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Representation of successor-type proof-theoretically regular ordinals via limits

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Abstract
In this paper, we extend a result in [Ta04], that is, we show that every successor-type proof-theoretically regular ordinal has its own representation as a limit of a sequence consisting of certain canonical elements.

1 Introduction
In our previous paper [Ta04], we defined a set $\text{Reg}(\mathcal{T}(M))$ based on $\mathcal{T}(M)$, which was a primitive recursive well-ordered set defined by M.Rathjen to establish the proof theoretic ordinal of KRM. We call elements of $\text{Reg}(\mathcal{T}(M))$ "proof-theoretically regular ordinals based on $\mathcal{T}(M)$ (ptros)". In [Ta04], we also characterized some sort of ptros as proof-theoretical analogues of (hyper) inaccessible cardinals up to the least Mahlo cardinal. Since the characterization is based on $\text{Reg}(\mathcal{T}(M))$ as an analogue of the set of regular cardinals up to the least Mahlo cardinal, it is significant to characterize ptros and find the relationship between $\text{Reg}(\mathcal{T}(M))$ and the set of regular cardinals up to the least Mahlo cardinal. For these purpose, we are in the process of establishing a "canonical" fundamental sequence of each limit-type element of $\mathcal{T}(M)$. A coherent way to establish an appropriate fundamental sequence of each limit-type element of $\mathcal{T}(M)$ can be expected to be a coherent way to re-construct each element of $\mathcal{T}(M)$ as a more familiar concept, and hence, it turns out to provide a desirable characterization of ptros as proof-theoretical analogues of regular cardinals.

In this paper, we extend a result in [Ta04] (cf. Theorem 2.11 in this paper). The result gives a fundamental sequence of the least "successor-type" ptro $\psi^{\Omega_{1}}_{M}(\Omega_{1})$, by which $\psi^{\Omega_{1}}_{M}(\Omega_{1})$ can be characterized as the least fixed point of the function enumerating strongly critical ordinals. We here give a similar sequence $\{\gamma_{n}\}_{n\in\omega}$ of every successor-type ptro $\gamma$. Compared with the previous result in [Ta04], the proof of the property that $\gamma = \lim_{n\in\omega}\gamma_{n}$ needs some special attentions. Therefore, for (a certain type of) a given ordinal $\delta$ less than $\gamma$, we construct a labeled tree informing us the number $n \in \omega$ with $\delta < \gamma_{n}$.

In Section 2, we explain several definitions and results in [Ta04]. In Section 3, we show the extended version of the result above.

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2 Preliminaries

In this paper, $M$ denotes the least Mahlo cardinal, and $\varphi$ the Veblen function. For more details, one can refer to [Bu92], [Ra98], [Ra99] or [Ta04].

**Definition 2.1** (Rathjen98,99). For given ordinals $\alpha$ and $\beta$, we define a set $C^M(\alpha, \beta)$ called a Skolem’s hull as well as functions $\chi^\alpha$ and $\psi^\alpha_M$ called collapsing functions, as follows:

(M1) $\beta \cup \{0, M\} \subseteq C^M(\alpha, \beta)$;
(M2) $\gamma = \gamma_1 + \gamma_2 \& \gamma_1, \gamma_2 \in C^M(\alpha, \beta) \Rightarrow \gamma \in C^M(\alpha, \beta)$;
(M3) $\gamma = \varphi \gamma_1 \gamma_2 \& \gamma_1, \gamma_2 \in C^M(\alpha, \beta) \Rightarrow \gamma \in C^M(\alpha, \beta)$;
(M4) $\gamma = \Omega \gamma_1 \& \gamma_1 \in C^M(\alpha, \beta) \Rightarrow \gamma \in C^I(\alpha, \beta)$;
(M5) $\gamma = \chi^\xi(\delta) \& \xi, \delta \in C^M(\alpha, \beta) \& \xi < \alpha \& \xi \in C^M(\xi, \gamma) \& \delta < M \Rightarrow \gamma \in C^M(\alpha, \beta)$;
(M6) $\gamma = \psi^\xi_M(\kappa) \& \xi, \kappa \in C^M(\alpha, \beta) \& \xi < \alpha \& \xi \in C^M(\xi, \gamma) \Rightarrow \gamma \in C^M(\alpha, \beta)$;

$\chi^\alpha(\delta)$ denotes the $\delta^{th}$ regular cardinal $\pi < M$ with $C^M(\alpha, \pi) \cap M = \pi$; $\psi^\alpha_M(\kappa)$ denotes $\min\{\rho < \kappa: C^M(\alpha, \rho) \cap \kappa = \rho \land \kappa \in C^M(\alpha, \rho)\}$.

**Definition 2.2**

(i) $\gamma =_{nf} \alpha + \beta : \Leftrightarrow \gamma = \alpha + \beta \& \gamma > \alpha \geq \beta \& \beta$ is an additive principal number.

(ii) $\gamma =_{nf} \varphi \alpha \beta : \Leftrightarrow \gamma = \varphi \alpha \beta \& \alpha, \beta < \gamma$.

(iii) $\gamma =_{nf} \Omega \alpha : \Leftrightarrow \gamma = \Omega \alpha \& \alpha < \gamma$.

(iv) $\gamma =_{nf} \psi^\alpha_M(\kappa) : \Leftrightarrow \gamma = \psi^\alpha_M(\kappa) \& \alpha \in C^I(\alpha, \gamma)$.

(v) $\gamma =_{nf} \chi^\alpha(\beta) : \Leftrightarrow \gamma = \chi^\alpha(\beta) \& \beta < \gamma \& \alpha \in C^M(\alpha, \gamma)$.

**Definition 2.3** (Rathjen95,98). We define a set $T(M)$ called an elementary ordinal representation system for $\text{KPM}$ and the degree $d(\alpha) < \omega$ of each element $\alpha$ of $T(M)$, as follows:

(i) $0, M \in T(M)$ \& $d(0) = d(M) = 0$;

(ii) $\gamma =_{nf} \alpha + \beta \& \alpha, \beta \in T(M)$

$\Rightarrow (\gamma \in T(M) \& d(\gamma) = \max\{d(\alpha), d(\beta)\} + 1)$;

(iii) $\gamma =_{nf} \varphi \alpha \beta \& \alpha, \beta \in T(M) \& (\gamma < M \text{ or } \alpha = 0)$

$\Rightarrow (\gamma \in T(M) \& d(\gamma) = \max\{d(\alpha), d(\beta)\} + 1)$;

(iv) $\gamma =_{nf} \Omega \alpha < M \& \alpha > 0 \& \alpha \in T(M)$

$\Rightarrow (\gamma \in T(M) \& d(\gamma) = d(\alpha) + 1)$;

(v) $\gamma =_{nf} \chi^\xi(\alpha) \& \xi, \alpha \in T(M)$

$\Rightarrow (\gamma \in T(M) \& d(\gamma) = d(\alpha) + 1)$;

(vi) $\gamma =_{nf} \psi^\alpha_M(\kappa) \& \kappa, \alpha \in T(M)$

$\Rightarrow (\gamma \in T(M) \& d(\gamma) = \max\{d(\kappa), d(\alpha)\} + 1)$.

**Theorem 2.4** (Rathjen91, Buchholz92). (1) Each element of $T(M)$ has a unique representation with $0, M, +, \varphi, \Omega, \chi, \psi_M$.

(2) $|KPM| \leq \psi^{C^M+1}_M(\Omega_1) = T(M) \cap \Omega_1$, where $|KPM|$ denotes the proof theoretic ordinal of $\text{KPM}$.
Definition 2.5 An ordinal $\gamma$ is called a **proof-theoretically regular ordinal based on** $\mathcal{T}(M)$ if $\gamma$ is (expressed by) an element of $\mathcal{T}(M)$ having the form of $\psi^\kappa_M(\Omega_1)$ with $\kappa \in \text{Reg}$, where $\text{Reg}$ denotes the set of regular cardinals.

Definition 2.6 A ptro $\gamma$ is called a **successor-type** ptro if $\gamma$ has an element $\theta \in \mathcal{T}(M)$ satisfying that $\gamma$ is the least ptro larger than $\theta$.

Definition 2.7 An ordinal $\gamma$ is called a **proof-theoretically inaccessible ordinal based on** $\mathcal{T}(M)$ if $\gamma$ is an element of $\text{Reg}(\mathcal{T}(M))$ as well as the supremum of $\text{Reg}(\mathcal{T}(M)) \cap \gamma$, where $\text{Reg}(\mathcal{T}(M))$ denotes the set of ptros based on $\mathcal{T}(M)$.

Theorem 2.8 (Takaki 04). All ptros are classified into the following two types:
(i) Successor-type ptros, which are of the form $\psi^\Omega_M(\Omega_{\alpha+1})$ or $\psi^\Omega_M(\Omega_1)$;
(ii) Proof-theoretically inaccessible ordinals, which are of the form $\psi^\chi\alpha(\beta)(\Omega_1)$ or $\psi^\Omega_M(\Omega_1)$.

Definition 2.9 For each $n \in \omega$, we define $\Psi_n$ by:
$$\Psi_n = \begin{cases} 0 & \text{if } n = 0; \\ \psi_{M}^{\Psi_{n-1}}(\Omega_1) & \text{if } n > 0. \end{cases}$$

Lemma 2.10 For each $n \in \omega$, $\Psi_n \in \mathcal{T}(M)$ and $\Psi_n < \Psi_{n+1}$.

The purpose of this paper is to extend the following theorem.

Theorem 2.11 (cf. Theorem 4 in [Ta04]). $\psi^\Omega_M(\Omega_1) = \lim_{n \in \omega} \Psi_n$.

3 Representation of successor-type ptros

Definition 3.1 Let $\alpha$ and $\beta$ be elements of $\mathcal{T}(M)$. Then, for each $n \in \omega$, we define an ordinal $\Psi_n^\beta(\alpha)$, as follows:
$$\Psi_n^\beta(\alpha) = \begin{cases} \beta & \text{if } n = 0; \\ \psi_{M}^{\Psi_{n-1}^\beta(\alpha)}(\Omega_{\alpha+1}) & \text{otherwise}. \end{cases}$$

In particular, $\Psi_n(\alpha) := \Psi_n^0(\alpha)$

$\Psi_n^\beta(\alpha)$ also satisfies properties of $\Psi_n$.

Lemma 3.2 For each $\alpha, \beta \in \mathcal{T}(M)$, if
$$\beta < \psi_{M}^{\Omega_{\alpha+1}}(\Omega_1) \quad \text{and} \quad \forall \xi \left( \alpha < \xi \Rightarrow \beta \in C^M(\beta, \xi) \right)$$
then, for each $n \in \omega$,
$$\Psi_n^\beta(\alpha) \in \mathcal{T}(M) \quad \text{and} \quad \Psi_n^\beta(\alpha) < \Psi_{n+1}^\beta(\alpha). \quad (1)$$

In particular, for each $\alpha \in \mathcal{T}(M)$ and $n < \omega$,
$$\Psi_n(\alpha) \in \mathcal{T}(M) \quad \text{and} \quad \Psi_n(\alpha) < \Psi_{n+1}(\alpha).$$
Proof. This lemma is shown by checking the properties in (1) as well as
\[ \forall \xi \ ( \alpha < \xi \Rightarrow \Psi_n^\beta(\alpha) \in C^M(\Psi_n^\beta(\alpha), \xi) ), \]
by using induction on \( n \).

Now we give a representation of each successor-type ptro via \( \Psi_n(\alpha) \) and the concept of limit.

**Theorem 3.3** For each \( \alpha \) with \( \psi_M^{\Omega_{\alpha+1}}(\Omega_1) \in \mathcal{T}(M) \),
\[ \psi_M^{\Omega_{\alpha+1}}(\Omega_1) = \lim_{n \in \omega} \psi_M^{\Psi_n(\alpha)}(\Omega_1). \] (2)

Proof. Since in [Ta04] we dealt with the case where \( \alpha = 0 \), it suffices to show (2) in the case where \( \alpha > 0 \).

[1] One can show that \( \psi_M^{\Omega_{\alpha+1}}(\Omega_1) \geq \lim_{n \in \omega} \psi_M^{\Psi_n(\alpha)}(\Omega_1) \), by the following two claims.

**Claim 1** (cf. Lemmas 9.(3) and 11 in [Ta04]). For each \( \alpha \) and \( \beta \), \( \psi_M^\beta(\Omega_{\alpha+1}) \) is defined and \( \Omega_\alpha < \psi_M^\beta(\Omega_{\alpha+1}) < \Omega_{\alpha+1} \).

**Claim 2** (cf. Lemma 10 in [Ta04]). For each \( \alpha_1 \), \( \alpha_2 \) and \( \pi \in \text{Reg} \), if \( \psi_M^{\alpha_1}(\pi) \) and \( \psi_M^{\alpha_2}(\pi) \) are defined and if \( \alpha_1 \leq \alpha_2 \), then \( \psi_M^{\alpha_1}(\pi) \leq \psi_M^{\alpha_2}(\pi) \).

[2] In order to show that \( \psi_M^{\Omega_{\alpha+1}}(\Omega_1) \leq \lim_{n \in \omega} \psi_M^{\Psi_n(\alpha)}(\Omega_1) \), we show that, for each \( \gamma < \psi_M^{\Omega_{\alpha+1}}(\Omega_1) \), there exists an \( n \in \omega \) with \( \gamma \leq \psi_M^{\Psi_n(\alpha)}(\Omega_1) \), by using induction on \( d(\gamma) \).

Since it is easy to check the property above in any case except the case where \( \gamma = \psi_M^\xi(\pi) \), we let \( \gamma = \psi_M^\xi(\pi) \) in what follows.

For the given \( \xi \) (and \( \alpha \)), we now define a labeled binary tree \( T_2(\xi) \) (more precisely, \( T_2(\xi, \alpha) \)).

**Definition 3.4** We define a labeled binary tree \( T_2(\xi) \) to satisfy the following property (i).

(i) For each node \( s \in T_2(\xi) \), we denote the label of \( s \) by \( l_s \). Then, the label \( l_s \) of each node in \( T_2(\xi) \) is an element of \( \mathcal{T}(M) \) satisfying:

(i.i) \( l_s \) is a subterm of \( \xi \);
(i.ii) \( l_s \leq \xi \);
(i.iii) \( l_s \in C^M(\xi; \psi_M^\xi(\Omega_1)) \).

\[ ^1 \text{More precisely, we should assume that } \gamma =_{\text{nf}} \psi_M^\xi(\pi). \text{ However, we use only the symbol } "=\text{" unless we need special attention.} \]
(ii) We define each node of $T_2(\xi)$ and its label, by using recursion on the distance from the root of $T_2(\xi)$, as follows.

(ii.0) If $s \in T_2(\xi)$ is the root, then $l_s$ is $\xi$.

Let $s$ be a node of $T_2(\xi)$. Then, we define the successors (successor nodes) of $s$ as well as their labels, according to the following conditions of $l_s$.

(ii.i) If $l_s = 0$, then $s$ is a leaf, that is, $s$ has no successor node.

(ii.ii) If $l_s = \delta + \eta$ or $l_s = \phi \delta \eta$, then $s$ has successors $s_1$ and $s_2$, and $l_{s_1} := \delta$, $l_{s_2} := \eta$.

(ii.iii) If $l_s = \Omega_\beta$ and $l_s = \chi(\eta)$, then $s$ is a leaf.

(ii.iv) Let $l_s = \psi^\delta(\tau)$. In this case, $\tau \leq \Omega_{\alpha+1}$ since $l_s \leq \xi$.

(ii.iv.i) If $\tau < \Omega_{\alpha+1}$, then $s$ is a leaf.

(ii.iv.ii) If $\tau = \Omega_{\alpha+1}$, then $s$ has a successor $s_1$ and $l_{s_1} := \delta$.

Claim 3 $T_2(\xi)$ is well-defined to be a finite tree.

(Proof of Claim 3: In order to show that $T_2(\xi)$ is well-defined, we show that, for each node $s$ of $T_2(\xi)$, $l_s$ satisfies the properties (i.i)∼(i.iii) above, by using induction on the distance from the root to $s$.

If $s$ is the root, it is trivial since $l_s = \xi$.

We let $l_s = \psi^\delta(\Omega_{\alpha+1})$ and show that $\delta$ satisfies (i.i)∼(i.iii), as follows. By induction hypothesis, $l_s$ is a subterm of $\xi$, $l_s \leq \xi$ and $l_s \in C^M(\xi, \gamma)$. Then, $\delta$ is also a subterm of $\xi$. On the other hand, $l_s > \Omega_1 \geq \gamma$. So, we have $\delta \in C^M(\xi, \gamma)$ and $\delta \leq \xi$ from Definition 2.1.(M5) and $l_s \in C^M(\xi, \gamma)$.

Any other case is similar to the case above.

Moreover, for each node $s \in T_2(\xi)$ and each successor $s'$ of $s$, it holds that $d(s) > d(s')$. So, $T_2(\xi)$ is finite. □)

Definition 3.5 (1) A node $s$ of $T_2(\xi)$ (=$T_2(\xi, \alpha)$) is said to be critical when $l_s = \psi^\delta(\Omega_{\alpha+1})$ for some $\delta$. CN denotes the set of critical nodes (of $T_2(\xi)$).

(2) For each path $p$ of each subtree of $T_2(\xi)$, the number of critical nodes in $p$ is called the weight of $p$. Moreover, for each subtree $T$ of $T_2(\xi)$, the maximum number of weights of all paths of $T$ is called the weight of $T$, and denoted by $\mathrm{w}(T)$. Furthermore, for each node $s$ of $T_2(\xi)$, the weight of the subtree of $T_2(\xi)$ with root $s$ is called the weight of $s$, and denoted by $\mathrm{w}(s)$.

(3) For each subtree $T$ of $T_2(\xi)$, the maximum length of all paths of $T$ is called the height of $T$. Moreover, for each node $s$ of $T_2(\xi)$, the height of the subtree of $T_2(\xi)$ with root $s$ is called the depth of $s$, and denoted by $\mathrm{d}(s)$.

Claim 4 For each node $s$ of $T_2(\xi)$, it holds that $l_s < \Psi_{\mathrm{d}(s)+1}(\alpha)$.

(Proof of Claim 4: We show the claim by induction on the depth of $s$.

(i) If $s$ is a leaf, then $l_s \leq \Omega_\alpha$. So, since $\Omega_\alpha < \Psi_n(\alpha)$ for each $n > 0$, we have $l_s < \Psi_1(\alpha)$. )
(ii) Assume that $s$ is not any leaf. Then, $l_s = \text{nf } \delta + \eta$, $l_s = \text{nf } \varphi \delta \eta$, or $l_s = \text{nf } \psi_M^\delta(\Omega_{\alpha+1})$.

Let $l_s = \text{nf } \psi_M^\delta(\Omega_{\alpha+1})$. Then, $l_s \in \text{CN}$ and $s$ has one successor $s_1$ with $l_{s_1} = \delta$. Since $\text{wt}(s_1) = \text{wt}(s) - 1$ and $\text{dp}(s_1) < \text{dp}(s)$, the induction hypothesis implies that $l_{s_1} < \Psi_{\text{wt}(s)}(\alpha)$. On the other hand, since $l_s \in \mathcal{T}(M)$ and $\Psi_{\text{wt}(s)+1}(\alpha) \in \mathcal{T}(M)$, we have $l_s < \Psi_{\text{wt}(s)+1}(\alpha)$ (cf. Lemma 16 in [Ta04]).

Any other case is similar to or easier than the case above.  

By Claim 4, we have $\xi < \Psi_{\text{wt}(T_2(\xi))}+1(\alpha)$, and hence, by Claim 2,

$$\gamma \leq \psi_M^{\Psi_{\text{wt}(T_2(\xi))}+1(\alpha)}(\Omega_1).$$

So, the proof of Theorem 3.3 is completed.

We can also expect that each $\psi_M^{\Psi_n(\alpha)}(\Omega_1)$ has itself as its regular expression, that is, $\psi_M^{\Psi_n(\alpha)}(\Omega_1) \in \mathcal{T}(M)$. Unfortunately, we have not yet completed the proof of the property. However, it is not hard to show this property for each $\alpha$ less than a certain ordinal. For example, one can easily show the following proposition.

**Proposition 3.6** For each $\alpha \in \mathcal{T}(M)$ and $n \in \omega$, if $\alpha \in C^M(\Psi_n(\alpha), \psi_M^{\Psi_n(\alpha)}(\Omega_1))$, then

$$\psi_M^{\Psi_n(\alpha)}(\Omega_1) \in \mathcal{T}(M) \text{ and } \psi_M^{\Psi_n(\alpha)}(\Omega_1) < \psi_M^{\Psi_{n+1}(\alpha)}(\Omega_1).$$

By Theorem 3.3 and Proposition 3.6, each successor-type ptro $\psi_M^{\Omega_{\alpha+1}}(\Omega_1)$ has a fundamental sequence $\{\psi_M^{\Psi_n(\alpha)}(\Omega_1)\}_{n \in \omega}$ if $\alpha \in C^M(\Psi_n(\alpha), \psi_M^{\Psi_n(\alpha)}(\Omega_1))$.

**Reference**


