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<tr>
<td>著者</td>
<td>Takaki, Osamu</td>
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Representation of successor-type proof-theoretically regular ordinals via limits

O.Takaki (高木理)*
Faculty of Science, Kyoto Sangyo Univ. (京都産業大学・理学部)

Abstract
In this paper, we extend a result in [Ta04], that is, we show that every successor-type proof-theoretically regular ordinal has its own representation as a limit of a sequence consisting of certain canonical elements.

1 Introduction
In our previous paper [Ta04], we defined a set $\text{Reg}(\mathcal{T}(M))$ based on $\mathcal{T}(M)$, which was a primitive recursive well-ordered set defined by M.Rathjen to establish the proof theoretic ordinal of KRM. We call elements of $\text{Reg}(\mathcal{T}(M))$ "proof-theoretically regular ordinals based on $\mathcal{T}(M)$ (ptros)". In [Ta04], we also characterized some sort of ptros as proof-theoretical analogues of (hyper) inaccessible cardinals up to the least Mahlo cardinal. Since the characterization is based on $\text{Reg}(\mathcal{T}(M))$ as an analogue of the set of regular cardinals up to the least Mahlo cardinal, it is significant to characterize ptros and find the relationship between $\text{Reg}(\mathcal{T}(M))$ and the set of regular cardinals up to the least Mahlo cardinal. For these purpose, we are in the process of establishing a "canonical" fundamental sequence of each limit-type element of $\mathcal{T}(M)$. A coherent way to establish an appropriate fundamental sequence of each limit-type element of $\mathcal{T}(M)$ can be expected to be a coherent way to re-construct each element of $\mathcal{T}(M)$ as a more familiar concept, and hence, it turns out to provide a desirable characterization of ptros as proof-theoretical analogues of regular cardinals.

In this paper, we extend a result in [Ta04] (cf. Theorem 2.11 in this paper). The result gives a fundamental sequence of the least "successor-type" ptro $\psi_M^{\Omega_1}(\Omega_1)$, by which $\psi_M^{\Omega_1}(\Omega_1)$ can be characterized as the least fixed point of the function enumerating strongly critical ordinals. We here give a similar sequence $\{\gamma_n\}_{n \in \omega}$ of every successor-type ptro $\gamma$. Compared with the previous result in [Ta04], the proof of the property that $\gamma = \lim_{n \in \omega} \gamma_n$ needs some special attentions. Therefore, for (a certain type of) a given ordinal $\delta$ less than $\gamma$, we construct a labeled tree informing us the number $n \in \omega$ with $\delta < \gamma_n$.

In Section 2, we explain several definitions and results in [Ta04]. In Section 3, we show the extended version of the result above.

*email address: tkk@cc.kyoto-su.ac.jp
2 Preliminaries

In this paper, $M$ denotes the least Mahlo cardinal, and $\varphi$ the veblin function. For more details, one can refer to [Bu92], [Ra98], [Ra99] or [Ta04].

Definition 2.1 (Rathjen98,99). For given ordinals $\alpha$ and $\beta$, we define a set $C^M(\alpha, \beta)$ called a Skolem’s hull as well as functions $\chi^\alpha$ and $\psi^\alpha_M$ called collapsing functions, as follows:

(M1) $\beta \cup \{0, M\} \subset C^M(\alpha, \beta)$;
(M2) $\gamma = \gamma_1 + \gamma_2$ & $\gamma_1, \gamma_2 \in C^M(\alpha, \beta)$ $\Rightarrow$ $\gamma \in C^M(\alpha, \beta)$;
(M3) $\gamma = \varphi \gamma_1 \gamma_2$ & $\gamma_1, \gamma_2 \in C^M(\alpha, \beta)$ $\Rightarrow$ $\gamma \in C^M(\alpha, \beta)$;
(M4) $\gamma = \Omega_{\gamma_1}$ & $\gamma_1 \in C^M(\alpha, \beta)$ $\Rightarrow$ $\gamma \in C^I(\alpha, \beta)$;
(M5) $\gamma = \chi^\xi(\delta)$ & $\xi, \delta \in C^M(\alpha, \beta)$ & $\xi < \alpha$ & $\xi \in C^M(\xi, \gamma)$ & $\delta < M$ $\Rightarrow$ $\gamma \in C^M(\alpha, \beta)$
(M6) $\gamma = \psi^\xi_M(\kappa)$ & $\xi, \kappa \in C^M(\alpha, \beta)$ & $\xi < \alpha$ & $\kappa \in C^M(\xi, \gamma)$ $\Rightarrow$ $\gamma \in C^M(\alpha, \beta)$;
$\chi^\alpha(\delta) \simeq$ the $\delta^{th}$ regular cardinal $\pi < M$ with $C^M(\alpha, \pi) \cap M = \pi$;
$\psi^\alpha_M(\kappa) \simeq \min \{ \rho < \kappa : C^M(\alpha, \rho) \cap \kappa = \rho \land \kappa \in C^M(\alpha, \rho) \}.$

Definition 2.2
(i) $\gamma =_{nf} \alpha + \beta : \Leftrightarrow \gamma = \alpha + \beta$ & $\gamma > \alpha \geq \beta$ & $\beta$ is an additive principal number.
(ii) $\gamma =_{nf} \varphi \alpha \beta : \Leftrightarrow \gamma = \varphi \alpha \beta$ & $\alpha, \beta < \gamma$.
(iii) $\gamma =_{nf} \Omega_{\alpha} : \Leftrightarrow \gamma = \Omega_{\alpha}$ & $\alpha < \gamma$.
(iv) $\gamma =_{nf} \psi^\alpha_M(\kappa) : \Leftrightarrow \gamma = \psi^\alpha_M(\kappa)$ & $\alpha \in C^I(\alpha, \gamma)$.
(v) $\gamma =_{nf} \chi^\alpha(\beta) : \Leftrightarrow \gamma = \chi^\alpha(\beta)$ & $\beta < \gamma$ & $\alpha \in C^M(\alpha, \beta)$.

Definition 2.3 (Rathjen95,98). We define a set $T(M)$ called an elementary ordinal representation system for KPM and the degree $d(\alpha) < \omega$ of each element $\alpha$ of $T(M)$, as follows:
(i) $0, M \in T(M)$ & $d(0) = d(M) = 0$;
(ii) ($\gamma =_{nf} \alpha + \beta$ & $\alpha, \beta \in T(M)$ )
$\Rightarrow$ ($\gamma \in T(M)$ & $d(\gamma) = \max\{d(\alpha), d(\beta)\} + 1$ );
(iii) ($\gamma =_{nf} \varphi \alpha \beta$ & $\alpha, \beta \in T(M)$ & ($\gamma < M$ or $\alpha = 0$))
$\Rightarrow$ ($\gamma \in T(M)$ & $d(\gamma) = \max\{d(\alpha), d(\beta)\} + 1$ );
(iv) ($\gamma =_{nf} \Omega_{\alpha} < M$ & $\alpha > 0$ & $\alpha \in T(M)$ )
$\Rightarrow$ ($\gamma \in T(M)$ & $d(\gamma) = d(\alpha) + 1$ );
(v) ($\gamma =_{nf} \chi^\xi(\alpha)$ & $\xi, \alpha \in T(M)$ )
$\Rightarrow$ ($\gamma \in T(M)$ & $d(\gamma) = d(\alpha) + 1$ );
(vi) ($\gamma =_{nf} \psi^\alpha_M(\kappa)$ & $\kappa, \alpha \in T(M)$ )
$\Rightarrow$ ($\gamma \in T(M)$ & $d(\gamma) = \max\{d(\kappa), d(\alpha)\} + 1$ ).

Theorem 2.4 (Rathjen91, Buchholz92). (1) Each element of $T(M)$ has a unique representation with $0, M, +, \varphi, \Omega, \chi, \psi_M$.
(2) $|\text{KPM}| \leq \psi^{C^M+1}_M(\Omega_1) = T(M) \cap \Omega_1$, where $|\text{KPM}|$ denotes the proof theoretic ordinal of KPM.
Definition 2.5 An ordinal \( \gamma \) is called a proof-theoretically regular ordinal based on \( T(M) \) if \( \gamma \) is (expressed by) an element of \( T(M) \) having the form of \( \psi^\kappa_M(\Omega_1) \) with \( \kappa \in \text{Reg} \), where \( \text{Reg} \) denotes the set of regular cardinals.

Definition 2.6 A ptro \( \gamma \) is called a successor-type ptro if \( \gamma \) has an element \( \theta \in T(M) \) satisfying that \( \gamma \) is the least ptro larger than \( \theta \).

Definition 2.7 An ordinal \( \gamma \) is called a proof-theoretically inaccessible ordinal based on \( T(M) \) if \( \gamma \) is an element of \( \text{Reg}(T(M)) \) as well as the supremum of \( \text{Reg}(T(M)) \cap \gamma \), where \( \text{Reg}(T(M)) \) denotes the set of ptros based on \( T(M) \).

Theorem 2.8 (Takaki 04). All ptros are classified into the following two types:
(i) Successor-type ptros, which are of the form \( \psi^\alpha_M(\Omega_{\alpha+1}) \) or \( \psi^0_M(\Omega_1) \);
(ii) Proof-theoretically inaccessible ordinals, which are of the form \( \psi^\chi_M(\Omega_1) \) or \( \psi^M_M(\Omega_1) \).

Definition 2.9 For each \( n \in \omega \), we define \( \Psi_n \) by:
\[
\Psi_n = \begin{cases} 
0 & \text{if } n = 0; \\
\psi^{\Psi_{n-1}}_M(\Omega_1) & \text{if } n > 0.
\end{cases}
\]

Lemma 2.10 For each \( n \in \omega \), \( \Psi_n \in T(M) \) and \( \Psi_n < \Psi_{n+1} \).

The purpose of this paper is to extend the following theorem.

Theorem 2.11 (cf. Theorem 4 in [Ta04]). \( \psi^\Omega_M(\Omega_1) = \lim_{n \in \omega} \Psi_n \).

3 Representation of successor-type ptros

Definition 3.1 Let \( \alpha \) and \( \beta \) be elements of \( T(M) \). Then, for each \( n \in \omega \), we define an ordinal \( \Psi^n_\beta(\alpha) \), as follows:
\[
\Psi^n_\beta(\alpha) = \begin{cases} 
\beta & \text{if } n = 0; \\
\psi^{\Psi_{n-1}^\beta(\alpha)}_M(\Omega_{\alpha+1}) & \text{otherwise}.
\end{cases}
\]
In particular, \( \Psi_n(\alpha) := \Psi^n_0(\alpha) \).

\( \Psi^n_\beta(\alpha) \) also satisfies properties of \( \Psi_n \).

Lemma 3.2 For each \( \alpha, \beta \in T(M) \), if
\[
\beta < \psi^\alpha_M(\Omega_{\alpha+1}) \quad \text{and} \quad \forall \xi \left( \alpha < \xi \Rightarrow \beta \in C^M(\beta, \xi) \right)
\]
then, for each \( n \in \omega \),
\[
\Psi_n^\beta(\alpha) \in T(M) \quad \text{and} \quad \Psi_n^\beta(\alpha) < \Psi_{n+1}^\beta(\alpha).
\]
In particular, for each \( \alpha \in T(M) \) and \( n < \omega \),
\[
\Psi_n(\alpha) \in T(M) \quad \text{and} \quad \Psi_n(\alpha) < \Psi_{n+1}(\alpha).
\]

\( \Psi_n(\alpha) \in T(M) \) and \( \Psi_n(\alpha) < \Psi_{n+1}(\alpha) \).
Proof. This lemma is shown by checking the properties in (1) as well as

$$\forall \xi \ ( \alpha < \xi \Rightarrow \Psi_{n}^{\beta}(\alpha) \in C^{M}(\Psi_{n}^{\beta}(\alpha), \xi) ),$$

by using induction on $n$. $\square$

Now we give a representation of each successor-type ptro via $\Psi_{n}(\alpha)$ and the concept of limit.

**Theorem 3.3** For each $\alpha$ with $\psi_{M}^{\alpha+1}(\Omega_{1}) \in \mathcal{T}(M)$,

$$\psi_{M}^{\alpha+1}(\Omega_{1}) = \lim_{n \in \omega} \psi_{M}^{\Psi_{n}(\alpha)}(\Omega_{1}). \quad (2)$$

**Proof.** Since in [Ta04] we dealt with the case where $\alpha = 0$, it suffices to show (2) in the case where $\alpha > 0$.

[1] One can show that $\psi_{M}^{\alpha+1}(\Omega_{1}) \geq \lim_{n \in \omega} \psi_{M}^{\Psi_{n}(\alpha)}(\Omega_{1})$, by the following two claims.

**Claim 1** (cf. Lemmas 9.(3) and 11 in [Ta04]). For each $\alpha$ and $\beta$, $\psi_{M}^{\beta}(\Omega_{\alpha+1})$ is defined and $\Omega_{\alpha} < \psi_{M}^{\beta}(\Omega_{\alpha+1}) < \Omega_{\alpha+1}$.

**Claim 2** (cf. Lemma 10 in [Ta04]). For each $\alpha_{1}$, $\alpha_{2}$ and $\pi \in \text{Reg}$, if $\psi_{M}^{\alpha_{1}}(\pi)$ and $\psi_{M}^{\alpha_{2}}(\pi)$ are defined and if $\alpha_{1} \leq \alpha_{2}$, then $\psi_{M}^{\alpha_{1}}(\pi) \leq \psi_{M}^{\alpha_{2}}(\pi)$.

[2] In order to show that $\psi_{M}^{\alpha+1}(\Omega_{1}) \leq \lim_{n \in \omega} \psi_{M}^{\Psi_{n}(\alpha)}(\Omega_{1})$, we show that, for each $\gamma < \psi_{M}^{\alpha+1}(\Omega_{1})$, there exists an $n \in \omega$ with $\gamma \leq \psi_{M}^{\Psi_{n}(\alpha)}(\Omega_{1})$, by using induction on $d(\gamma)$.

Since it is easy to check the property above in any case except the case where $\gamma = \psi_{M}^{\xi}(\pi)$, we let $\gamma = \psi_{M}^{\xi}(\pi)$ in what follows.

For the given $\xi$ (and $\alpha$), we now define a labeled binary tree $T_{2}(\xi)$ (more precisely, $T_{2}(\xi, \alpha)$).

**Definition 3.4** We define a labeled binary tree $T_{2}(\xi)$ to satisfy the following property (i).

(i) For each node $s \in T_{2}(\xi)$, we denote the label of $s$ by $l_{s}$. Then, the label $l_{s}$ of each node in $T_{2}(\xi)$ is an element of $\mathcal{T}(M)$ satisfying:

  (i.i) $l_{s}$ is a subterm of $\xi$;
  (i.ii) $l_{s} \leq \xi$;
  (i.iii) $l_{s} \in C^{M}(\xi, \psi_{M}^{\xi}(\Omega_{1}))$.

\[ More\ precisely,\ we\ should\ assume\ that\ \gamma =_{\text{nf}} \psi_{M}^{\xi}(\pi).\ \text{However,\ we\ use\ only\ the\ symbol}\ \text{"=\"}\ \text{unless\ we\ need\ special\ attention.}\]
(ii) We define each node of $T_2(\xi)$ and its label, by using recursion on the distance from the root of $T_2(\xi)$, as follows.

(ii.0) If $s \in T_2(\xi)$ is the root, then $l_s$ is $\xi$.

Let $s$ be a node of $T_2(\xi)$. Then, we define the successors (successor nodes) of $s$ as well as their labels, according to the following conditions of $l_s$.

(ii.i) If $l_s = 0$, then $s$ is a leaf, that is, $s$ has no successor node.

(ii.ii) If $l_s = \delta + \eta$ or $l_s = \varphi \delta \eta$, then $s$ has successors $s_1$ and $s_2$, and

\[ l_{s_1} := \delta, \ l_{s_2} := \eta. \]

(ii.iii) If $l_s = \Omega_\beta$ and $l_s = \chi^\delta(\eta)$, then $s$ is a leaf.

(ii.iv) Let $l_s = \psi_M^\delta(\tau)$. In this case, $\tau \leq \Omega_{\alpha+1}$ since $l_s \leq \xi$.\n
(ii.iv.i) If $\tau < \Omega_{\alpha+1}$, then $s$ is a leaf.

(ii.iv.ii) If $\tau = \Omega_{\alpha+1}$, then $s$ has a successor $s_1$ and $l_{s_1} := \delta$.

**Claim 3** $T_2(\xi)$ is well-defined to be a finite tree.

(Proof of Claim 3: In order to show that $T_2(\xi)$ is well-defined, we show that, for each node $s$ of $T_2(\xi)$, $l_s$ satisfies the properties (i.i)~(i.iii) above, by using induction on the distance from the root to $s$.

If $s$ is the root, it is trivial since $l_s = \xi$.

We let $l_s = \psi_M^\delta(\Omega_{\alpha+1})$ and show that $\delta$ satisfies (i.i)~(i.iii), as follows. By induction hypothesis, $l_s$ is a subterm of $\xi$, $l_s \leq \xi$ and $l_s \in C^M(\xi, \gamma)$. Then, $\delta$ is also a subterm of $\xi$. On the other hand, $l_s > \Omega_1 > \gamma$. So, we have $\delta \in C^M(\xi, \gamma)$ and $\delta < \xi$ from Definition 2.1.(M5) and $l_s \in C^M(\xi, \gamma)$.

Any other case is similar to the case above.

Moreover, for each node $s \in T_2(\xi)$ and each successor $s'$ of $s$, it holds that $d(s) > d(s')$. So, $T_2(\xi)$ is finite.\)

**Definition 3.5** (1) A node $s$ of $T_2(\xi)$ ($=T_2(\xi, \alpha)$) is said to be *critical* when $l_s = \psi_M^\delta(\Omega_{\alpha+1})$ for some $\delta$. CN denotes the set of critical nodes (of $T_2(\xi)$).

(2) For each path $p$ of each subtree of $T_2(\xi)$, the number of critical nodes in $p$ is called the *weight* of $p$. Moreover, for each subtree $T$ of $T_2(\xi)$, the maximum number of weights of all paths of $T$ is called the *weight* of $T$, and denoted by $\text{wt}(T)$. Furthermore, for each node $s$ of $T_2(\xi)$, the weight of the subtree of $T_2(\xi)$ with root $s$ is called the *weight* of $s$, and denoted by $\text{wt}(s)$.

(3) For each subtree $T$ of $T_2(\xi)$, the maximum length of all paths of $T$ is called the *height* of $T$. Moreover, for each node $s$ of $T_2(\xi)$, the height of the subtree of $T_2(\xi)$ with root $s$ is called the *depth* of $s$, and denoted by $\text{dp}(s)$.

**Claim 4** For each node $s$ of $T_2(\xi)$, it holds that $l_s < \Psi_{\text{wt}(s)+1}(\alpha)$.

(Proof of Claim 4: We show the claim by induction on the depth of $s$.

(i) If $s$ is a leaf, then $l_s \leq \Omega_{\alpha}$. So, since $\Omega_{\alpha} < \Psi_n(\alpha)$ for each $n > 0$, we have $l_s < \Psi_1(\alpha)$.\)
(ii) Assume that $s$ is not any leaf. Then, $l_s = \text{n}_f \delta + \eta$, $l_s = \text{n}_f \varphi \delta \eta$, or $l_s = \text{n}_f \psi^\delta_M (\Omega_{\alpha+1})$.

Let $l_s = \text{n}_f \psi^\delta_M (\Omega_{\alpha+1})$. Then, $l_s \in \text{CN}$ and $s$ has one successor $s_1$ with $l_{s_1} = \delta$. Since $\text{wt}(s_1) = \text{wt}(s) - 1$ and $\text{dp}(s_1) < \text{dp}(s)$, the induction hypothesis implies that $l_{s_1} < \Psi_{\text{wt}(s)}(\alpha)$. On the other hand, since $l_s \in \mathcal{T}(M)$ and $\Psi_{\text{wt}(s)+1}(\alpha) \in \mathcal{T}(M)$, we have $l_s < \Psi_{\text{wt}(s)+1}(\alpha)$ (cf. Lemma 16 in [Ta04]).

Any other case is similar to or easier than the case above. \(\square\)

By Claim 4, we have $\xi < \Psi_{\text{wt}(T_2(\xi))+1}(\alpha)$, and hence, by Claim 2,

$$\gamma \leq \psi^\xi_M (\Omega_1).$$

So, the proof of Theorem 3.3 is completed. \(\square\)

We can also expect that each $\psi^\alpha_M (\Omega_1)$ has itself as its regular expression, that is, $\psi^\alpha_M (\Omega_1) \in \mathcal{T}(M)$. Unfortunately, we have not yet completed the proof of the property. However, it is not hard to show this property for each $\alpha$ less than a certain ordinal. For example, one can easily show the following proposition.

**Proposition 3.6** For each $\alpha \in \mathcal{T}(M)$ and $n \in \omega$, if $\alpha \in C^M (\Psi_n(\alpha), \psi^\Psi_n(\alpha)(\Omega_1))$, then

$$\psi^\alpha_M (\Omega_1) \in \mathcal{T}(M) \text{ and } \psi^\alpha_M (\Omega_1) < \psi^{\alpha+1}_M (\Omega_1).$$

By Theorem 3.3 and Proposition 3.6, each successor-type ptro $\psi^{\Omega_{\alpha+1}}_M (\Omega_1)$ has a fundamental sequence $\{\psi^\alpha_M (\Omega_1)\}_{\alpha \in \omega}$ if $\alpha \in C^M (\Psi_n(\alpha), \psi^\Psi_n(\alpha)(\Omega_1))$.

**Reference**


