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Representation of successor-type proof-theoretically regular ordinals via limits

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Abstract
In this paper, we extend a result in [Ta04], that is, we show that every successor-type proof-theoretically regular ordinal has its own representation as a limit of a sequence consisting of certain canonical elements.

1 Introduction

In our previous paper [Ta04], we defined a set \( \text{Reg}(\mathcal{T}(M)) \) based on \( \mathcal{T}(M) \), which was a primitive recursive well-ordered set defined by M. Rathjen to establish the proof theoretic ordinal of KRM. We call elements of \( \text{Reg}(\mathcal{T}(M)) \) “proof-theoretically regular ordinals based on \( \mathcal{T}(M) \) (ptros)”. In [Ta04], we also characterized some sort of ptros as proof-theoretical analogues of (hyper) inaccessible cardinals up to the least Mahlo cardinal. Since the characterization is based on \( \text{Reg}(\mathcal{T}(M)) \) as an analogue of the set of regular cardinals up to the least Mahlo cardinal, it is significant to characterize ptros and find the relationship between \( \text{Reg}(\mathcal{T}(M)) \) and the set of regular cardinals up to the least Mahlo cardinal. For these purpose, we are in the process of establishing a “canonical” fundamental sequence of each limit-type element of \( \mathcal{T}(M) \). A coherent way to establish an appropriate fundamental sequence of each limit-type element of \( \mathcal{T}(M) \) can be expected to be a coherent way to re-construct each element of \( \mathcal{T}(M) \) as a more familiar concept, and hence, it turns out to provide a desirable characterization of ptros as proof-theoretical analogues of regular cardinals.

In this paper, we extend a result in [Ta04] (cf. Theorem 2.11 in this paper). The result gives a fundamental sequence of the least “successor-type” ptro \( \psi^\Omega_1(\Omega_1) \), by which \( \psi^\Omega_1(\Omega_1) \) can be characterized as the least fixed point of the function enumerating strongly critical ordinals. We here give a similar sequence \( \{\gamma_n\}_{n \in \omega} \) of every successor-type ptro \( \gamma \). Compared with the previous result in [Ta04], the proof of the property that \( \gamma = \lim_{n \in \omega} \gamma_n \) needs some special attentions. Therefore, for (a certain type of) a given ordinal \( \delta \) less than \( \gamma \), we construct a labeled tree informing us the number \( n \in \omega \) with \( \delta < \gamma_n \).

In Section 2, we explain several definitions and results in [Ta04]. In Section 3, we show the extended version of the result above.

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2 Preliminaries

In this paper, $M$ denotes the least Mahlo cardinal, and $\varphi$ the veblin function. For more details, one can refer to [Bu92], [Ra98], [Ra99] or [Ta04].

**Definition 2.1** (Rathjen98,99). For given ordinals $\alpha$ and $\beta$, we define a set $C^M(\alpha, \beta)$ called a Skolem's hull as well as functions $\chi^\alpha$ and $\psi^\alpha_M$ called collapsing functions, as follows:

(M1) $\beta \cup \{0, M\} \subset C^M(\alpha, \beta)$;
(M2) $\gamma = \gamma_1 + \gamma_2 \& \gamma_1, \gamma_2 \in C^M(\alpha, \beta) \Rightarrow \gamma \in C^M(\alpha, \beta)$;
(M3) $\gamma = \varphi_1 \gamma_2 \& \gamma_1, \gamma_2 \in C^M(\alpha, \beta) \Rightarrow \gamma \in C^M(\alpha, \beta)$;
(M4) $\gamma = \Omega_{\gamma_1} \& \gamma_1 \in C^M(\alpha, \beta) \Rightarrow \gamma \in C^I(\alpha, \beta)$;
(M5) $\gamma = \chi^\xi(\delta) \& \xi, \delta \in C^M(\alpha, \beta) \& \xi < \alpha \& \xi \in C^M(\xi, \gamma) \& \delta < M \Rightarrow \gamma \in C^M(\alpha, \beta)$.
(M6) $\gamma = \psi^\xi_M(\kappa) \& \xi, \kappa \in C^M(\alpha, \beta) \& \xi < \alpha \& \xi \in C^M(\xi, \gamma) \Rightarrow \gamma \in C^M(\alpha, \beta)$;

$\chi^\alpha(\delta) \simeq$ the $\delta^{th}$ regular cardinal $\pi < M$ with $C^M(\alpha, \pi) \cap M = \pi$;

$\psi^\alpha_M(\kappa) \simeq \min\{\rho < \kappa : C^M(\alpha, \rho) \cap \kappa = \rho \land \kappa \in C^M(\alpha, \rho)\}$.

**Definition 2.2**

(i) $\gamma =_{nf} \alpha + \beta : \Leftrightarrow \gamma = \alpha + \beta \& \gamma > \alpha \geq \beta \& \beta$ is an additive principal number.
(ii) $\gamma =_{nf} \varphi \alpha \beta : \Leftrightarrow \gamma = \varphi \alpha \beta \& \alpha, \beta < \gamma$.
(iii) $\gamma =_{nf} \Omega \alpha : \Leftrightarrow \gamma = \Omega \alpha \& \alpha < \gamma$.
(iv) $\gamma =_{nf} \psi^\alpha_M(\kappa) : \Leftrightarrow \gamma = \psi^\alpha_M(\kappa) \& \alpha \in C^I(\alpha, \gamma)$.
(v) $\gamma =_{nf} \chi^\alpha(\beta) : \Leftrightarrow \gamma = \chi^\alpha(\beta) \& \beta < \gamma \& \alpha \in C^M(\alpha, \gamma)$.

**Definition 2.3** (Rathjen95,98). We define a set $T(M)$ called an elementary ordinal representation system for KPM and the degree $d(\alpha) < \omega$ of each element $\alpha$ of $T(M)$, as follows:

(i) $0, M \in T(M)$ & $d(0) = d(M) = 0$;
(ii) $(\gamma =_{nf} \alpha + \beta \& \alpha, \beta \in T(M))$

$\Rightarrow (\gamma \in T(M) \& d(\gamma) = \max\{d(\alpha), d(\beta)\} + 1)$;
(iii) $(\gamma =_{nf} \varphi \alpha \beta \& \alpha, \beta \in T(M) \& (\gamma < M \text{ or } \alpha = 0))$

$\Rightarrow (\gamma \in T(M) \& d(\gamma) = \max\{d(\alpha), d(\beta)\} + 1)$;
(iv) $(\gamma =_{nf} \Omega \alpha < M \& \alpha > 0 \& \alpha \in T(M))$

$\Rightarrow (\gamma \in T(M) \& d(\gamma) = d(\alpha) + 1)$;
(v) $(\gamma =_{nf} \chi^\alpha(\kappa) \& \xi, \alpha \in T(M))$

$\Rightarrow (\gamma \in T(M) \& d(\gamma) = d(\alpha) + 1)$;
(vi) $(\gamma =_{nf} \psi^\alpha_M(\kappa) \& \kappa, \alpha \in T(M))$

$\Rightarrow (\gamma \in T(M) \& d(\gamma) = \max\{d(\kappa), d(\alpha)\} + 1)$.

**Theorem 2.4** (Rathjen91, Buchholz92). (1) Each element of $T(M)$ has a unique representation with $0, M, +, \varphi, \Omega, \chi, \psi_M$.
(2) $\vert \text{KPM} \vert \leq \psi^{\Sigma_{M+1}}(\Omega_1) = T(M) \cap \Omega_1$, where $\vert \text{KPM} \vert$ denotes the proof theoretic ordinal of KPM.
Definition 2.5 An ordinal $\gamma$ is called a proof-theoretically regular ordinal based on $\mathcal{T}(M)$ if $\gamma$ is (expressed by) an element of $\mathcal{T}(M)$ having the form of $\psi_{M}^{\kappa}(\Omega_{1})$ with $\kappa \in \text{Reg}$, where $\text{Reg}$ denotes the set of regular cardinals.

Definition 2.6 A ptro $\gamma$ is called a successor-type ptro if $\gamma$ has an element $\theta \in \mathcal{T}(M)$ satisfying that $\gamma$ is the least ptro larger than $\theta$.

Definition 2.7 An ordinal $\gamma$ is called a proof-theoretically inaccessible ordinal based on $\mathcal{T}(M)$ if $\gamma$ is an element of $\text{Reg}(\mathcal{T}(M))$ as well as the supremum of $\text{Reg}(\mathcal{T}(M)) \cap \gamma$, where $\text{Reg}(\mathcal{T}(M))$ denotes the set of ptros based on $\mathcal{T}(M)$.

Theorem 2.8 (Takaki 04). All ptros are classified into the following two types:
(i) Successor-type ptros, which are of the form $\psi_{M}^{\Omega_{\alpha+1}}(\Omega_{1})$ or $\psi_{M}^{\omega}(\Omega_{1})$;
(ii) Proof-theoretically inaccessible ordinals, which are of the form $\psi_{M}^{\chi^{\omega}(\beta)}(\Omega_{1})$ or $\psi_{M}^{\omega}(\Omega_{1})$.

Definition 2.9 For each $n \in \omega$, we define $\Psi_{n}$ by:
$$\Psi_{n} = \begin{cases} 0 & \text{if } n = 0; \\ \psi_{M}^{\Psi_{n-1}^{\beta}(\alpha)}(\Omega_{\alpha+1}) & \text{if } n > 0. \end{cases}$$

Lemma 2.10 For each $n \in \omega$, $\Psi_{n} \in \mathcal{T}(M)$ and $\Psi_{n} < \Psi_{n+1}$.

The purpose of this paper is to extend the following theorem.

Theorem 2.11 (cf. Theorem 4 in [Ta04]). $\psi_{M}^{\Omega_{1}}(\Omega_{1}) = \lim_{n \in \omega} \Psi_{n}$.

3 Representation of successor-type ptros

Definition 3.1 Let $\alpha$ and $\beta$ be elements of $\mathcal{T}(M)$. Then, for each $n \in \omega$, we define an ordinal $\Psi_{n}^{\beta}(\alpha)$, as follows:
$$\Psi_{n}^{\beta}(\alpha) = \begin{cases} \beta & \text{if } n = 0; \\ \psi_{M}^{\Psi_{n-1}^{\beta}(\alpha)}(\Omega_{\alpha+1}) & \text{otherwise.} \end{cases}$$

In particular, $\Psi_{n}(\alpha) := \Psi_{n}^{0}(\alpha)$
$$\Psi_{n}^{\beta}(\alpha)$$ also satisfies properties of $\Psi_{n}$.

Lemma 3.2 For each $\alpha, \beta \in \mathcal{T}(M)$, if $\beta < \psi_{M}^{\beta}(\Omega_{\alpha+1})$ and $\forall \xi \ (\alpha < \xi \Rightarrow \beta \in C^{M}(\beta, \xi))$, then, for each $n \in \omega$,
$$\Psi_{n}^{\beta}(\alpha) \in \mathcal{T}(M) \quad \text{and} \quad \Psi_{n}^{\beta}(\alpha) < \Psi_{n+1}^{\beta}(\alpha).$$

In particular, for each $\alpha \in \mathcal{T}(M)$ and $n < \omega$,
$$\Psi_{n}(\alpha) \in \mathcal{T}(M) \quad \text{and} \quad \Psi_{n}(\alpha) < \Psi_{n+1}(\alpha).$$
Proof. This lemma is shown by checking the properties in (1) as well as

\[ \forall \xi \left( \alpha < \xi \Rightarrow \Psi_n^\beta(\alpha) \in C^M(\Psi_n^\beta(\alpha), \xi) \right), \]

by using induction on \( n \).

\end{proof}

Now we give a representation of each successor-type ptro via \( \Psi_n(\alpha) \) and the concept of limit.

**Theorem 3.3** For each \( \alpha \) with \( \psi_M^{\Omega_{\alpha+1}}(\Omega_1) \in T(M) \),

\[ \psi_M^{\Omega_{\alpha+1}}(\Omega_1) = \lim_{n \in \omega} \psi_M^{\Psi_n(\alpha)}(\Omega_1). \tag{2} \]

**Proof.** Since in [Ta04] we dealt with the case where \( \alpha = 0 \), it suffices to show (2) in the case where \( \alpha > 0 \).

[1] One can show that \( \psi_M^{\Omega_{\alpha+1}}(\Omega_1) \geq \lim_{n \in \omega} \psi_M^{\Psi_n(\alpha)}(\Omega_1) \), by the following two claims.

**Claim 1** (cf. Lemmas 9.(3) and 11 in [Ta04]). For each \( \alpha \) and \( \beta \), \( \psi_M^\beta(\Omega_{\alpha+1}) \) is defined and \( \Omega_{\alpha} < \psi_M^\beta(\Omega_{\alpha+1}) < \Omega_{\alpha+1} \).

**Claim 2** (cf. Lemma 10 in [Ta04]). For each \( \alpha_1, \alpha_2 \) and \( \pi \in \text{Reg} \), if \( \psi_M^{\alpha_1}(\pi) \) and \( \psi_M^{\alpha_2}(\pi) \) are defined and if \( \alpha_1 \leq \alpha_2 \), then \( \psi_M^{\alpha_1}(\pi) \leq \psi_M^{\alpha_2}(\pi) \).

[2] In order to show that \( \psi_M^{\Omega_{\alpha+1}}(\Omega_1) \leq \lim_{n \in \omega} \psi_M^{\Psi_n(\alpha)}(\Omega_1) \), we show that, for each \( \gamma < \psi_M^{\Omega_{\alpha+1}}(\Omega_1) \), there exists an \( n \in \omega \) with \( \gamma \leq \psi_M^{\Psi_n(\alpha)}(\Omega_1) \), by using induction on \( d(\gamma) \).

Since it is easy to check the property above in any case except the case where \( \gamma = \psi_M^\xi(\pi) \), we let \( \gamma = \psi_M^\xi(\pi) \) in what follows.

For the given \( \xi \) (and \( \alpha \)), we now define a labeled binary tree \( T_2(\xi) \) (more precisely, \( T_2(\xi, \alpha) \)).

**Definition 3.4** We define a labeled binary tree \( T_2(\xi) \) to satisfy the following property (i).

(i) For each node \( s \in T_2(\xi) \), we denote the label of \( s \) by \( l_s \). Then, the label \( l_s \) of each node in \( T_2(\xi) \) is an element of \( T(M) \) satisfying:

(i.i) \( l_s \) is a subterm of \( \xi \);

(i.ii) \( l_s \leq \xi \);

(i.iii) \( l_s \in C^M(\xi, \psi_M^\xi(\Omega_1)) \).

1 More precisely, we should assume that \( \gamma =_{nf} \psi_M^\xi(\pi) \). However, we use only the symbol "=" unless we need special attention.
(ii) We define each node of $T_2(\xi)$ and its label, by using recursion on the distance from the root of $T_2(\xi)$, as follows.

(ii.0) If $s \in T_2(\xi)$ is the root, then $l_s$ is $\xi$.

Let $s$ be a node of $T_2(\xi)$. Then, we define the successors (successor nodes) of $s$ as well as their labels, according to the following conditions of $l_s$.

(ii.i) If $l_s = 0$, then $s$ is a leaf, that is, $s$ has no successor node.

(ii.ii) If $l_s = \delta + \eta$ or $l_s = \varphi \delta \eta$, then $s$ has successors $s_1$ and $s_2$, and $l_{s_1} := \delta$, $l_{s_2} := \eta$.

(ii.iii) If $l_s = \Omega_\beta$ and $l_s = \chi^\delta(\eta)$, then $s$ is a leaf.

(ii.iv) Let $l_s = \psi^\delta_M(\tau)$. In this case, $\tau \leq \Omega_{\alpha+1}$ since $l_s \leq \xi$.

(ii.iv.i) If $\tau < \Omega_{\alpha+1}$, then $s$ is a leaf.

(ii.iv.ii) If $\tau = \Omega_{\alpha+1}$, then $s$ has a successor $s_1$ and $l_{s_1} := \delta$.

**Claim 3** $T_2(\xi)$ is well-defined to be a finite tree.

(Proof of Claim 3): In order to show that $T_2(\xi)$ is well-defined, we show that, for each node $s$ of $T_2(\xi)$, $l_s$ satisfies the properties (i.i)~(i.iii) above, by using induction on the distance from the root to $s$.

If $s$ is the root, it is trivial since $l_s = \xi$.

We let $l_s = \psi^\delta_M(\Omega_{\alpha+1})$ and show that $\delta$ satisfies (i.i)~(i.iii), as follows. By induction hypothesis, $l_s$ is a subterm of $\xi$, $l_s \leq \xi$ and $l_s \in C^M(\xi, \gamma)$. Then, $\delta$ is also a subterm of $\xi$. On the other hand, $l_s > \Omega_1 > \gamma$. So, we have $\delta \in C^M(\xi, \gamma)$ and $\delta < \xi$ from Definition 3.1.(M5) and $l_s \in C^M(\xi, \gamma)$.

Any other case is similar to the case above.

Moreover, for each node $s \in T_2(\xi)$ and each successor $s'$ of $s$, it holds that $d(s) > d(s')$. So, $T_2(\xi)$ is finite.

**Definition 3.5** (1) A node $s$ of $T_2(\xi)$ (=$T_2(\xi, \alpha)$) is said to be critical when $l_s = \psi^\delta_M(\Omega_{\alpha+1})$ for some $\delta$. CN denotes the set of critical nodes (of $T_2(\xi)$).

(2) For each path $p$ of each subtree of $T_2(\xi)$, the number of critical nodes in $p$ is called the weight of $p$. Moreover, for each subtree $T$ of $T_2(\xi)$, the maximum number of weights of all paths of $T$ is called the weight of $T$, and denoted by $\text{wt}(T)$. Furthermore, for each node $s$ of $T_2(\xi)$, the weight of the subtree of $T_2(\xi)$ with root $s$ is called the weight of $s$, and denoted by $\text{wt}(s)$.

(3) For each subtree $T$ of $T_2(\xi)$, the maximum length of all paths of $T$ is called the height of $T$. Moreover, for each node $s$ of $T_2(\xi)$, the height of the subtree of $T_2(\xi)$ with root $s$ is called the depth of $s$, and denoted by $\text{dp}(s)$.

**Claim 4** For each node $s$ of $T_2(\xi)$, it holds that $l_s < \Psi_{\text{wt}(s)+1}(\alpha)$.

(Proof of Claim 4): We show the claim by induction on the depth of $s$.

(i) If $s$ is a leaf, then $l_s \leq \Omega_\alpha$. So, since $\Omega_\alpha < \Psi_n(\alpha)$ for each $n > 0$, we have $l_s < \Psi_1(\alpha)$. 

Assume that \( s \) is not any leaf. Then, \( l_s =_{nf} \delta + \eta, l_s =_{nf} \varphi \delta \eta, \) or \( l_s =_{nf} \psi_M^\delta(\Omega_{\alpha+1}) \).

Let \( l_s =_{nf} \psi_M^\delta(\Omega_{\alpha+1}) \). Then, \( l_s \in \mathcal{C}N \) and \( s \) has one successor \( s_1 \) with \( l_{s_1} = \delta \).

Since \( \mathrm{wt}(s_1) = \mathrm{wt}(s) - 1 \) and \( \mathrm{dp}(s_1) < \mathrm{dp}(s) \), the induction hypothesis implies that \( l_{s_1} < \Psi_{\mathrm{wt}(s)}(\alpha) \). On the other hand, since \( l_s \in \mathcal{T}(M) \) and \( \Psi_{\mathrm{wt}(s)+1}(\alpha) \in \mathcal{T}(M) \), we have \( l_s < \Psi_{\mathrm{wt}(s)+1}(\alpha) \) (cf. Lemma 16 in [Ta04]).

Any other case is similar to or easier than the case above. \( \square \)

By Claim 4, we have \( \xi < \Psi_{\mathrm{wt}(T_2(\xi)) + 1}(\alpha) \), and hence, by Claim 2,

\[
\gamma \leq \psi_M^{\Psi_{\mathrm{wt}(T_2(\xi)) + 1}(\alpha)}(\Omega_1).
\]

So, the proof of Theorem 3.3 is completed. \( \square \)

We can also expect that each \( \psi_M^{\Psi_n(\alpha)}(\Omega_1) \) has itself as its regular expression, that is, \( \psi_M^{\Psi_n(\alpha)}(\Omega_1) \in \mathcal{T}(M) \). Unfortunately, we have not yet completed the proof of the property. However, it is not hard to show this property for each \( \alpha \) less than a certain ordinal. For example, one can easily show the following proposition.

**Proposition 3.6** For each \( \alpha \in \mathcal{T}(M) \) and \( n \in \omega \), if \( \alpha \in C^M(\Psi_n(\alpha), \psi_M^{\Psi_n(\alpha)}(\Omega_1)) \), then

\[
\psi_M^{\Psi_n(\alpha)}(\Omega_1) \in \mathcal{T}(M) \quad \text{and} \quad \psi_M^{\Psi_n(\alpha)}(\Omega_1) < \psi_M^{\Psi_{n+1}(\alpha)}(\Omega_1).
\]

By Theorem 3.3 and Proposition 3.6, each successor-type ptro \( \psi_M^{\Omega_{\alpha+1}}(\Omega_1) \) has a fundamental sequence \( \{\psi_M^{\Psi_n(\alpha)}(\Omega_1)\}_{n \in \omega} \) if \( \alpha \in C^M(\Psi_n(\alpha), \psi_M^{\Psi_n(\alpha)}(\Omega_1)) \).

**Reference**


