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Algebraic generalization of rough sets

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Abstract

In this paper we give an affirmative answer to the problem whether there is a relation R on the set $\mathcal{A}(\mathcal{B})$ of atoms of a complete atomic Boolean algebra $\mathcal{B}$ such that properties of a map $\varphi : \mathcal{A}(\mathcal{B}) \to \mathcal{B}$ can be inherited to the relation R, that is,

1. φ^* : extensive $\Longleftrightarrow$ R : reflexive
2. φ^* : symmetric $\Longleftrightarrow$ R : symmetric
3. φ^* : closed $\Longleftrightarrow$ R : transitive

1 Introduction

Since the presentation of rough sets by Pawlak ([4]), the theory of rough sets is applied to many practical fields in data base theory, especially, in data-mining. But the fundamental and mathematical base of rough sets is not enough to apply the theory to other many research fields. Recently, many papers about mathematical fundation of rough sets are published, but almost papers treated only the finite case of the universe U in approximation spaces. Under this restriction, it is very hard to understand the essential properties of rough sets. In [2, 3], J.Jarvinen obtained important and fundamental results about rough sets using complete atomic Boolean algebras. He treated rough sets abstractly. He constructed an operator $R_- : \mathcal{P}(U) \to \mathcal{P}(U)$ from a relation R on U and then extended it to a map $\varphi : \mathcal{A}(\mathcal{B}) \to \mathcal{B}$ from the set $\mathcal{A}(\mathcal{B})$ of atoms of a complete atomic Boolean algebra $\mathcal{B}$ to $\mathcal{B}$. He proved that properties of the relation R was inherited to those of φ. In this paper we extend his results. Morever, we consider the converse problem whether there is a relation R on $\mathcal{A}(\mathcal{B})$ such that properties of a map $\varphi : \mathcal{A}(\mathcal{B}) \to \mathcal{B}$ is inherted by the relation R. We will give an affirmative answer to the problem.

2 Preliminaries

At first we define operators R_- and R_+ on approximation spaces according to [2, 3]. Considering properties of approximation spaces (U, R) in rough sets, we construct subsets $R(x) = \{y \in U \mid xRy\}$ and operators

$$R_- : \mathcal{P}(U) \to \mathcal{P}(U).$$

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After that we define lower (upper) appoximation sets respectively such as :

$$R_-(X) = \{a \in U \,|\, R(a) \subseteq X\}$$
$$R_+(X) = \{a \in U \,|\, R(a) \cap X \neq \emptyset\}$$

Since $\mathcal{P}(U)$ is a complete Boolean algebra under the usual operations $\cap, \cup, ^c$ and a singleton set $\{x\}$ $(x \in U)$ is identified with an atom of $\mathcal{P}(U)$, the Boolean algebra $\mathcal{P}(U)$ can be considered as a complete atomic Boolean algebra.

Note: The fact that $\mathcal{B}$ is a complete atomic Boolean algebra means that any map $v : U \to \mathcal{P}(U)$ can be extended uniquely to the map from $\mathcal{P}(U) \to \mathcal{P}(U)$. Thus, for the operator R_- induced by a relation R, if we only define the value of $R_-(x)$ $(x \in U)$ then the map $R_- : \mathcal{P}(U) \to \mathcal{P}(U)$ is determined uniquely.

Let $\mathcal{B}$ be a complete atomic Boolean lattice and $\mathcal{A}(\mathcal{B})$ a set of atoms. For a map $\varphi : \mathcal{A}(\mathcal{B}) \to \mathcal{B}$, operators $^\triangledown, ^\triangle$ are defined as follows ([3]):

$$x^\triangledown = \bigvee\{a \in \mathcal{A}(\mathcal{B}) \,|\, \varphi(a) \leq x\}$$
$$x^\triangle = \bigvee\{a \in \mathcal{A}(\mathcal{B}) \,|\, \varphi(a) \wedge x \neq 0\}$$

These correspond to respectively

$$R_-(X) = \{a \in U \,|\, R(a) \subseteq X\}$$
$$R_+(X) = \{a \in U \,|\, R(a) \cap X \neq \emptyset\}.$$

In [3] it is proved that

Proposition 1. *For every $a \in \mathcal{A}(\mathcal{B})$, $x \in \mathcal{B}$,*

1. $a \leq x^\triangledown \iff \varphi(a) \leq x$
2. $a \leq x^\triangle \iff \varphi(a) \wedge x \neq 0$
3. $0^\triangle = 0,\ 1^\triangledown = 1$
4. $x \leq y \implies x^\triangledown \leq y^\triangledown, x^\triangle \leq y^\triangle$
5. $\bigvee S^\triangle = (\bigvee S)^\triangle$, *hence in particular,* $(x \vee y)^\triangle = x^\triangle \vee y^\triangle$
6. $\bigwedge S^\triangledown = (\bigwedge S)^\triangledown$, *hence in particular,* $(x \wedge y)^\triangledown = x^\triangledown \wedge y^\triangledown$
7. $^\triangle, ^\triangledown$: *dual, that is,*

$$(x^\triangledown)' = (x')^\triangle,\ (x^\triangle)' = (x')^\triangledown$$

For $\varphi : \mathcal{A}(\mathcal{B}) \to \mathcal{B}$, three kinds of maps are defined:

$$\varphi : \text{extensive} \iff x \leq \varphi(x)$$
$$\varphi : \text{symmetric} \iff x \leq \varphi(y) \text{ implies } y \leq \varphi(x)$$
$$\varphi : \text{closed} \iff y \leq \varphi(x) \text{ implies } \varphi(y) \leq \varphi(x)$$

Since x and y are atoms, we see that a symmetric map φ can be represented by

$$\varphi : \text{symmetric} \iff x \wedge \varphi(y) = 0 \text{ iff } y \wedge \varphi(x) = 0.$$

In the following we adopt this definition, that is, a map φ is symmetric if and only $x\wedge\varphi(y) = 0$ implies $y \wedge \varphi(x) = 0$ and vice versa.

Considering the relation between the properties of R and those of map φ defined above, he also proved that

$$\begin{aligned} \varphi : \text{extensive} &\iff R : \text{reflexive} \\ \varphi : \text{symmetric} &\iff R : \text{symmetric} \\ \varphi : \text{closed} &\iff R : \text{transitive} \end{aligned}$$

For $x = \bigvee_\lambda a_\lambda$ $(a_\lambda \in \mathcal{A}(\mathcal{B}))$, we define a map $\varphi^* : \mathcal{B} \to \mathcal{B}$ by

$$\varphi^* = \bigvee \varphi(a_\lambda).$$

It follows from this definition that

Proposition 2. *1. φ^* : well-defined*

2. φ^* : order-preserving, i.e.,

$$x \leq y \implies \varphi^*(x) \leq \varphi^*(y)$$

A map $\psi : \mathcal{B} \to \mathcal{B}$ is called *extensive* if $x \leq \psi(x)$ for all $x \in B$. Then extensiveness of φ and φ^* are the same.

Proposition 3. *φ : extensive (i.e., $a \leq \varphi(a)$ for all $a \in \mathcal{A}(\mathcal{B})$)*
$\iff \varphi^*$: extensive

Proof. ($\Longleftarrow$) Take $x = a \in \mathcal{A}(\mathcal{B})$.

($\Longrightarrow$) Let $x = \bigvee_\lambda a_\lambda$ $(a_\lambda \in \mathcal{A}(\mathcal{B}))$. For every a_λ, since φ is extensive, we have

$$a_\lambda \leq \varphi(a_\lambda)$$

and hence

$$x = \bigvee a_\lambda \leq \bigvee \varphi(a_\lambda) = \varphi^*(x).$$

□

A map $\psi : \mathcal{B} \to \mathcal{B}$ is called *symmetric* when

$$x \wedge \psi^*(y) = 0 \text{ iff } y \wedge \psi^*(x) = 0 \text{ for all } x, y \in \mathcal{B}.$$

Then symmetices of φ and φ^* are the same.

Proposition 4. *φ : symmetric (i.e., $a \wedge \varphi(b) = 0$ iff $b \wedge \varphi(a) = 0$ for all $a, b \in \mathcal{A}(\mathcal{B})$)*
$\iff \varphi^*$: symmetric

Proof. ($\Longleftarrow$) It is obvious if we take $x = a, y = b$ $(a, b \in \mathcal{A}(\mathcal{B}))$.

($\Longrightarrow$) Let $x = \bigvee_\lambda a_\lambda$ and $y = \bigvee_\mu b_\mu$ $(a_\lambda, b_\mu \in \mathcal{A}(\mathcal{B}))$. Suppose $x \wedge \varphi^*(y) = 0$. It is sufficient to verify $y \wedge \varphi^*(x) = 0$.

Since $x \wedge \varphi^*(y) = 0$, we have

$$\bigvee_\lambda a_\lambda \wedge \bigvee_\mu \varphi(b_\mu) = 0$$

and hence

$$\bigvee_{\lambda,\mu}(a_\lambda \wedge \varphi(b_\mu)) = 0.$$

This means that for all λ, μ

$$a_\lambda \wedge \varphi(b_\mu) = 0.$$

Since φ is symmetric,

$$b_\mu \wedge \varphi(a_\lambda) = 0.$$

Thus, we have

$$\bigvee_{\lambda,\mu}(b_\mu \wedge \varphi(a_\lambda)) = \bigvee b_\mu \wedge \bigvee \varphi(a_\lambda) = 0.$$

This means that

$$y \wedge \varphi^*(x) = 0.$$

□

Corollary 1. φ^* *: symmetric* $\iff$ $\varphi^*(\varphi^* x)' \leq x'$ *for all* $x \in \mathcal{B}$

There is another characterization of symmetry by use of φ.

Proposition 5. φ *: symmetric* $\iff$ $\varphi a = a^\triangle$ *for all* $a \in \mathcal{A}(\mathcal{B})$

Proof. ($\Longrightarrow$) Prop.3.9 in [3].

($\Longleftarrow$) Suppose that φ is not symmetric. There are elements $a, b \in \mathcal{A}(\mathcal{B})$ such that

$$a \leq \varphi b \text{ but } b \not\leq \varphi a.$$

Since $\varphi b = b^\triangle$, we have

$$a \leq b^\triangle.$$

Thus

$$\varphi a \wedge b \neq 0.$$

Since b is an atom, this implies

$$\varphi a \wedge b = b.$$

So we conclude that $b \leq \varphi a$. But this is a contradiction. Hence φ is symmetric. □

A map $\psi : \mathcal{B} \to \mathcal{B}$ is called *closed* if $x \leq \psi(y)$ implies $\psi(x) \leq \psi(y)$ for all $x, y \in \mathcal{B}$.

Proposition 6. φ *: closed (i.e.,* $a \leq \varphi b$ implies $\varphi a \leq \varphi b$ for all $a, b \in \mathcal{A}(\mathcal{B})$*)*
$\iff$ φ^* *: closed*

Proof. ($\Longleftarrow$) Take $x = a, y = b$ $(a, b \in \mathcal{A}(\mathcal{B}))$.

($\Longrightarrow$) Let $x = \bigvee_\lambda a_\lambda$ and $y = \bigvee_\mu b_\mu$ $(a_\lambda, b_\mu \in \mathcal{A}(\mathcal{B}))$. Suppose that $x \leq \varphi^* y$. For every λ, we have

$$a_\lambda \leq \bigvee_\lambda a_\lambda \leq \varphi^* y = \bigvee_\mu \varphi b_\mu.$$

There is an element b_μ such that

$$a_\lambda \leq \varphi b_\mu.$$

Indeed, since $a_\lambda \leq \bigvee \varphi b_\mu$ and a_λ is an atom, we have

$$a_\lambda = a_\lambda \wedge \bigvee \varphi b_\mu = \bigvee_\mu (a_\lambda \wedge \varphi b_\mu).$$

There is an element b_μ such that

$$a_\lambda = a_\lambda \wedge \varphi b_\mu.$$

This implies that $a_\lambda \leq \varphi b_\mu$ for some μ. Since φ is closed,

$$\varphi a_\lambda \leq \varphi b_\mu \leq \bigvee \varphi b_\mu = \varphi^* y \text{ for all } \lambda.$$

Thus we have

$$\varphi^* x = \bigvee \varphi a_\lambda \leq \varphi^* y.$$

□

Corollary 2. *φ^* : closed $\iff$ $\varphi^*(\varphi^* x) \leq \varphi^* x$ for all $x \in \mathcal{B}$*

3 Relations derived from operators

We generalized the map $\varphi^* : \mathcal{B} \to \mathcal{B}$ from the map $\varphi : \mathcal{A}(\mathcal{B}) \to \mathcal{B}$. This is a generalization of the operator $R_- : \mathcal{P}(U) \to \mathcal{P}(U)$ induced by the relation R on a set U. Then we have proved that the original properties of R are inherited as follows:

$$\begin{aligned} R : \text{reflexive} &\iff \varphi \text{ (or } \varphi^*) : \text{extensive} \\ R : \text{symmetric} &\iff \varphi \text{ (or } \varphi^*) : \text{symmetric} \\ R : \text{transivite} &\iff \varphi \text{ (or } \varphi^*) : \text{closed} \end{aligned}$$

It is a natural question whether we can define a relation R on $\mathcal{A}(\mathcal{B})$ such that it reflects properties of a map $\varphi^* : \mathcal{B} \to \mathcal{B}$ which is an extension of a map $\varphi : \mathcal{A}(\mathcal{B}) \to \mathcal{B}$. If we can answer "YES" to the question, since φ^* can be represented by R completely, then we have several methods to develop the theory of generalized rough sets.

Let $\varphi : \mathcal{A}(\mathcal{B}) \to \mathcal{B}$ be any map and $\varphi^* : \mathcal{B} \to \mathcal{B}$ a uniquely extended map of φ. It is clear that the map φ^* is order-preserving and $\varphi^* \restriction_{\mathcal{A}(\mathcal{B})} = \varphi$.

We define a relation R on $\mathcal{A}(\mathcal{B})$ as follows: For all $a, b \in \mathcal{A}(\mathcal{B})$,

$$aRb \iff a \leq \varphi(b)$$

We can show that

Proposition 7. *φ^* : extensive $\iff$ R : reflexive*

Proof. Suppose that φ^* is extensive. For any $a \in \mathcal{A}(\mathcal{B})$, since φ^* is extensive, we have $a \leq \varphi^*(a) = \varphi(a)$ and hence R is reflexive.

Conversely, assume that φ^* is not extensive. Since $x \not\leq \varphi^*(x)$ for some $x \in \mathcal{B}$, there exists $a \in \mathcal{A}(\mathcal{B})$ such that

$$a \leq x \quad \text{but} \quad a \not\leq \varphi^*(x).$$

Since R is reflexive, $a \leq \varphi(a)$ implies

$$\varphi(a) \not\leq \varphi^*(x).$$

On the other hand, $a \leq x$ means that $\varphi^*(a) = \varphi(a) \leq \varphi^*(x)$. But this is a contradiction. Thus, φ^* is extensive. □

Proposition 8. *φ^* : symmetric $\iff$ R : symmetric*

Proof. If R is not symmetric, then there exist $a, b \in \mathcal{A}(\mathcal{B})$ such that aRb but *not* bRa. This means that

$$a \leq \varphi(b) \quad \text{but} \quad b \not\leq \varphi(a)$$

and hence that $b \wedge \varphi(a) = 0$. Since $b \leq (\varphi(a))'$ and φ^* is order preserving, we have

$$\varphi(b) = \varphi^*(b) \leq \varphi^*((\varphi(a))') \leq a'.$$

Hence $\varphi(b) \wedge a = 0$. But from $a \leq \varphi(b)$, we get $\varphi(b) \wedge a = a$. This is a contradiction. Thus, R is symmetric.

Conversely, assume that φ^* is not symmetric. There exist $x, y \in \mathcal{B}$ such that

$$x \wedge \varphi^*(y) = 0 \quad \text{but} \quad y \wedge \varphi^*(x) \neq 0.$$

Since $\mathcal{B}$ is atomic, there exist $a, a_\lambda \in \mathcal{A}(\mathcal{B})$ such that

$$a \leq y, \; a \leq \varphi^*(x) = \bigvee \varphi(a_\lambda).$$

It follows that $a \leq \varphi(a_\lambda)$ for some λ and aRa_λ. Since R is symmetric, this implies $a_\lambda Ra$, that is, for some λ,

$$a_\lambda \leq \varphi(a).$$

On the other hand, we have $\varphi(a) = \varphi^*(a) \leq \varphi^*(y)$ by $a \leq y$. Thus,

$$0 = x \wedge \varphi^*(y) \geq a_\lambda \wedge \varphi(a) = a_\lambda.$$

But this is a contradiction. Hence φ^* is symmetric. □

Proposition 9. φ^* *: closed* $\Longleftrightarrow$ R *: transitive*

Proof. Suppose that aRb and bRc for $a, b, c \in \mathcal{A}(\mathcal{B})$. This means that

$$a \leq \varphi(b) \quad \text{and} \quad b \leq \varphi(c).$$

Since φ^* is closed, we have $\varphi(b) = \varphi^*(b) \leq \varphi(c)$ and $a \leq \varphi(c)$. Hence, R is transitive.

Conversely, assume that φ^* is not closed. There exist $x, y \in \mathcal{B}$ such that

$$x \leq \varphi^*(y) \quad \text{but} \quad \varphi^*(x) \not\leq \varphi^*(y).$$

If we take $x = \bigvee a_\lambda$ and $y = \bigvee b_\mu$ $(a_\lambda, b_\mu \in \mathcal{A}(\mathcal{B}))$, since $\varphi^*(x) \not\leq \varphi^*(y)$, then there exists $a \in \mathcal{A}(\mathcal{B})$ such that

$$a \leq \varphi^*(x) = \bigvee \varphi(a_\lambda) \quad \text{but} \quad a \not\leq \varphi^*(y) = \bigvee \varphi(b_\mu).$$

It follows that $a \leq \varphi(a_{\lambda_0})$ for some λ_0 but $a \not\leq \varphi(b_\mu)$ for all μ.

On ther other hand, since $a_{\lambda_0} \leq \bigvee a_\lambda = x \leq \varphi^*(y) = \bigvee \varphi(b_\mu)$, there exists μ_0 for that λ_0 such that

$$a_{\lambda_0} \leq \varphi(b_{\mu_0}).$$

This implies aRa_{λ_0} and $a_{\lambda_0}Rb_{\mu_0}$. Since R is transitive, we have aRb_{μ_0}, that is, $a \leq \varphi(b_{\mu_0})$. But this is a contradiction. Hence φ^* is closed. □

Summing up the above, we have the following theorem.

Theorem 1. *Let $\mathcal{B}$ be a complete atomic Boolean algebra and $\mathcal{A}(\mathcal{B})$ be the set of all atoms of $\mathcal{B}$. For any map $\varphi : \mathcal{A}(\mathcal{B}) \to \mathcal{B}$, there exists a relation R on $\mathcal{A}(\mathcal{B})$ such that*

$$\begin{aligned} \varphi^* &: \text{extensive} \iff R : \text{reflexive} \\ \varphi^* &: \text{symmetric} \iff R : \text{symmetric} \\ \varphi^* &: \text{closed} \iff R : \text{transitive} \end{aligned}$$

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