot |\nabla v_\epsilon|^2 \cdot (M_{u,\infty} + \epsilon)^{2q-4} \int |\nabla (u_\epsilon + \epsilon)^{m+1}|^2 \, dx \]

\[+ m \int (u_\epsilon + \epsilon)^m \cdot |\Delta v_\epsilon|^2 \, dx. \]

(3.4)

By integrating with respect to time variable,

$$\frac{2m}{(m+1)^2} \cdot \int_0^T \int \left| \left( u_\epsilon + \epsilon \right)^{m+1} \right|^2 \, dx \, dt + \frac{1}{2} \sup_{0 < t < T} \int |\nabla (u_\epsilon + \epsilon)|^2 \, dx \]

\[= \frac{1}{2} \int |\nabla (u_0 + \epsilon)|^2 \, dx \]

\[+ \frac{4m(q-1)^2}{(m+1)^2} \cdot |\nabla v_\epsilon|^2 \cdot (M_{u,\infty} + \epsilon)^{2q-4} \int_0^T \int |\nabla (u_\epsilon + \epsilon)^{m+1}|^2 \, dx \, dt \]

\[+ m (M_{u,\infty} + \epsilon)^{m+2q-3} \int_0^T \int |\Delta v_\epsilon|^2 \, dx \, dt. \]

(3.5)

On the other hand, by the multiplication (1) in (KS)$_\epsilon$ by $u_\epsilon$ and the integration with respect to $x$ and $t$,

$$\int_0^T \int |\nabla (u_\epsilon + \epsilon)^{m+1}|^2 \, dx \, dt \]

\[\leq \frac{(m+1)^2}{8m} \left( \frac{1}{q^2} \int_0^T \int u_\epsilon^{2q} \, dx \, dt + \frac{\epsilon^2}{(q-1)^2} \int_0^T \int u_\epsilon^{2q-2} \, dx \, dt + 2 \int_0^T \int |\Delta u_\epsilon|^2 \, dx \, dt \right) \]

(3.6)

$$+ \frac{(m+1)^2}{8m} ||u_0||^2_{L^2}. \]

From (3.5) and (3.6), we see that for $q \geq 2$ there exists a positive constant $C$ (which is independent of $\epsilon),\]

$$\int_0^T \int \left| \frac{\partial}{\partial t} \left( u_\epsilon \right)^{m+1} \right|^2 \, dx \, dt \]

\[\leq \frac{(m+1)^2}{8m} \left( \frac{1}{q^2} \int_0^T \int u_\epsilon^{2q} \, dx \, dt + \frac{\epsilon^2}{(q-1)^2} \int_0^T \int u_\epsilon^{2q-2} \, dx \, dt + 2 \int_0^T \int |\Delta u_\epsilon|^2 \, dx \, dt \right) \]

(3.7)

$$\leq C. \]

Thus we find that $u_\epsilon^m \in L^{\infty}(0,T;H^1(\mathbb{R}^N)) \cap H^1(0,T;L^2(\mathbb{R}^N)).$ Hence, we can extract a subsequence such that

(3.8) \[u_\epsilon^m \to \xi \text{ strongly in } C((0,T);L^2(\mathbb{R}^N)).\]

This gives

$$u_\epsilon^m(x,t) \to \xi(x,t) \quad \text{a.a } x \in \mathbb{R}^N, \ t \in (0,T).$$

A function $g(u) = u^{\frac{1}{m}}$ is continuous with respect to $u.

Thus, we see that

(3.9) \[u_\epsilon(x,t) \to \xi^{\frac{1}{m}}(x,t) \quad \text{a.a } x \in \mathbb{R}^N, \ t \in (0,T),\]
Since the sequence \( \{u_{\epsilon_{n}}\} \) is bounded in \( L^{2}(0, T; L^{2}(\mathbb{R}^{N})) \), we conclude by Lions's Lemma that

\[
(3.10) \quad u_{\epsilon_{n}} \rightharpoonup \xi \quad \text{weakly} \quad \text{in} \ L^{2}(0, T; L^{2}(\mathbb{R}^{N})).
\]

By (3.1), (3.8) and (3.10),

\[
(3.11) \quad u_{\epsilon_{n}}^{m} \rightarrow u^{m} \quad \text{strongly} \quad \text{in} \ C((0, T); L^{2}(\mathbb{R}^{N})),
\]

which proves (3.2).

Next, we multiply (1) in (KS)\(_{\epsilon}\) by \( u_{\epsilon}^{m} \) and integrate with respect to the space variable over \( \mathbb{R}^{N} \). Then we get

\[
(3.12) \quad \frac{1}{m+1} \cdot \frac{d}{dt} \int u_{\epsilon}^{m+1} \, dx \leq -\frac{1}{2} \int |\nabla(u_{\epsilon} + \epsilon)^{m}|^{2} \, dx + \frac{1}{2} \cdot \|u_{\epsilon} + \epsilon\|_{L^{\infty}(Q_{T})}^{2(q-1)} \cdot \|\nabla v_{\epsilon}\|_{L^{2}(Q_{T})}^{2}. \tag{3.12}
\]

Integrating (3.12) with respect to \( t \), by (2.1) in Proposition 2.1 and (A.3), we have

\[
(3.13) \quad \frac{1}{m+1} \int u_{\epsilon}^{m+1} \, dx + \frac{1}{2} \cdot \int_{0}^{T} \int |\nabla u_{\epsilon}^{m}|^{2} \, dx \, dt \leq \frac{1}{m+1} \int u_{0\epsilon}^{m+1} \, dx + \frac{1}{2} \|u_{\epsilon} + \epsilon\|_{L^{\infty}(Q_{T})}^{2(q-1)} \cdot \|\nabla v_{\epsilon}\|_{L^{2}(Q_{T})}^{2} \leq C. \tag{3.13}
\]

From (3.2) and (3.13), we obtain (3.3).

By the standard argument, in both cases \( \tau = 0 \) and \( \tau = 1 \), we see that there exists a positive constant \( C \) which is independent of \( \epsilon \),

\[
(3.14) \quad \int_{0}^{T} \int |(v_{\epsilon})_{t}|^{2} \, dx \, dt + \sup_{0 < t < T} \int |\nabla v_{\epsilon}|^{2} \, dx \leq C.
\]

Hence, we can extract a subsequence \( \{v_{\epsilon_{n}}\} \) such that

\[
(3.15) \quad v_{\epsilon_{n}} \rightarrow v \quad \text{strongly} \quad \text{in} \ C((0, T); L^{2}(\mathbb{R}^{N})),
\]

\[
(3.16) \quad \nabla v_{\epsilon_{n}} \rightharpoonup \chi = \nabla v \quad \text{weakly} \quad \text{in} \ L^{2}(0, T; L^{2}(\mathbb{R}^{N})).
\]

By the standard argument, we complete the proof of Theorem 1.1 and 1.2.

4 Proof of Theorem 1.3

As for the proof of Theorem 1.3, we refer to [9].

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References


