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Global structure of solutions for the 1-D Ginzburg-Landau equation

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1 Introduction.

This is a joint work with Prof. Y. Morita and Prof. S. Yotsutani (Ryukoku University). In this article we are dealing with a simplified model of the superconductivity in a thin uniform superconducting ring. The energy functional in a one-dimensional form of such a model is given by

$$E(\psi) := \int_0^{2\pi} \frac{1}{2}|D_h\psi|^2 + \frac{\lambda}{4}(1-|\psi|^2)^2\,dx, \qquad D_h := \frac{d}{dx} - ih(x),$$

where ψ is a complex-valued order parameter ($|\psi|^2$ expresses the density of superconducting electrons), λ is a positive parameter, and $h(x)$ is a periodic C^1 function. Note that $h(x)$ is the projection of magnetic potential of an applied magnetic field to the tangent direction of a parametrized ring (see [8], [9]). We consider this functional on a space of 2π-periodic functions in $H^1_{\mathrm{loc}}(\mathbb{R})$. Then the Euler-Lagrange equation of this functional is given by

$$\begin{cases} D_h^2\psi + \lambda(1-|\psi|^2)\psi = 0, & x \in \mathbb{R}, \\ \psi(x+2\pi) = \psi(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

which is the Ginzburg-Landau equation of this model. One feature of this equation is that it is transformed into the equation

$$\begin{cases} u_{xx} + \lambda(1-|u|^2)u = 0, & x \in \mathbb{R}, \\ u(x+2\pi)\exp(2\pi\mu i) = u(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

by the change of variable

$$\psi = u\exp\left(i\int_0^x h(s)ds\right), \tag{1.3}$$

where

$$\mu := \frac{1}{2\pi}\int_0^{2\pi} h(s)ds. \tag{1.4}$$

Our goal is to completely solve (1.2) for each $\mu \in \mathbb{R}$ and $\lambda > 0$. We will also discuss the global structure of solutions to (1.2) for the parameters λ and μ.

We here give a remark on μ in (1.2). For each $\tilde{\mu} \in \mathbb{R}$, let $\tilde{\mu}_0$ be a constant such that $\tilde{\mu}_0 \in [-1/2, 1/2]$ and $\tilde{\mu}-\tilde{\mu}_0 \in \mathbb{Z}$. Since $\exp(2\pi\tilde{\mu}_0 i) = \exp(2\pi\tilde{\mu}i)$, all the solutions to (1.2) for $\mu = \tilde{\mu}_0$ are also all solutions to (1.2) for $\mu = \tilde{\mu}$. We then realize that it suffices to solve (1.2) for $\mu \in [-1/2, 1/2]$ instead of $\mu \in \mathbb{R}$. However we assume $\mu \in \mathbb{R}$ in this paper for a simple expression of each solution to (1.1) which is given by (1.3). We also note that given solution $u(x)$ of (1.2) the symmetry of the equation allows $u(x)e^{ic}$ and $u(x+c)$ to be solutions for any constant $c \in \mathbb{R}$. However we will not mention about this fact explicitly unless we need to state clearly.

As for a specific case $\mu \in \mathbb{Z}$, we note that a complete global bifurcation diagram for λ is obtained in the previous paper [4]. We will extend this study to the present case (see also [1] and [5]). However the bifurcation structure exhibits more complex in the presence of an additional parameter μ. Nonetheless we can see the global bifurcation structure by solving the equation (1.2) for any $\mu \in \mathbb{R}$ and $\lambda > 0$. The approach developed in [4] fortunately works in the present situation so that a small modification of the argument can provide an explicit expression of every solution. In consequence one can observe how the secondary bifurcating solution deforms as μ varies until it disappears through another bifurcation.

To achieve it, we first classify all the solutions to (1.2) according to their configuration. In what follows the idea of the classification is quite simple but crucial for drawing the whole bifurcation diagram. Here we exclude the trivial solution $u = 0$ and modify the classification found in [4] a little for convenience of dealing with the present problem. Thus all the nontrivial solutions to (1.2) are classified into three types as

(I) Solutions with zero.

(IIa) Solutions with constant amplitude.

(IIb) Solutions with nonconstant and nonvanishing amplitude.

Note that this classification also works in (1.1).

We here characterize solutions in each class. As will be discussed in the next section, the solution of Type (I) is written in the form

$$u(x) = \exp(ic)\phi(x)$$

where c is a constant in $\mathbb{R}$ and ϕ is a real-valued function. Thus the parameter μ must satisfy $2\mu \in \mathbb{Z}$ if solutions of Type (I) exist. In other words, solutions of Type (I) do not exist if $2\mu \notin \mathbb{Z}$. More precisely we will prove the following. There exist solutions of Type (I) which have even zeros in $[0, 2\pi)$ if and only if 2μ is even and $\lambda > 1$, otherwise, there exist solutions which have odd zeros in $[0, 2\pi)$ if and only if 2μ is odd and $\lambda > 1/4$.

Next we consider the solution of Type (IIa). It is easy to obtain the following nontrivial (constant amplitude) solution to (1.2)

$$u^{c}_{\lambda,\mu,m} := \sqrt{1-(m-\mu)^2/\lambda}\,\exp(i(m-\mu)x) \tag{1.5}$$

for each $m \in \mathbb{Z}$. This solution exists if and only if (μ, λ) satisfies

$$\lambda > \lambda_{\mu,m} := (m-\mu)^2.$$

It gives a solution to (1.1) as

$$\psi_m := u^{c}_{\lambda,\mu,m}(x) \exp\left(i \int_0^x h(s)ds\right)$$

where μ is defined in (1.4). For each m, this solution emerges from the trivial solution 0 when (μ, λ) crosses the curve $\lambda = \lambda_{\mu,m}$. The study of [10] tells a local bifurcation structure of (1.2) by using a standard local bifurcation analysis. As a result they showed a secondary bifurcation, that is, bifurcations from the nontrivial solution take place at

$$\lambda = \lambda_{\mu,m,n} := 3(m-\mu)^2 - n^2/2, \quad (n \in \mathbb{N}). \tag{1.6}$$

Besides the local bifurcation structure, we are interested in a global one of (1.2). Among other things it is interesting to show how the configuration of the secondary bifurcating solution changes as the parameters varies.

Finally we deal with solutions of Type (IIb). It is much more difficult than the other case. We will discuss it in § 3 and show that a Type (IIb) solution arises through a secondary bifurcation which exists in regions

$$D^{-}_{m,n} := \{(\mu, \lambda) : \mu < m - n/2,\ \lambda > \lambda_{\mu,m,n}\}, \tag{1.7}$$
$$D^{+}_{m,n} := \{(\mu, \lambda) : \mu > m + n/2,\ \lambda > \lambda_{\mu,m,n}\}, \tag{1.8}$$

for arbitrarily given $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. For fixed $\mu < m - n/2$ (resp. $\mu > m + n/2$), as λ increase in a neighborhood of the curve $\lambda = \lambda_{\mu,m,n}$, a secondary bifurcating branch emanates from a branch of a Type (IIa) solution at $\lambda = \lambda_{\mu,m,n}$. The secondary bifurcating branch is composed of a Type (IIb) solution. Similarly, for fixed $\lambda > n^2/4$, as μ increase in a neighborhood $D^-_{m,n} \cup D^+_{m-n,n}$, a secondary bifurcating branch emanates from a branch of a Type (IIa) solution at the curve $\lambda = \lambda_{\mu,m,n}$. A Type (IIb) solution for $(\mu, \lambda) \in D^-_{m,n}$ is the component of the secondary bifurcating branch. As μ increase through $\mu = m - n/2$, the Type (IIb) solution changes into another Type (IIb) solution for $(\mu, \lambda) \in D^+_{m-n,n}$ via a Type (I) solution. The branch ends up by connecting itself with the branch of a Type (IIa) solution at $\lambda = \lambda_{\mu,m-n,n}$.

2 Type (I) solutions.

In this section we treat the Type (I) solutions to (1.2). We will show that, for each $n \in \mathbb{N}$, there exists a solution to (1.2) which has n zero points on $[0, 2\pi)$ if and only if

$$\lambda > n^2/4, \qquad \mu = m + n/2 \quad (\forall m \in \mathbb{Z}). \tag{2.1}$$

Each solution is written in the form

$$u = u^{\mathrm{s}}_{\lambda,n}(x + \omega)\exp(ic), \quad u^{\mathrm{s}}_{\lambda,n} := k\sqrt{2/(1+k^2)}\,\mathrm{sn}(nK(k)x/\pi, k)$$

where $k \in (0, 1)$ is a unique solution to

$$\sqrt{1+k^2}K(k) = \pi\sqrt{\lambda}/n, \tag{2.2}$$

c and ω are arbitrary constant of $\mathbb{R}$, $\mathrm{sn}(x, k)$ is the Jacobi elliptic function whose inverse is given by

$$\mathrm{sn}^{-1}(u, k) = \int_0^u \frac{1}{\sqrt{1-\tau^2}\sqrt{1-k^2\tau^2}}d\tau,$$

and $K(k)$ is a complete elliptic integral

$$K(k) := \int_0^1 \frac{1}{\sqrt{1-\tau^2}\sqrt{1-k^2\tau^2}}d\tau.$$

Recall that $\mathrm{sn}(x, k)$ is extended to $\mathbb{R}$ with period $4K(k)$ and it is not difficult to show $\phi = u^{\mathrm{s}}_{\lambda,n}$ is a solution to the real-valued equation

$$\phi_{xx} + \lambda(1 - \phi^2)\phi = 0. \tag{2.3}$$

To achieve the above results, we first show that if a nontrivial solution to (1.2) has a zero point, then 2μ is an integer and the solution is a real-valued function multiplied by a complex constant. Consider a nontrivial solution $u(x)$ which vanishes at $x = x_0$. Denote $u(x) = u_1(x) + iu_2(x)$. Then (1.2) allows the expression

$$\begin{cases} (u_1)_{xx} + Q(x)u_1 = 0, & x \in \mathbb{R}, \\ (u_2)_{xx} + Q(x)u_2 = 0, & x \in \mathbb{R}, \end{cases} \quad Q(x) := \lambda(1 - |u(x)|^2).$$

Since $u_j(x_0) = u_j(x_0 + 2\pi) = 0$ $(j = 1, 2)$, each u_j is an eigenfunction of the operator

$$L := \frac{d^2}{dx^2} + Q(x), \quad D(L) = \{u \in H^2(x_0, x_0 + 2\pi) : u(x_0) = u(x_0 + 2\pi) = 0\}$$

corresponding to zero eigenvalue if $u_j \not\equiv 0$. It thus follows that $c_1 u_1 = c_2 u_2$ for some constants $c_1, c_2 \in \mathbb{R}$ $((c1, c2) \neq (0, 0))$ from the Sturm-Liouville theorem, which tells that the dimension of each eigenspace is one. Put $\phi := \sqrt{1 + c_1^2/c_2^2}\, u_1$ (or $\sqrt{1 + c_2^2/c_1^2}\, u_2$). Then the solution u is written in

$$u(x) = \phi(x) \exp(ic)$$

for a constant $c \in \mathbb{R}$. Thus the second condition of (1.2) implies

$$\phi(x + 2\pi) \exp(2\pi\mu i) = \phi(x). \tag{2.4}$$

Since $\phi(x)$ is real valued, 2μ must be an integer.

We next verify that any solution ϕ of (2.3) with (2.4) is written in the form $\phi = u^{\mathrm{s}}_{\lambda,n}$ up to translation and (2.1) is a necessary and sufficient condition of existence. It follows from an elementary argument of ordinary differential equations.

Let ϕ be a nontrivial solution to (2.3) with (2.4) which has zero points. Without loss of generality we may assume $\phi(0) = \phi(2\pi) = 0$ and $\phi_x(0) > 0$ because ϕ is not the trivial solution. Thus there exists a point $x_1 \in (0, 2\pi)$ such that

$$\phi_x(x_1) = 0, \qquad \phi_x(x) > 0 \quad \forall x \in [0, x_1).$$

Put $\alpha = \phi(x_1)$. Then the equation (2.3) implies

$$\frac{d\phi}{dx} = \sqrt{\lambda(\alpha^2 - \phi^2)(\beta^2 - \phi^2)/2} \quad \forall x \in [0, x_1]$$

where

$$\alpha^2 + \beta^2 = 2, \qquad 0 < \alpha < \beta. \tag{2.5}$$

Changing variable $\xi(x) := \phi(x)/\alpha$, we have

$$x = \frac{\sqrt{2}}{\beta\sqrt{\lambda}} \int_0^{\phi(x)/\alpha} \frac{1}{\sqrt{(1-\xi^2)(1-(\alpha/\beta)^2\xi^2)}}\, d\xi, \quad \forall x \in [0, x_1]$$

and hence

$$x_1 = \sqrt{2}K(k)/\beta\sqrt{\lambda}, \quad k := \alpha/\beta. \tag{2.6}$$

Therefore ϕ is written in the form

$$\phi(x) = \alpha\,\mathrm{sn}\,(K(k)x/x_1, k) \tag{2.7}$$

and this equality is satisfied on the whole $\mathbb{R}$. Since ϕ satisfies (2.4) and the Jacobi elliptic function $\mathrm{sn}(\cdot, k)$ also satisfies

$$\mathrm{sn}(x + 2K(k)n, k) = (-1)^n\,\mathrm{sn}(x, k), \quad x \in \mathbb{R}, \quad n \in \mathbb{N},$$

there exist $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ such that

$$x_1 = \pi/n, \quad \mu = m + n/2. \tag{2.8}$$

On the other hand, since α and β satisfy (2.5) and k is defined by $k = \alpha/\beta$, α and β are written in the form

$$\alpha = k\sqrt{2/(1+k^2)}, \quad \beta = \sqrt{2/(1+k^2)}.$$

Thus (2.7) implies the expression of $u^{\mathrm{s}}_{\lambda,n}$ and (2.6) changes into

$$x_1 = \sqrt{1+k^2}K(k)/\sqrt{\lambda}. \tag{2.9}$$

Since (2.8) and (2.9), k must satisfy (2.2).

Therefore if the equation (2.3) with (2.4) has a nontrivial solution with n zero points in $[0, 2\pi)$, the solution is $u^{\mathrm{s}}_{\lambda,n}$ (up to translation) and $k \in (0,1)$ satisfies (2.2) and $\mu = m + n/2$ ($m \in \mathbb{Z}$). On the other hand, it is clear that, for each $n \in \mathbb{N}$, $u^{\mathrm{s}}_{\lambda,n}$ solves the equation (2.3) with (2.4) if $k \in (0,1)$ satisfies (2.2) and $\mu = m + n/2$ ($m \in \mathbb{Z}$).

Now let us consider (2.2). Since

$$K(0) = \frac{\pi}{2}, \qquad \frac{dK}{dk} > 0, \qquad \lim_{k \to 1} K(k) = \infty,$$

(2.2) has a unique solution if and only if $n^2 < 4\lambda$. Accordingly we can assert that any solution ϕ of (2.3) with (2.4) is written in the form $\phi = u^{\mathrm{s}}_{\lambda,n}$ and the solution exists if and only if (2.1) is satisfied.

3 Type (IIb) solutions.

In this section we consider a solution with nonconstant amplitude but non-vanishing everywhere. The method developed in the previous paper [4] can still work in this present 2-parameters case.

Since $|u(x)| > 0$, we can write $u = w(x)\exp(i\theta(x))$ where $w(x) > 0$. Putting it into the equation (1.2) yields

$$w_{xx} - \theta_x^2 w + \lambda(1-w^2)w = 0 \quad x \in \mathbb{R}, \qquad (w^2\theta_x)_x = 0 \quad x \in \mathbb{R}. \tag{3.1}$$

Then the periodic condition in (1.2) is reduced to

$$\theta(x+2\pi) + 2\pi\mu = \theta(x) + 2m\pi, \tag{3.2}$$

for an integer m. Integrating the equation $(w^2\theta_x)_x = 0$, we have that $\theta_x = b/w^2$ for a constant $b \in \mathbb{R}$. Integrating this equality again and using (3.2), we obtain

$$2(m-\mu)\pi = b\int_0^{2\pi} \frac{1}{w(x)^2}\,dx.$$

Thus the equation (3.1) is reduced to

$$\begin{cases} w_{xx} - \dfrac{b^2}{w^3} + \lambda(1-w^2)w = 0, & x \in \mathbb{R}, \\ b = 2(m-\mu)\pi \Big/ \displaystyle\int_0^{2\pi} \frac{dx}{w(x)^2}, & \\ w(x+2\pi) = w(x), & x \in \mathbb{R}, \\ w(x) > 0, & x \in \mathbb{R}. \end{cases} \tag{3.3}$$

Then a solution ψ of (1.1) is obtained by solving the above equation and it is written in the form

$$\psi = w(x)\exp\left\{2(m-\mu)\pi i \int_0^x \frac{1}{w(s)^2}ds \Big/ \int_0^{2\pi} \frac{1}{w(s)^2}ds + i\int_0^x h(s)\,ds\right\}$$

We note that (3.3) has a constant solution

$$w = \sqrt{1-(m-\mu)^2/\lambda}$$

if $\lambda > (m-\mu)^2$. This gives a solution of Type (IIa). Since w stands for the amplitude of a solution ψ, we exclude this constant solution. Finding all the solution of (3.3), we also give an attention to a solution of a higher mode, that is, a $2\pi/n$-periodic solution of (3.3) for $n \in \mathbb{N}$. Consequently all Type

(IIb) solutions are obtained by solving the following system of equations for $w(x)$ and b:

$$\begin{cases} w_{xx} - \dfrac{b^2}{w^3} + \lambda(1-w^2)w = 0, & x \in \mathbb{R}, \\ w(x) > 0 & x \in \mathbb{R}, \\ T_w = 2\pi/n \end{cases} \tag{3.4}$$

and

$$b = 2(m-\mu)\pi \Big/ \int_0^{2\pi} \frac{1}{w(x)^2} dx, \tag{3.5}$$

for each $m \in \mathbb{Z}$, $n \in \mathbb{N}$, $\mu \in \mathbb{R}$, and $\lambda > 0$, where T_w denotes the fundamental period of $w(x)$.

The following result establishes not only the existence but also the configuration of every secondary bifurcating solution.

THEOREM 3.1 *For each $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, if and only if (μ, λ) belongs to $D^+_{m,n} \cup D^-_{m,n}$, which is defined in (1.7) and (1.8), there exists a solution*

$$\begin{aligned} u^{\circ}_{\lambda,\mu,m,n} &:= w(x)\exp(i\theta(x)) \\ w(x) &:= \sqrt{\frac{2}{3} + \frac{2n^2K(k)^2}{\lambda\pi^2}\left\{k^2 \operatorname{sn}^2\left(\frac{nK(k)}{\pi}x, k\right) - \frac{k^2+1}{3}\right\}}, \\ \theta(x) &:= 2(m-\mu)\pi\left(\int_0^{2\pi}\frac{dy}{w(y)^2}\right)^{-1}\int_0^x \frac{dy}{w(y)^2}, \end{aligned} \tag{3.6}$$

where $k \in (0,1)$ is a unique solution of

$$\begin{cases} 2(m-\mu)^2K(k)^2 - \lambda\gamma(k)\Pi\left(\beta(k)/\alpha(k) - 1, k\right)^2 \beta(k)/\alpha(k) = 0, \\ \alpha(k) > 0 \end{cases}$$

and α, β, γ, and Π are defined as

$$\Pi(\nu, k) := \int_0^1 \frac{1}{(1+\nu\tau^2)\sqrt{1-\tau^2}\sqrt{1-k^2\tau^2}}d\tau, \tag{3.7}$$

$$\alpha(k) := \frac{2}{3} - \frac{2n^2K(k)^2(k^2+1)}{3\lambda\pi^2}, \tag{3.8}$$

$$\beta(k) := \frac{2}{3} - \frac{2n^2K(k)^2(1-2k^2)}{3\lambda\pi^2}, \tag{3.9}$$

$$\gamma(k) := \frac{2}{3} - \frac{2n^2K(k)^2(k^2-2)}{3\lambda\pi^2}. \tag{3.10}$$

For $(\mu,\lambda)\in D^+_{m,n}\cup D^-_{m,n}$, *the solution* $u^{\mathrm{o}}_{\lambda,\mu,m,n}$ *satisfies*

$$u^{\mathrm{o}}_{\lambda,\mu,m,n}-u^{\mathrm{c}}_{\lambda,\mu,m}\to 0 \quad \text{uniformly on } \mathbb{R} \quad \text{as } \lambda\to\lambda_{\mu,m,n}, \tag{3.11}$$

$$u^{\mathrm{o}}_{\lambda,\mu,m,n}-u^{\mathrm{c}}_{\lambda,\mu,m}\to 0 \quad \text{uniformly on } \mathbb{R} \quad \text{as } \mu\to m\pm\sqrt{\lambda/3+n^2/6}, \tag{3.12}$$

$$u^{\mathrm{o}}_{\lambda,\mu,m,n}\pm iu^{\mathrm{s}}_{\lambda,n}\to 0 \quad \text{uniformly on } \mathbb{R} \quad \text{as } \mu\to m\pm n/2. \tag{3.13}$$

Moreover, for given $\mu\in\mathbb{R}$ *and* $\lambda>0$, *every solution of* (1.2) *except for* $u\equiv 0$ *is given by one of*

$$\begin{array}{ll} e^{ic}u^{\mathrm{s}}_{\lambda,n}(x+\omega) & \text{for } n\in\mathbb{N} \text{ such as } n^2/4<\lambda,\ n/2-\mu\in\mathbb{Z}, \\ e^{ic}u^{\mathrm{c}}_{\lambda,\mu,m}(x) & \text{for } m\in\mathbb{Z} \text{ such as } (m-\mu)^2<\lambda, \\ e^{ic}u^{\mathrm{o}}_{\lambda,\mu,m,n}(x+\omega) & \text{for } (m,n)\in\mathbb{Z}\times\mathbb{N} \text{ such as } n^2/4<3(m-\mu)^2-n^2/2<\lambda, \end{array}$$

where c *and* ω *are arbitrarily taken real numbers.*

COROLLARY 3.1 *Let* $\lambda>0$. *For given* 2π*-periodic* C^1*-function* h, *define* μ *as* (1.4). *Then each nontrivial solution to* (1.1) *is one of the following*

$$\begin{array}{ll} u^{s}_{\lambda,n}(x+\omega)\exp\left(i\int_0^x h(s)ds+ic\right) & \text{for } n\in\mathbb{N} \text{ such as } n^2/4<\lambda,\ n/2-\mu\in\mathbb{Z}, \\ u^{c}_{\lambda,\mu,m}(x)\exp\left(i\int_0^x h(s)ds+ic\right) & \text{for } m\in\mathbb{Z} \text{ such as } (m-\mu)^2<\lambda, \end{array}$$

and

$$u^{o}_{\lambda,\mu,m,n}(x+\omega)\exp\left(i\int_0^x h(s)ds+ic\right)$$
$$\text{for } (m,n)\in\mathbb{Z}\times\mathbb{N} \text{ such as } n^2/4<3(m-\mu)^2-n^2/2<\lambda,$$

where c *and* ω *are real numbers.*

4 Appendix.

We will give a sketch of the proof of Theorem 3.1. The readers can refer to [4] for the detailed argument.

We solve (3.4) (without considering (3.5)). Since w is a nonconstant periodic function in C^2, there exist $x_1,x_2\in\mathbb{R}$ such that $x_1<x_2$ and

$$w_x(x_1)=w_x(x_2)=0,\quad w_x(x)>0\quad (\forall x\in(x_1,x_2)). \tag{4.1}$$

Multiplying $2w_x$ to the equation in (3.4), we have

$$\frac{d}{dx}\left((w_x)^2 + \frac{b^2}{w^2} + \frac{\lambda}{2}\left(2w^2 - w^4\right)\right) = 0.$$

Thus

$$w_x(x)^2 = \frac{\lambda\{w(x)^2 - w(x_1)^2\}}{2w(x)^2 w(x_1)^2}\left(\{w(x)^2 + w(x_1)^2 - 2\}w(x)^2 w(x_1)^2 + \frac{2b^2}{\lambda}\right). \tag{4.2}$$

Put

$$\alpha = w(x_1)^2, \qquad \beta = w(x_2)^2,$$

and $x = x_2$ in (4.2). Then we obtain

$$(\beta + \alpha - 2)\alpha\beta + 2b^2/\lambda = 0,$$

which implies

$$\frac{2b^2}{\lambda} = \alpha\beta\gamma, \qquad \gamma := 2 - \alpha - \beta. \tag{4.3}$$

Introducing the new variable

$$v(x) := w(x)^2,$$

and substituting (4.3) into (4.2), we can easily verify

$$(v_x(x))^2 = 2\lambda(v(x) - \alpha)(v(x) - \beta)(v(x) - \gamma), \quad \forall x \in \mathbb{R}.$$

Since

$$0 < \alpha < v(x) < \beta, \quad v_x(x) > 0 \qquad (\forall x \in (x_1, x_2)),$$

the ordering $\gamma \geq \beta$ holds. In the sequel

$$\begin{cases} v_x(x) = \sqrt{2\lambda(v(x) - \alpha)(v(x) - \beta)(v(x) - \gamma)}, & \forall x \in [x_1, x_2], \\ 0 < \alpha < \beta \leq \gamma, \qquad \alpha + \beta + \gamma = 2. \end{cases} \tag{4.4}$$

Next we solve (4.4). By integration of (4.4)

$$x - x_1 = \frac{1}{\sqrt{2\lambda}}\int_\alpha^{v(x)} \frac{dy}{\sqrt{(y - \alpha)(y - \beta)(y - \gamma)}}, \quad \forall x \in [x_1, x_2]. \tag{4.5}$$

Changing the variable $y = \alpha + (\beta - \alpha)\tau^2$ in (4.5) an putting

$$k := \sqrt{(\beta - \alpha)/(\gamma - \alpha)}, \tag{4.6}$$

we see

$$\frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)}} = \frac{2d\tau}{\sqrt{(\gamma-\alpha)(1-\tau^2)(1-k^2\tau^2)}}.$$

Applying this to (4.5) yields

$$x - x_1 = \sqrt{\frac{2}{\lambda(\gamma-\alpha)}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{v(x)-\alpha}{\beta-\alpha}}, k\right), \quad \forall x \in [x_1, x_2]. \tag{4.7}$$

Thus on the interval $[x_1, x_2]$ it holds

$$w(x) = \sqrt{v(x)} = \sqrt{\alpha + (\beta-\alpha)\,\mathrm{sn}^2\left(\sqrt{\frac{\lambda(\gamma-\alpha)}{2}}(x-x_1), k\right)}. \tag{4.8}$$

Since this $w(x)$ is defined over $\mathbb{R}$ and periodic with period $2K(k)\sqrt{2/\lambda(\gamma-\alpha)}$ ($\mathrm{sn}^2(x,k)$ has a period $2K(k)$), $T_w = 2\pi/n$ implies

$$2K(k)\sqrt{\frac{2}{\lambda(\gamma-\alpha)}} = \frac{2\pi}{n}. \tag{4.9}$$

Combining (4.3), (4.6), and (4.9), we obtain the expressions (3.8), (3.9), and (3.10). In the sequel we obtained solutions of (3.4). In fact let $n \in \mathbb{N}$ and $\lambda > 0$. Then $(w(x), \lambda, b)$ satisfies (3.4) if and only if there exist $x_1 \in \mathbb{R}$ and $k \in (0,1)$ such that $\alpha = \alpha(k) > 0$ and

$$\begin{cases} w(x+x_1) = \sqrt{\alpha + (\beta-\alpha)\,\mathrm{sn}^2\left(\dfrac{nK(k)}{\pi}x, k\right)}, \\ b^2 = \dfrac{\lambda\alpha\beta\gamma}{2}. \end{cases} \tag{4.10}$$

Now we take the condition (3.5) into account. Since $T_w = 2\pi/n$ and a symmetry,

$$\int_0^{2\pi} \frac{dx}{w(x)^2} = \int_0^{2\pi} \frac{dx}{w(x+x_1)^2} = 2n\int_0^{\pi/n} \frac{dx}{w(x+x_1)^2}.$$

The similar argument used in the derivation of (4.5) and (4.7) leads us to

$$\begin{aligned} \int_0^{\pi/n} \frac{dx}{w(x+x_1)^2} &= \frac{1}{\sqrt{2\lambda}}\int_\alpha^\beta \frac{dy}{y\sqrt{(y-\alpha)(y-\beta)(y-\gamma)}} \\ &= \sqrt{\frac{2}{\lambda}}\frac{\Pi(\beta/\alpha-1, k)}{\alpha\sqrt{\gamma-\alpha}}, \end{aligned} \tag{4.11}$$

where Π is the complete elliptic integral defined in (3.7). Since $\alpha > 0$ and $\gamma - \alpha = 2n^2K(k)^2/\lambda\pi^2$, the equation (3.5) is written as

$$b = (m - \mu)\alpha K(k)/\Pi(\beta/\alpha - 1, k).$$

Substituting this into the second equation in (4.10), we can reduce our problem to solving the equation

$$2(m-\mu)^2K(k)^2 - \lambda\gamma\Pi\left(\beta/\alpha - 1, k\right)^2\beta/\alpha = 0 \tag{4.12}$$

under the constraint $\alpha > 0$. To simplify the notation in the rest of this paper, we denote the left hand of the above equation by $\rho(k, \lambda, \mu)$ for each $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, that is, we put

$$\rho(k, \lambda, \mu) := 2(m-\mu)^2K(k)^2 - \lambda\gamma\Pi\left(\beta/\alpha - 1, k\right)^2\beta/\alpha. \tag{4.13}$$

Summarizing the above argument, we can assert that for each given $n \in \mathbb{N}$, $m \in \mathbb{Z}$, $\mu \in \mathbb{R}$, $\lambda > 0$, and $x_1 \in \mathbb{R}$, every nonconstant solution of (3.4) with (3.5) is written as

$$w(x + x_1) = \sqrt{\alpha + (\beta - \alpha)\,\mathrm{sn}^2\left(\frac{nK(k)}{\pi}x, k\right)}, \quad b = \mathrm{sgn}(m-\mu)\sqrt{\frac{\lambda\alpha\beta\gamma}{2}}, \tag{4.14}$$

if the equation $\rho(k, \lambda, \mu) = 0$ has a solution $(k, \lambda) \in A$ where

$$A := \left\{(k, \lambda) \in (0,1) \times \mathbb{R}^+ \;:\; \alpha(k) > 0\right\}.$$

The following proposition guarantees the unique existence of a solution to $\rho = 0$.

PROPOSITION 4.1 *Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$.*

(i) *The equation (4.12) has a solution $(k, \lambda) = (k(\lambda, \mu), \lambda) \in A$ if*

$$(\mu, \lambda) \in D^-_{m,n} \cup D^+_{m,n}, \tag{4.15}$$

which are defined in (1.7) and (1.8). Moreover $k(\lambda, \mu)$ is unique for each $(\mu, \lambda) \in D^-_{m,n} \cup D^+_{m,n}$.

(ii) *Let $(\mu, \lambda) \in D^-_{m,n} \cup D^+_{m,n}$. Then*

$$k(\lambda, \mu) \to 0 \qquad \text{as } \lambda \to 3(\mu - m)^2 - n^2/2, \tag{4.16}$$

$$k(\lambda, \mu) \to 0 \qquad \text{as } \mu \to m \pm \sqrt{\lambda/3 + n^2/6}, \tag{4.17}$$

$$\alpha(k(\lambda, \mu)) \to 0 \qquad \text{as } \mu \to m \pm n/2. \tag{4.18}$$

(iii) *There is no solution to* (4.12) *in* A *if*

$$(\mu, \lambda) \notin \bigcup_{m \in \mathbb{Z},\, n \in \mathbb{N}} \left(D^-_{m,n} \cup D^+_{m,n}\right).$$

Proof of Proposition 4.1 (i). Let $k = k_\alpha(\lambda) \in (0,1)$ satisfy $\alpha(k) = 0$. It is easy to verify that $k = k_\alpha(\lambda)$ is uniquely determined for each $\lambda > n^2/4$ and A is written in the form

$$A = \{(k, \lambda) : 0 < k < k_\alpha(\lambda), \lambda > n^2/4\}.$$

We here remark that $A = \emptyset$ if $\lambda \le n^2/4$. By using the same manner as the proof of Proposition 3.1 (i) in [4], we can see

$$\rho(0, \lambda, \mu) = \frac{\pi^2}{6}\left(\frac{6(m-\mu)^2 - n^2}{2} - \lambda\right) \tag{4.19}$$

and

$$\lim_{k \uparrow k_\alpha(\lambda)} \rho(k, \lambda, \mu) = (4(m-\mu)^2 - n^2)\frac{K(k_\alpha(\lambda))^2}{2}. \tag{4.20}$$

Indeed, changing the variable $t = \sqrt{\nu+1}\tau/\sqrt{1-\tau^2}$ and $\nu = \tilde{\nu} - 1$ into (3.7), we have

$$\sqrt{\tilde{\nu}}\,\Pi(\tilde{\nu} - 1, k) = \int_0^\infty \frac{1}{1+t^2}\sqrt{\frac{\tilde{\nu} + t^2}{\tilde{\nu} + (1-k^2)t^2}}\,dt. \tag{4.21}$$

Clearly the integral kernel satisfies

$$0 < \frac{1}{1+t^2}\sqrt{\frac{\tilde{\nu} + t^2}{\tilde{\nu} + (1-k^2)t^2}} \le \frac{1}{1+t^2}\sqrt{\frac{1}{1 - k_\alpha(\lambda)^2}},$$
$$(\forall k \in [0, k_\alpha(\lambda)],\ \forall \tilde{\nu} \ge 0,\ \forall t \ge 0).$$

Thus it follows that

$$\sqrt{\tilde{\nu}}\,\Pi(\tilde{\nu} - 1, k) \to \pi/2 \quad (\tilde{\nu} \to \infty).$$

Since $\beta/\alpha \to \infty$ as $k \uparrow k_\alpha(\lambda)$, we obtain

$$\sqrt{\beta/\alpha}\,\Pi\left(\beta/\alpha - 1, k\right) \to \pi/2 \quad (k \uparrow k_\alpha(\lambda)) \tag{4.22}$$

and hence we obtain (4.20). Thus the function $\rho(\cdot, \cdot, \mu)$ is extended as a continuous function on $\overline{A} \setminus \{\lambda = n^2/4\}$. Consequently $\rho(0, \lambda, \mu)\rho(k_\alpha(\lambda), \lambda, \mu) < 0$ is satisfied if and only if

$$\frac{6(m-\mu)^2 - n^2}{2} < \lambda, \quad 0 < 4(m-\mu)^2 - n^2, \tag{4.23}$$

because the inequality $\lambda > n^2/4$ implies

$$\frac{6(m-\mu)^2 - n^2}{2} - \lambda < \frac{3(4(m-\mu)^2 - n^2)}{4}.$$

Therefore it follows from the continuity of $\rho(k,\lambda,\mu)$ that (4.12) has a solution $k = k(\lambda,\mu)$ if (μ,λ) satisfies (4.15) for each $n \in \mathbb{N}$ and $m \in \mathbb{Z}$.

The following lemma implies that a solution $k = k(\lambda,\mu)$ to (4.12) is unique for each (μ,λ) if it exists in $(0, k_\alpha(\lambda))$.

LEMMA 4.1 *If $k \in (0, k_\alpha(\lambda))$ satisfies $\rho(k,\lambda,\mu) = 0$, then*

$$\frac{\partial \rho}{\partial k}(k,\lambda,\mu) > 0.$$

The proof of Lemma 4.1 is performed literally in the same way as in [4], we omit it here (see the proof Lemma 3.5 in [4]). □

Proof of Proposition 4.1 (ii). We first show (4.16). It is clear that, for fixed μ which satisfies $(\mu - m)^2 > n^2/4$, the both $\rho(0,\lambda,\mu)$ and $\rho(k_\alpha(\lambda),\lambda,\mu)$ are strictly positive if $\lambda \in (n^2/4, \lambda_{\mu,m,n})$. From Lemma 4.1, it follows that

$$\rho(k,\lambda,\mu) > 0, \quad \forall k \in [0, k_\alpha(\lambda)], \quad \forall \lambda \in (n^2/4, \lambda_{\mu,m,n}) \tag{4.24}$$

and hence

$$\rho(k, \lambda_{\mu,m,n}, \mu) = \lim_{\lambda \uparrow \lambda_{\mu,m,n}} \rho(k,\lambda,\mu) \geq 0, \quad \forall k \in [0, k_\alpha(\lambda_{\mu,m,n})].$$

By using (4.19), (4.20), and Lemma 4.1 again, we can conclude that

$$\begin{cases} \rho(k, \lambda_{\mu,m,n}, \mu) > 0, & \forall k \in (0, k_\alpha(\lambda_{\mu,m,n})], \\ \rho(0, \lambda_{\mu,m,n}, \mu) = 0. \end{cases} \tag{4.25}$$

Let $\{\lambda_\sigma\}$ be any sequence satisfying $\lambda_\sigma \downarrow \lambda_{\mu,m,n}$ as $\sigma \to \infty$. Since $k(\lambda,\mu)$ is bounded and ρ is continuous, there exists a subsequence $\{\lambda_{\sigma'}\} \subset \{\lambda_\sigma\}$ such that a limit k_* of $k(\lambda_{\sigma'},\mu)$ as $\sigma' \to \infty$ exists in $[0, k_\alpha(\lambda_{\mu,m,n})]$ and $\rho(k_*, \lambda_{\mu,m,n}, \mu) = 0$. Thus the limit k_* must be 0 by (4.25). This concludes the proof of (4.16).

We next prove (4.17). That is similar to the above argument. Let $\lambda > n^2/4$ be fixed. If μ satisfies $(\mu - m)^2 > \lambda/3 + n^2/6$

$$\rho(k,\lambda,\mu) > 0, \quad \forall k \in [0, k_\alpha(\lambda)].$$

Because the both $\rho(0,\lambda,\mu)$ and $\rho(k_\alpha(\lambda),\lambda,\mu)$ are strictly positive and Lemma 4.1 is applied. Thus

$$\rho(k,\lambda,m\pm\sqrt{\lambda/3+n^2/6})\geq 0,\quad \forall k\in[0,k_\alpha(\lambda)].$$

Combining (4.19), (4.20), Lemma 4.1, and the above inequality, we obtain (4.17) by a similar argument to the proof of (4.16).

Now we observe (4.18), that is, we will prove

$$\lim_{\mu\to m\pm n/2} k(\lambda,\mu)=k_\alpha(\lambda) \tag{4.26}$$

for $(\mu,\lambda)\in D^+_{m,n}\cup D^-_{m,n}$. Let $\lambda>n^2/4$ be fixed. The both $\rho(0,\lambda,\mu)$ and $\rho(k_\alpha(\lambda),\lambda,\mu)$ are strictly negative if μ satisfies

$$m-n/2<\mu<m+n/2.$$

Lemma 4.1 implies

$$\rho(k,\lambda,\mu)<0,\quad \forall k\in[0,k_\alpha(\lambda)],\quad \forall\mu\in(m-n/2,m+n/2). \tag{4.27}$$

Thus

$$\rho(k,\lambda,m\pm n/2)\leq 0,\quad \forall k\in[0,k_\alpha(\lambda)]. \tag{4.28}$$

Combining (4.19), (4.20), (4.28), and Lemma 4.1, we obtain

$$\begin{cases}\rho(k,\lambda,m\pm n/2)<0, & \forall k\in[0,k_\alpha(\lambda)),\\ \rho(k_\alpha(\lambda),\lambda,m\pm n/2)=0, & \end{cases} \tag{4.29}$$

Therefore it follows that $k(\lambda,\mu)\to k_\alpha(\lambda)$ as $\mu\to m\pm n/2$ from a similar argument to the proof of (4.16).

□

Proof of Proposition 4.1 (iii). Let $m\in\mathbb{Z}$ and $n\in\mathbb{N}$ be fixed. As mentioned in the proof of (i), $A=\emptyset$ if $\lambda\leq n^2/4$. Thus it suffices to prove $\rho(k,\lambda,\mu)\neq 0$ for $\forall k\in(0,k_\alpha(\lambda))$ if $(\mu,\lambda)\in\{(\mu,\lambda):\lambda>n^2/4\}\setminus(D^+_{m,n}\cup D^-_{m,n})$. Since (4.24) and (4.25), it is clear that $\rho(k,\lambda,\mu)>0$ for $\forall k\in(0,k_\alpha(\lambda))$ if $n^2/4<\lambda\leq\lambda_{\mu,m,n}$. On the other hand, it follows from (4.27) and (4.29) that $\rho(k,\lambda,\mu)<0$ for $\forall k\in(0,k_\alpha(\lambda))$ if $m-n/2\leq\mu\leq m+n/2$ and $\lambda>n^2/4$. Therefore (iii) was proved. □

Proof of Theorem 3.1. As mentioned above we proved that the nonconstant amplitude solution is written in the form (4.14) and Proposition 4.1 directly implies the existence condition and the nonexistence condition of the nonconstant amplitude solutions which are stated in Theorem 3.1. Substituting (3.8), (3.9), and $x_1 = 0$ into (4.14), we obtain (3.6). Now we verify (3.11), (3.12), and (3.13). In the rest of the proof, $w = w(x)$ denotes (4.14) with $x_1 = 0$ for simplicity.

We first prove (3.11). It follows from (4.16) that if $(\mu, \lambda) \in D^-_{m,n} \cup D^+_{m,n}$ and $\lambda \to \lambda_{\mu,m,n}$, then $k(\lambda, \mu) \to 0$. Thus

$$\alpha \to 2/3 - n^2/6\lambda_{\mu,m,n} = 1 - (\mu - m)^2/\lambda_{\mu,m,n} \qquad \text{as } \lambda \to \lambda_{\mu,m,n}, \tag{4.30}$$

$$\beta - \alpha \to 0 \qquad \text{as } \lambda \to \lambda_{\mu,m,n}. \tag{4.31}$$

For each $\ell \in \mathbb{Z}$, if $x \in [2\pi\ell, 2\pi(\ell+1)]$ then

$$\begin{aligned}\theta(x) - (m-\mu)x &= (m-\mu)\left\{2\pi \int_0^x \frac{1}{w(y)^2}dy \Big/ \int_0^{2\pi} \frac{1}{w(y)^2}dy - x\right\} \\ &= (m-\mu)\left\{2\pi \int_0^{x-2\pi\ell} \frac{1}{w(y)^2}dy \Big/ \int_0^{2\pi} \frac{1}{w(y)^2}dy + 2\pi\ell - x\right\}.\end{aligned}$$

Since

$$\sqrt{\alpha(k(\lambda,\mu))} \le w(x) \le \sqrt{\beta(k(\lambda,\mu))},$$

a simple calculation implies

$$\frac{\alpha-\beta}{\beta}(x-2\pi\ell) \le 2\pi \int_0^{x-2\pi\ell} \frac{1}{w(y)^2}dy \Big/ \int_0^{2\pi} \frac{1}{w(y)^2}dy + 2\pi\ell - x \le \frac{\beta-\alpha}{\alpha}(x-2\pi\ell)$$

and hence we obtain

$$\sup_{x\in[2\pi\ell,2\pi(\ell+1)]} |\theta(x) - (m-\mu)x| \le \frac{2\pi|m-\mu|(\beta-\alpha)}{\alpha}, \quad \forall \ell \in \mathbb{Z}.$$

It is clear that

$$\begin{aligned}\left|u^{\mathrm{o}}_{\lambda,\mu,m,n}(x) - u^{\mathrm{c}}_{\lambda,\mu,m}(x)\right| &= \left|w(x)\exp\left(i\left(\theta(x) - (m-\mu)x\right)\right) - \sqrt{1-(m-\mu)^2/\lambda}\right| \\ &\le |w(x)|\left|\exp\left(i\left(\theta(x)-(m-\mu)x\right)\right) - 1\right| \\ &\quad + \left|w(x) - \sqrt{1-(m-\mu)^2/\lambda}\right|\end{aligned}$$

and

$$\sup_{x\in\mathbb{R}} \left|w(x) - \sqrt{1-(m-\mu)^2/\lambda}\right| \to 0 \quad \text{as } \lambda \to \lambda_{\mu,m,n}$$

for (4.30) and (4.31). It is also clear that

$$\begin{aligned}|\exp(i(\theta(x)-(m-\mu)x))-1|^2 &= \{\cos(\theta(x)-(m-\mu)x)-1\}^2 \\ &\quad+\sin^2(\theta(x)-(m-\mu)x) \\ &\le 4\sin^4(\pi|m-\mu|(\beta-\alpha)/\alpha) \\ &\quad+\sin^2(2\pi|m-\mu|(\beta-\alpha)/\alpha)\end{aligned}$$

and hence

$$\sup_{x\in\mathbb{R}}|\exp(\theta(x)-(m-\mu)x)-1|\to 0 \quad \text{as } \lambda\to\lambda_{\mu,m,n}$$

for (4.31). Therefore (3.11) follows.

Similarly, (3.12) follows from (4.17).

Next we prove (3.13). Let $(\mu,\lambda)\in D^-_{m,n}\cup D^+_{m,n}$. First, we consider the limit of $w=w(x)$ as $\mu\to m\pm n/2$. Since $\alpha(k_\alpha(\lambda))=0$ and (4.18) implies $k(\lambda,\mu)\to k_\alpha(\lambda)$ as $\mu\to m\pm n/2$,

$$\beta-\alpha\to 2k_\alpha(\lambda)^2/(k_\alpha(\lambda)^2+1) \quad (\mu\to m\pm n/2)$$

and hence

$$w\to k_\alpha(\lambda)\sqrt{2/(k_\alpha(\lambda)^2+1)}\,|\mathrm{sn}\,(nK(k_\alpha(\lambda))x/\pi,k_\alpha(\lambda))| \tag{4.32}$$

uniformly for x as $\mu\to m\pm n/2$. On the other hand, $k=k_\alpha(\lambda)$ satisfies (2.2). Thus $u^{\mathrm{s}}_{\lambda,n}$ is written in the form

$$u^{\mathrm{s}}_{\lambda,n}(x)=k_\alpha(\lambda)\sqrt{2/(1+k_\alpha(\lambda)^2)}\,\mathrm{sn}(nK(k_\alpha(\lambda))x/\pi,k_\alpha(\lambda)).$$

Therefore we obtain the following: For any $\varepsilon>0$, there exists $\delta_0=\delta_0(\varepsilon)>0$ such that

$$\sup_{x\in\mathbb{R}}\big|w(x)-|u^{\mathrm{s}}_{\lambda,n}(x)|\big|\le\varepsilon \tag{4.33}$$

for $\forall\mu\in(m-n/2-\delta_0,m-n/2)\cup(m+n/2,m+n/2+\delta_0)$. We here remark that $u^{\mathrm{s}}_{\lambda,n}(x)$ can be defined independently of μ, however it is not a solution to (1.2) if $2\mu\notin\mathbb{Z}$. Since $u^{\mathrm{s}}_{\lambda,n}(x)$ is zero at $x=2\ell\pi/n$ $(\ell\in\mathbb{Z})$, we can verify that, for any $\varepsilon>0$, there exist $d_1=d_1(\varepsilon)$ and $\delta_1=\delta_1(\varepsilon)$ such that

$$\max_{\ell\in\mathbb{Z}}\sup_{|x-2\ell\pi/n|\le d_1}|w(x)|\le\varepsilon, \tag{4.34}$$

$$\max_{\ell\in\mathbb{Z}}\sup_{|x-2\ell\pi/n|\le d_1}|u^{\mathrm{s}}_{\lambda,n}(x)|\le\varepsilon \tag{4.35}$$

for $\forall\mu \in (m-n/2-\delta_1, m-n/2) \cup (m+n/2, m+n/2+\delta_1)$.

Next we deal with θ. Combining (3.8), (3.10), and (4.11), we have

$$\int_0^{2\pi} \frac{1}{w(x)^2}dx = \frac{2\pi\Pi(\beta/\alpha-1,k)}{\alpha K(k)}.$$

We also obtain that for each $\ell \in \mathbb{Z}$ if $x \in (2\pi\ell/n, 2\pi(\ell+1)/n)$ then

$$\int_0^x \frac{1}{w(s)^2}ds = \frac{(2\ell+1)\pi\Pi(\beta/\alpha-1,k)}{n\alpha K(k)} + \int_{(2\ell+1)\pi/n}^x \frac{1}{w(s)^2}ds,$$

else if $x = 2\pi\ell/n$ then

$$\int_0^x \frac{1}{w(s)^2}ds = \frac{2\ell\pi\Pi(\beta/\alpha-1,k)}{n\alpha K(k)}.$$

Thus

$$\theta(x) = \begin{cases} \dfrac{(m-\mu)(2\ell+1)\pi}{n} + \dfrac{(m-\mu)\alpha K(k)}{\Pi(\beta/\alpha-1,k)} \displaystyle\int_{(2\ell+1)\pi/n}^x \frac{1}{w(s)^2}ds \\ \qquad\qquad\qquad \text{if } x \in (2\pi\ell/n, 2\pi(\ell+1)/n), \quad (\ell \in \mathbb{Z}), \\ \dfrac{(m-\mu)\ell\pi}{n} \quad \text{if } x = 2\pi\ell/n, \quad (\ell \in \mathbb{Z}). \end{cases} \tag{4.36}$$

The limit of θ as $\mu \to m \pm n/2$ is characterized as follows. Since (4.22), it holds that

$$\alpha/\Pi(\beta/\alpha-1,k) \to 0 \quad (k \uparrow k_\alpha(\lambda)). \tag{4.37}$$

Let $d_2 > 0$. It is also clear that

$$\left|\int_{(2\ell+1)\pi/n}^x \frac{1}{w(s)^2}ds\right| \le \frac{\pi}{n}\frac{1}{w(d_2)^2}, \quad \forall x \in [2\pi\ell/n + d_2, 2\pi(\ell+1)/n - d_2].$$

Thus it follows from (4.32), (4.36), (4.37), and the above inequality that for any $\varepsilon > 0$ there exists $\delta_2 = \delta_2(\varepsilon, d_2) > 0$ such that

$$\sup_{x\in(2\pi\ell/n+d_2, 2\pi(\ell+1)/n-d_2)} |\theta(x) - (2\ell+1)\pi/2| \le \varepsilon \qquad \forall\mu \in (m-n/2-\delta_2, m-n/2), \tag{4.38}$$

$$\sup_{x\in(2\pi\ell/n+d_2, 2\pi(\ell+1)/n-d_2)} |\theta(x) + (2\ell+1)\pi/2| \le \varepsilon \qquad \forall\mu \in (m+n/2, m+n/2+\delta_2). \tag{4.39}$$

Now we estimate

$$|u^{\mathrm{o}}_{\lambda,\mu,m,n}(x) \pm iu^{\mathrm{s}}_{\lambda,n}(x)|^2 \\ = w(x)^2\cos(\theta(x))^2 + \left\{w(x)\sin(\theta(x)) \pm u^{\mathrm{s}}_{\lambda,n}(x)\right\}^2. \tag{4.40}$$

For $\varepsilon > 0$, put

$$\delta = \delta(\varepsilon) := \min\{\delta_0(\varepsilon), \delta_1(\varepsilon), \delta_2(\varepsilon, d_1(\varepsilon))\}. \tag{4.41}$$

Then it is clear that, if $\mu \in (m-n/2-\delta, m-n/2) \cup (m+n/2, m+n/2+\delta)$,

$$\begin{aligned}\sup_{x\in\mathbb{R}} w(x)^2\cos^2(\theta(x)) &\le \max_{\ell\in\mathbb{Z}} \sup_{|x-2\ell\pi/n|\le d_1} w(x)^2\cos^2(\theta(x)) \\ &\quad + \max_{\ell\in\mathbb{Z}} \sup_{x\in(2\pi\ell/n+d_1, 2\pi(\ell+1)/n-d_1)} w(x)^2\cos^2(\theta(x)) \\ &\le \varepsilon^2 + \max_{\ell\in\mathbb{Z}} \beta\cos^2((2\ell+1)\pi/2+\varepsilon) \\ &\le \varepsilon^2 + 2\sin^2(\varepsilon). \end{aligned} \tag{4.42}$$

Next we estimate the second term of the right hand side of (4.40). Since (4.34) and (4.35), we have the following estimate in neighborhoods of zero points of $u^{\mathrm{s}}_{\lambda,n}$:

$$\max_{\ell\in\mathbb{Z}} \sup_{|x-2\ell\pi/n|\le d_1} \left\{w(x)\sin(\theta(x)) \pm u^{\mathrm{s}}_{\lambda,n}(x)\right\}^2 \le 4\varepsilon^2$$

if $\mu \in (m-n/2-\delta, m-n/2) \cup (m+n/2, m+n/2+\delta)$. In the complement to the neighborhoods of the zero points of $u^{\mathrm{s}}_{\lambda,n}$, the second term of the right hand side of (4.40) is estimated as follows: Let $\mu \in (m-n/2-\delta, m-n/2)$ and $x \in (2\pi\ell/n + d_1, 2\pi(\ell+1)/n - d_1)$. Then

$$\begin{aligned}&|w(x)\sin(\theta(x)) - u^{\mathrm{s}}_{\lambda,n}(x)| \\ &\le \left|w(x) - |u^{\mathrm{s}}_{\lambda,n}(x)|\right| |\sin(\theta(x))| + \left|u^{\mathrm{s}}_{\lambda,n}(x)\right| \left|\sin(\theta(x)) - (-1)^\ell\right| \\ &\le \varepsilon + k_\alpha(\lambda)\sqrt{2/(1+k_\alpha(\lambda)^2)} \left|\sin(\theta(x)) - (-1)^\ell\right| \\ &\le \varepsilon + \sqrt{2}\left|\sin(\theta(x)) - (-1)^\ell\right|. \end{aligned}$$

Since $\theta(x)$ has an estimate (4.38) and $\sin(\theta(x)) - (-1)^\ell$ is estimated as

$$\begin{aligned}\left|\sin(\theta(x)) - (-1)^\ell\right| &= \left|(-1)^\ell\{\cos(\theta(x) - (2\ell+1)\pi/2) - 1\}\right| \\ &= 2\sin^2((\theta(x) - (2\ell+1)\pi/2)/2) \\ &\le 2\sin^2(\varepsilon/2),\end{aligned}$$

we obtain

$$\sup_{x\in(2\pi\ell/n+d_1,2\pi(\ell+1)/n-d_1)} |w(x)\sin(\theta(x)) - u^{\mathrm{s}}_{\lambda,n}(x)| \le \varepsilon + 2\sqrt{2}\sin^2(\varepsilon/2). \tag{4.43}$$

Similarly, if $\mu \in (m+n/2, m+n/2+\delta)$, it holds that

$$\sup_{x\in(2\pi\ell/n+d_1,2\pi(\ell+1)/n-d_1)} |w(x)\sin(\theta(x)) + u^{\mathrm{s}}_{\lambda,n}(x)| \le \varepsilon + 2\sqrt{2}\sin^2(\varepsilon/2). \tag{4.44}$$

Combining (4.42) and (4.43), we obtain that there exists $C > 0$ such that for any $\varepsilon > 0$

$$\sup_{x\in\mathbb{R}} \left|u^{\mathrm{o}}_{\lambda,\mu,m,n}(x) - iu^{\mathrm{s}}_{\lambda,n}(x)\right| \le C\varepsilon, \quad \forall\mu \in (m-n/2-\delta, m-n/2).$$

On the other hand, (4.42) and (4.44) imply that, for any $\varepsilon > 0$,

$$\sup_{x\in\mathbb{R}} \left|u^{\mathrm{o}}_{\lambda,\mu,m,n}(x) + iu^{\mathrm{s}}_{\lambda,n}(x)\right| \le C\varepsilon, \quad \forall\mu \in (m+n/2, m+n/2+\delta).$$

Therefore it completes the proof of (3.13). $\square$

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