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Combinatorial structure of group divisible designs and their constructions

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Abstract Group divisible (GD) designs can be classified into the three types (1) singular, (2) semi-regular and (3) regular, where the first case holds iff $r=\lambda_1$. Since $r\geq\lambda_1$ always holds, the case $r>\lambda_1$ will cover the other two types. In this paper, at first, the combinatorial structure of GD designs with $r\geq\lambda_1+1$ is discussed. Next, the combinatorial structure of GD designs is discussed from another point of view of assuming local structure in each group. We characterize GD designs by a new point of view and provide some constructions of regular GD designs based on the characterization.

Keywords: Group divisible design; Balanced incomplete block design; Hadamard tournament; Strongly regular graph

1. Definition and classification of group divisible designs

Let V be a finite set and $\mathcal{B}$ be a collection of subsets of the same size of V. A pair $(V,\mathcal{B})$ is called a *block design*, or simply a *design*. Elements of V and $\mathcal{B}$ are called *points* and *blocks*, respectively. Let $v=|V|$ and $b=|\mathcal{B}|$. In discussing the combinatorial problems on designs, we adopt the terminology "points" instead of treatments used usually. For a block design $(V,\mathcal{B})$, let $V=\{p_1,p_2,\cdots,p_v\}$ and $\mathcal{B}=\{B_1,B_2,\cdots,B_b\}$, and the $v\times b$ matrix $N=(n_{ij})$, called an *incidence matrix* of a block design $(V,\mathcal{B})$, is defined as $n_{ij}=1$ when $p_i\in B_j$, and $n_{ij}=0$ when $p_i\notin B_j$. The *complement* of a design with the incidence matrix N is the design with the incidence matrix $\bar{N}$ which is obtained by exchanging 0's and 1's in N.

Now a group divisible (GD) design is defined. Let $v=mn$ $(m,n\geq 2)$, b, r, k, λ_1, λ_2 be positive integers. A *GD design* with parameters $v=mn$, b, r, k, λ_1, λ_2 is a triplet $(V,\mathcal{B},\mathcal{G})$, where V is a v-set of *points*, $\mathcal{B}$ is a collection of b k-subsets, called *blocks*, of V and $\mathcal{G}=\{G_1,\cdots,G_m\}$ is a partition of V into m groups of n points each such that any two distinct points in the same group occur together in exactly λ_1 blocks of $\mathcal{B}$, while those in different groups occur together in exactly λ_2 blocks of $\mathcal{B}$. Here, r is the number of blocks containing a given point. Note that r is a constant not depending on the point chosen. Among parameters of a GD design, it holds that

$$bk=vr, \tag{1.1}$$

$$\lambda_1(n-1)+\lambda_2n(m-1)=r(k-1). \tag{1.2}$$

When λ_1 equals λ_2, a GD design is called a *balanced incomplete block* (*BIB*) *design* with parameters v, b, r, k, λ $(=\lambda_1=\lambda_2)$, which satisfy (1.1) and

$$r(k-1)=\lambda(v-1). \tag{1.3}$$

When $v=b$, a design is said to be *symmetric*.

Let N be an incidence matrix of a GD design and N' be the transpose of N. In the analysis of the design of experiment, the eigenvalues of the matrix NN' play an important role. For the incidence matrix N of a GD design with parameters v, b, r, k, λ_1 and λ_2, the determinant of NN' is given by

$$|NN'| = rk(r-\lambda_1)^{m(n-1)}(rk-v\lambda_2)^{m-1}$$

and the eigenvalues of NN' are rk, $r-\lambda_1$, $rk-v\lambda_2$ with multiplicities 1, $m(n-1)$ and $m-1$, respectively (see, for example, Raghavarao [14, pp.127-128]).

Bose and Connor [7] classified GD designs into three types in terms of the eigenvalues of NN' as follows:

(1) *Singular* if $r-\lambda_1=0$,

(2) *Nonsingular* if $r-\lambda_1>0$

(2a) *Semi-regular* if $rk-v\lambda_2=0$

(2b) *Regular* if $rk-v\lambda_2>0$.

Now, we briefly explain properties for each type of GD designs. We refer the reader to [5] and [11] for relevant design-theoretic terminology.

(i) Singular group divisible designs

Singular GD designs are characterized by the relation $r=\lambda_1$. It is obvious that the existence of a BIB design with parameters v^*, b^*, r^*, k^*, λ^* is equivalent to the existence of a singular GD design with parameters $v=v^*n$, $b=b^*$, $r=r^*$, $k=k^*n$, $\lambda_1=r^*$, $\lambda_2=\lambda^*$ for every n, since the n points in the same group of a singular GD design must occur in the same block by the relation $r=\lambda_1$. Hence, a singular GD design is derivable from a corresponding BIB design.

(ii) Semi-regular group divisible designs

Semi-regular GD designs are characterized by the relations $r>\lambda_1$ and $rk=v\lambda_2$. In this case, the matrix NN' has the eigenvalue 0 with multiplicity $m-1$, and then $v-m+1=\text{rank}(NN')=\text{rank}(N)\le b$. Hence, it follows that $b\ge v-m+1$ holds for a semi-regular GD design.

Bose and Connor [7] showed that the block size k is divisible by m for a semi-regular GD design and that, if $k=cm$, every block contains c points from each group (see, for example, Raghavarao [14, p.132, Theorem 8.5.6]).

Especially in case of $c=1$, a semi-regular GD design is referred to as a *transversal design*. A transversal design with parameters

$$v=mn, b=n^2\lambda_2, r=n\lambda_2, k=m, \lambda_1=0, \lambda_2, m, n \tag{1.4}$$

is equivalent to an "orthogonal array of strength 2." Moreover, the existence of a transversal design with parameters given in (1.4) implies the existence of the semi-regular GD design with parameters $v=m_1n$, $b=n^2\lambda_2$, $r=n\lambda_2$, $k=m_1$, $\lambda_1=0$, λ_2, m_1, n, where m_1 can be any integer satisfying $0<m_1<m$. A transversal design is also obtained by considering the dual structure of an "affine resolvable BIB design" (see, for example, Raghavarao [14]). For constructions of a semi-regular GD design with $c>1$, see [10].

(iii) Regular group divisible designs

Regular GD designs are characterized by the property $r>\lambda_1$ and $rk>v\lambda_2$. By considering the rank of NN', it follows that $v\le b$ holds in the case of a regular GD design similarly to the case of a BIB design, which is called *Fisher's inequality*. A design is said to be *symmetric*, if $v=b$.

Several methods of constructing GD designs are given by Bose et al. [9]. There are also methods of constructing regular GD designs from known BIB designs. For example, by omitting the blocks containing a given point θ of a BIB design with parameters v^*, b^*, r^*, k^*, $\lambda^* = 1$, we obtain a GD design with parameters $v = v^* - 1$, $b = b^* - r^*$, $r = r^* - 1$, $k = k^*$, $m = r^*$, $n = k^* - 1$, $\lambda_1 = 0$, $\lambda_2 = 1$.

Here, we will state several known constructions of regular GD designs in connection with our constructions in this thesis. In view of the existence of the "affine resolvable BIB designs" with parameters $v^* = s^2$, $b^* = s(s+1)$, $r^* = s+1$, $k^* = s$, $\lambda^* = 1$ when s is a prime or a prime power, it can be seen that a series of symmetric regular GD designs with parameters $v = b = s^2 - 1$, $r = k = s$, $m = s+1$, $n = s-1$, $\lambda_1 = 0$, $\lambda_2 = 1$ always exists.

A geometrical method of constructing symmetric regular GD designs is given by Sprott [20]. When s is a prime or a prime power, there exists a regular symmetric GD design with parameters $v = b = s(s-1)(s^2+s+1)$, $m = s^2+s+1$, $n = s(s-1)$, $r = k = s^2$, $\lambda_1 = 0$, $\lambda_2 = 1$.

We can also construct a GD design by the method of differences. The scope of the method of differences can be further extended by using the concepts of the partial cycle of Rao [15]. Arasu and Pott [4] gave constructions of symmetric GD designs using "divisible difference sets."

Many studies for symmetric GD designs have been taken by Ryser [16, 17], Bose [6], Bose and Shrikhande [8] , Sathe and Pradhan [18], and so on. Our results in this paper also present constructions of GD designs including a symmetric case.

2. Group divisible designs with $r \geq \lambda_1 + 1$

For a singular GD design $r = \lambda_1$ holds, while in case of a nonsingular GD design $r > \lambda_1$ holds. It may be natural to investigate the case of $r = \lambda_1 + 1$, since it may have some interconnecting property (the next saturated case) between singular and nonsingular cases.

In this section, we will characterize the combinatorial structure of GD designs with $r = \lambda_1 + 1$, and that of GD designs with $r = \lambda_1 + 2$ and $n = 3, 4$. All the results in this section are due to [19], [13], [1] and [2].

To state the results, we will give some basic notations. We denote the identity matrix of order s, an $s \times t$ matrix all of whose elements are unity and an $s \times t$ matrix all of whose elements are zero, by I_s, $J_{s\times t}$ and $O_{s\times t}$, respectively. In particular, let $J_s = J_{s\times s}$ and $O_s = O_{s\times s}$. Moreover, let $\mathbf{1}_n = J_{1\times n}$ and $\mathbf{0}_n = O_{1\times n}$. Hence the above $\bar{A} = \mathbf{1}_v'\mathbf{1}_b - A = J_{v\times b} - A$. Here $\mathbf{1}_n'$ means the transpose of $\mathbf{1}_n$. $A \otimes B$ denotes the kronecker product of matrices A and B.

A symmetric BIB design with parameters v, $k = (v-1)/2$, $\lambda = (v-3)/4$ is called a *Hadamard design*. For a tournament, i.e., a complete simple digraph, with the $v \times v$ adjacency matrix N, if N is the incidence matrix of a Hadamard design, then the tournament is called a *Hadamard tournament of order v* (see [11]). A simple undirected graph is called a *strongly regular graph* if for any two distinct vertices i and j, there are p_{11}^1 or p_{11}^2 vertices which are connected to both of vertices i and j, according as i and j are connected or not. We refer the reader to [12] and [21] for relevant graph-theoretic terminology.

2.1. Group divisible designs with $r = \lambda_1 + 1$

The combinatorial property of a GD design with $r = \lambda_1 + 1$ was first investigated by Shimata and Kageyama [19] who showed that a GD design with $r = \lambda_1 + 1$ must be symmetric and regular. Jimbo and Kageyama [13] completely characterized a GD design with $r = \lambda_1 + 1$ in terms of Hadamard tournaments and strongly regular graphs.

In fact, in a GD design with parameters $v = mn$ $(m, n \geq 2) = b$, $r = k = \lambda_1 + 1$, λ_2, by the result given in [19], the $v \times v$ incidence matrix N of the GD design is divided into m^2 $n \times n$ submatrices such as $N = (N_{ij})$, where $N_{11} = N_{22} = \cdots = N_{mm} = I_n$ or $J_n - I_n$, and $N_{ij} = J_n$ or O_n for $i{\neq}j$. The incidence matrix N is completely characterized in terms of Hadamard tournaments and strongly regular graphs from the viewpoint of the construction as follows.

Theorem 2.1 (Jimbo and Kageyama [13]). *Let N be the incidence matrix of a regular GD design with $r = \lambda_1 + 1$ or of its complement such that $N_{ii} = I_n$ for any i.*

(i) *When $n \geq 3$ and $\lambda_2 \equiv 2 \pmod n$, the incidence matrix of the design is given by $N = I_m \otimes I_n + (J_m - I_m) \otimes J_n$ for general m and n, which leads to a symmetric regular GD design with parameters $v = b = mn$, $r = k = (m-1)n + 1$, $\lambda_1 = (m-1)n$, $\lambda_2 = (m-2)n + 2$.*

(ii) *When $n \geq 2$ and $\lambda_2 \equiv 1 \pmod n$ (i.e., $v = b = mn$, $r = k = n(m-1)/2 + 1$, $\lambda_1 = n(m-1)/2$, $\lambda_2 = n(m-3)/4 + 1)C$ the existence of the design is equivalent to the existence of a Hadamard tournament of order $m \equiv 3 \pmod 4$.*

(iii) *When $n = 2$ and λ_2 is even (i.e., $v = b = 2m$, $r = k = 2s + 1$, $\lambda_1 = 2s$, $\lambda_2 = 2s^2/(m-1))C$ the existence of the design is equivalent to the existence of a strongly regular graph with parameters $v = m$, $k = s$, $p_{11}^1 = x$, $p_{11}^2 = x + 1$, where $s^2 = (x+1)(m-1)$. Hence $\lambda_2 = 2(x+1)$.*

Remark. A regular GD design exists only when the parameters satisfy the conditions (i), (ii) or (iii).

Theorem 2.1 reveals that the inner structure of GD designs with $r = \lambda_1 + 1$ is characterized in terms of Hadamard tournaments and strongly regular graphs, as in Table 1. For Hadamard tournaments and strongly regular graphs, there are some available existence or non-existence results . Hence, the existence or nonexistence problem of GD designs with $r = \lambda_1 + 1$ can be reduced to those of Hadamard tournaments and strongly regular graphs.

Table 1: The combinatorial structure of GD designs with $r = \lambda_1 + 1$

	$n = 2$	$n \geq 3$
$\lambda_2 \equiv 1$ (mod n)	Hadamard tournament	Hadamard tournament
$\lambda_2 \equiv 2$ (mod n)	strongly regular graph	$N = I_m \otimes I_n$ $+(J_m - I_m) \otimes J_n$
$\lambda_2 \not\equiv 1, 2$ (mod n)	—	nonexistence

2.2. Group divisible designs with $r = \lambda_1 + 2$ and $n = 3, 4$

As the next interesting case we can consider a GD design with $r = \lambda_1 + 2$. However, a general characterization seems to be difficult. Then a partial case has been taken. GD designs with $r = \lambda_1 + 2$ and $n = 3$ are characterized by Adachi [1], who shows that in a GD design with parameters $v = 3m$ $(m \geq 2)$, b, r $(= \lambda_1 + 2 < b)$, k, λ_2, in which $r = \lambda_1 + 2$ and $n = 3$, the $v \times b$ incidence matrix N of such a GD design is, after an appropriate permutation of columns, divided into m^2 3×6 submatrices such as $N = (N_{ij})$, where $N_{11} = N_{22} = \cdots = N_{mm} = (I_3 : I_3)$ or $(J_3 - I_3 : J_3 - I_3)$, and N_{ij} $(i \neq j)$ is a matrix in $\mathcal{H} = \{\mathbf{1}_3' \otimes (x_1, x_2, x_3, \bar{x}_1, \bar{x}_2, \bar{x}_3) | x_i = 0, 1, \bar{x}_i = 1 - x_i, i = 1, 2, 3\} \cup \{O_{3\times 6}, J_{3\times 6}\}$. Moreover in this case, it holds that $b = 2v$. Therefore, we obtain that there are no symmetric GD designs satisfying $r = \lambda_1 + 2$ and $n = 3$.

As the next characterization, we will investigate the combinatorial structure of GD designs with $r = \lambda_1 + 2$ and $n = 4$.

Let N be the $v \times b$ incidence matrix of a GD design with parameters $v = 4m$ $(m \geq 2)$, b, r $(= \lambda_1 + 2 < b)$, k, λ_1, λ_2. Further let $\mathcal{H}_1 = \{\mathbf{1}_4' \otimes (x_1, x_2, x_3, x_4, \ \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) | \ x_i = 0, 1, \bar{x}_i = 1 - x_i, i = 1, 2, 3, 4\} \cup \{O_{4\times 8}, J_{4\times 8}\}$ and $\mathcal{H}_2 = \{\mathbf{1}_4' \otimes (x_1, x_2, x_3, x_1, x_2, x_3) | x_i = 0, 1, i = 1, 2, 3\}$. Then the following main theorem can be established.

Theorem 2.2 (Adachi, Kageyama and Jimbo [2]). *A GD design with $r = \lambda_1 + 2$ and $n = 4$ is regular and its $v \times b$ incidence matrix N is, after an appropriate permutation of rows and columns, divided into m^2 submatrices as follows:*

$$N = \begin{pmatrix} N_{11} & N_{12} & \cdots & N_{1m} \\ N_{21} & N_{22} & \cdots & N_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ N_{m1} & N_{m2} & \cdots & N_{mm} \end{pmatrix}, \tag{2.1}$$

where N_{ij} satisfy one of the following:

(i) *For $i \neq j$, N_{ij} are 4×8 matrices in $\mathcal{H}_1$ and*

$$N_{11} = N_{22} = \cdots = N_{mm} = (I_4 : I_4) \ \text{or} \ (J_4 - I_4 : J_4 - I_4).$$

In this case, $b = 2v$ holds.

(ii) *For $i \neq j$, N_{ij} are 4×6 matrices in $\mathcal{H}_2$ and*

$$N_{11} = N_{22} = \cdots = N_{mm} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \ (= S, \mathit{say}).$$

In this case, $b = 3v/2$ holds.

Remark. There are no symmetric GD designs satisfying $r = \lambda_1 + 2$ and $n = 4$.

The difficulty of a characterization of GD designs with $r = \lambda_1 + 2$ and general $n (\geq 5)$ is to have to take into account possible complicated inner structure with n rows corresponding to each group partitioning a set of v points. On account of such complexity, the structure will not be able to be characterized in a general form. Hence we may have to devise another approach to a characterization of GD designs with $r = \lambda_1 + 2$. Once the value of n is given, a characterization of GD designs in this set-up may be possible after tedious consideration. However, its task is troublesome and sometimes not feasible.

3. Group divisible designs without α-resolution class

In this section, we will characterize the combinatorial structure of GD designs without "α-resolution class" in each group. All the results in this section are due to [3].

3.1. Definition of an α-resolution class

In this subsection, we define an (r, λ)-design and an α-resolution class, which will be utilized when we consider some substructure in each group of GD designs.

For positive integers v, r, λ, an *(r, λ)-design* with parameters v, r, λ is a pair $(V, \mathcal{B})$ where V is a v-set of points and $\mathcal{B}$ is a collection of subsets of V such that every point of V occurs in r blocks of $\mathcal{B}$, and that any two distinct points of V occur together in exactly λ blocks of $\mathcal{B}$. In particular, when every block has the same size ($= k$, say), an (r, λ)-design is exactly a BIB design.

For a subcollection $\mathcal{B}'$ ($\subset \mathcal{B}$), if every point of V occurs in exactly α blocks ($1 \le \alpha \le r$) in $\mathcal{B}'$, then $\mathcal{B}'$ is called an *α-resolution class* of $(V, \mathcal{B})$. An α-resolution class is said to be *trivial* when $\alpha = r$, and *nontrivial* when $1 \le \alpha \le r - 1$. In this paper, an α-resolution class implies a nontrivial α-resolution class if it is not specified.

Here, we will give examples of α-resolution classes, in which one has nontrivial α-resolution class, while the other does not.

Example 3.1. The following design is a (3,1)-design with nontrivial 1-resolution classes.

$$S = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Example 3.2. The following design is a (3,1)-design with no nontrivial α-resolution class.

$$T = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

3.2. Combinatorial structure of these designs

Let N be the $v \times b$ incidence matrix of a GD design with parameters $v = mn$ $(m, n \ge 2)$, b, r $(< b)$, k, λ_1, λ_2. Any groups G_l $(l = 1, 2, \cdots, m)$ of the GD design have the $n \times b$ incidence matrices $B_l = (N_l^* : J : O)$ after appropriate permutations of columns, where N_l^* are the incidence matrices of (r_l^*, λ_l^*)-designs with parameters $v_l^* = n$, b_l^*, r_l^* $(< b_l^*)$, λ_l^* $(< r_l^*)$ and with block sizes less than n. In this paper, we suppose that all (r_l^*, λ_l^*)-designs with the incidence matrices N_l^* do not have any α-resolution classes, if not specified. We call such design a GD design without α-resolution classes in each group. Then the following two main theorems can be established.

Theorem 3.1 (Adachi, Jimbo and Kageyama [3]). *Suppose that a GD design without α-resolution classes in each group has parameters $v = mn$ $(m, n \ge 2)$, b, r $(< b)$, k, λ_1, λ_2. Then, the incidence* *matrix N of the GD design is, after an appropriate permutation of rows and columns, represented by*

$$N = \begin{pmatrix} N_1^* & O_{n\times b^*} & \cdots & O_{n\times b^*} \\ O_{n\times b^*} & N_2^* & \cdots & O_{n\times b^*} \\ \vdots & \vdots & \ddots & \vdots \\ O_{n\times b^*} & O_{n\times b^*} & \cdots & N_m^* \end{pmatrix} + D\otimes J_{n\times b^*}, \tag{3.1}$$

where all of N_l^* *are the incidence matrices of BIB designs with the same parameters* $v_l^* = n$, $b_l^* = b^*$, $r_l^* = r^*$, $k_l^* = k^*$, $\lambda_l^* = \lambda^*$, *and* $D = (d_{ij})$ *is an* $m\times m$ *matrix with entries* 0 *or* 1 *and* $d_{ii} = 0$ *for all* i.

Since N is the incidence matrix of a GD design, each row of D has the same number of 1's. Let s (≥ 1) be the number of 1's in each row of D. For convenience, we denote the first term of (3.1), by $\mathrm{diag}(N_1^*, N_2^*, \cdots, N_m^*)$.

Theorem 3.2 (Adachi, Jimbo and Kageyama [3]). *Let N be the incidence matrix* (3.1) *of a GD design without* α*-resolution classes in each group. Then the GD design is regular and N is characterized as follows:*

(i) *When* $b^* \neq 2r^*$ *and* $\lambda_2 \equiv 0 \pmod{b^*}$, *the incidence matrix of the GD design is given by* $N = \mathrm{diag}(N_1^*, N_2^*, \cdots, N_m^*)$ *for general m and n, that is,* $D = O_m$, *which leads to a GD design with parameters* $v = mn$, $b = mb^* = mnr^*/k^*$, $r = r^*$, $k = k^*$, $\lambda_1 = r^*(k^*-1)/(n-1)$, $\lambda_2 = 0$.

(ii) *When* $b^* \neq 2r^*$ *and* $\lambda_2 \equiv 2r^* \pmod{b^*}$, *the incidence matrix of the GD design is given by* $N = \mathrm{diag}(N_1^*, N_2^*, \cdots, N_m^*) + (J_m - I_m)\otimes J_{n\times b^*}$ *for general m and n, that is,* $D = J_m - I_m$, *which leads to a GD design with parameters* $v = mn$, $b = mb^* = mnr^*/k^*$, $r = r^*(mn - n + k^*)/k^*$, $k = k^* + (m-1)n$, $\lambda_1 = r^*\{(m-1)n(n-1) + k^*(k^*-1)\}/\{k^*(n-1)\}$, $\lambda_2 = r^*(mn - 2n + 2k^*)/k^*$.

(iii) *When* $\lambda_2 \equiv r^* \pmod{b^*}$, *D is the adjacency matrix of a Hadamard tournament of order* $m \equiv 3 \pmod 4$, *which leads to a GD design with parameters* $v = mn$, $b = mb^* = mnr^*/k^*$, $r = r^*(mn - n + 2k^*)/(2k^*)$, $k = k^* + (m-1)n/2$, $\lambda_1 = r^*\{(m-1)n(n-1) + 2k^*(k^*-1)\}/\{2k^*(n-1)\}$, $\lambda_2 = r^*(mn - 3n + 4k^*)/(4k^*)$. *In this case, the existence of the GD design is equivalent to that of a Hadamard tournament of order m.*

(iv) *When* $b^* = 2r^*$ *and* $\lambda_2 \equiv 0 \pmod{b^*}$, *D is the adjacency matrix of a strongly regular graph with parameters* $\tilde{v} = m$, $\tilde{k} = s$, $p_{11}^1 = x$, $p_{11}^2 = x+1$, *where* $s^2 = (x+1)(m-1)$, *which leads to a GD design with parameters* $v = mn$, $b = mb^* = 2mr^*$, $r = (2s+1)r^*$, $k = n(2s+1)/2$, $\lambda_1 = r^*\{4(n-1)s + n - 2\}/\{2(n-1)\}$, $\lambda_2 = 2(x+1)r^*$. *In this case, the existence of the GD design is equivalent to that of a strongly regular graph.*

Corollary 3.1. *Theorem* 2.1 *is a special case of Theorem* 3.2.

By Theorem 3.2, we see that the structure of GD designs without α-resolution classes in each group is characterized in terms of Hadamard tournaments and strongly regular graphs, as in Table 2.

4. Constructions of regular group divisible designs

By virtue of the characterization in Theorem 3.2, we obtain new constructions of regular GD designs including a symmetric case. The construction is made by presenting block structures of GD designs.

Table 2: The inner structure of GD designs without α-resolution classes in each group

	$b^* = 2r^*$	$b^* \neq 2r^*$
$\lambda_2 \equiv 0$ (mod b^*)	strongly regular graph	$N = \mathrm{diag}(N_1^*, \cdots, N_m^*)$
$\lambda_2 \equiv 2r^*$ (mod b^*)	strongly regular graph	$N = \mathrm{diag}(N_1^*, \cdots, N_m^*)$ $+(J_m - I_m) \otimes J_{n \times b^*}$
$\lambda_2 \equiv r^*$ (mod b^*)	Hadamard tournament	Hadamard tournament
$\lambda_2 \not\equiv 0, r^*, 2r^*$ (mod b^*)	nonexistence	nonexistence

Theorem 4.1. *The existence of a Hadamard tournament of order $m \equiv 3$ (mod 4) and a BIB design with parameters v^*, b^*, r^*, k^*, λ^* implies the existence of a regular GD design with parameters $v = mv^*$, $b = mb^*$, $r = r^* + b^*(m-1)/2$, $k = k^* + v^*(m-1)/2$, $\lambda_1 = \lambda^* + b^*(m-1)/2$, $\lambda_2 = r^* + b^*(m-3)/4$. As a special case, when the initial BIB design is symmetric, the GD design is also symmetric.*

Proof. Let D be the $m \times m$ adjacency matrix of the Hadamard tournament of order $m \equiv 3$ (mod 4). Let N^* be the $v^* \times b^*$ incidence matrix of the BIB design. We define the $mv^* \times mb^*$ matrix N as follows:

$$N = \begin{pmatrix} N^* & O_{v^* \times b^*} & \cdots & O_{v^* \times b^*} \\ O_{v^* \times b^*} & N^* & \cdots & O_{v^* \times b^*} \\ \vdots & \vdots & \ddots & \vdots \\ O_{v^* \times b^*} & O_{v^* \times b^*} & \cdots & N^* \end{pmatrix} + D \otimes J_{v^* \times b^*}. \tag{4.1}$$

Then there are $(m-1)/2$ 1's in each row of D. Thus any two points in the same group of the GD design are contained in $\lambda_1 = r^* + b^*(m-1)/2$ blocks. On the other hand, in any two rows of D, there are $(m-3)/4$ columns having 1's in those rows. Thus, $\lambda_2 = r^* + b^*(m-3)/4$, which implies that N is an incidence matrix of a GD design. Especially, if $v^* = b^*$, the GD design is also symmetric. □

Theorem 4.2. *The existence of a strongly regular graph with parameters $\tilde{v}$, $\tilde{k} = \sqrt{p_{11}^2(\tilde{v}-1)}$, p_{11}^1, $p_{11}^2 = p_{11}^1 + 1$ and a BIB design with parameters $v^* = 2k^*$, $b^* = 2r^*$, r^*, k^*, λ^* implies the existence of a regular GD design with parameters $v = \tilde{v}v^*$, $b = \tilde{v}b^*$, $r = r^* + \tilde{k}b^*$, $k = k^* + \tilde{k}v^*$, $\lambda_1 = \lambda^* + \tilde{k}b^*$, $\lambda_2 = p_{11}^2 b^*$.*

Proof. Let D be the $\tilde{v}\times\tilde{v}$ adjacency matrix of the strongly regular graph. Let N^* be the $v^*\times b^*$ incidence matrix of the BIB design. Then we define a $\tilde{v}v^*\times\tilde{v}v^*$ matrix N of a GD design in the same manner as in (4.1). Let $V_1 = \{x_1, x_2, \cdots, x_{\tilde{v}}\}$ be the set of vertices of D and $V_2 = \{y_1, y_2, \cdots, y_{v^*}\}$ be the set of points of N^*. Then the set of points of a GD design is defined by $V_1 \times V_2$. For any two points (x_i, y_j) and $(x_{i'}, y_j)$ in the same group occur together in $\lambda_1 = r^* + \tilde{k}b^*$. Now we take two points (x_i, y_j) and $(x_{i'}, y_{j'})$ in the distinct groups. If $(x_i, x_{i'})$ is an edge of D, then there are p_{11}^1 columns in D which has 1's in both of the rows corresponding to x_i and $x_{i'}$. In this case, $\lambda_2 = 2r^* + p_{11}^1 b^*$. While, if $(x_i, x_{i'})$ is not an edge, then there are p_{11}^2 columns in D which has 1's in both of the rows corresponding to x_i and $x_{i'}$, in this case, $\lambda_2 = p_{11}^2 b^*$. By the assumption, $2r^* + p_{11}^1 b^* = p_{11}^2 b^*$ holds, thus λ_2 is also constant. It is easy to show that the block size $k = k^* + \tilde{k}v^*$ is constant. Hence the theorem is proved. □

Now we consider a regular symmetric GD design which is obtained by Theorem 4.2. By the assumption of the theorem, $v^* = b^* = 2r^* = 2k^*$ holds. Thus by relation (1.3), $\lambda^*(2k^*-1) = k^*(k^*-1)$ must hold. By solving this equation for k^*, $4\lambda^{*2} + 1$ must be a square of an integer, which is possible only in the case of $v^* = b^* = 2r^* = 2k^* = 2$ and $\lambda^* = 0$. By this investigation we obtain the following corollary:

Corollary 4.1. *The existence of a strongly regular graph with parameters* $\tilde{v}$, $\tilde{k} = \sqrt{p_{11}^2(\tilde{v}-1)}$, p_{11}^1, $p_{11}^2 = p_{11}^1 + 1$ *implies the existence of a regular symmetric GD design with parameters* $v = b = 2\tilde{v}$, $r = k = 2\tilde{k} + 1$, $\lambda_1 = 2\tilde{k}$, $\lambda_2 = 2p_{11}^1$.

As an example of Corollary 4.1, a regular symmetric GD design with parameters $v = b = 20$, $r = k = 7$, $\lambda_1 = 6$, $\lambda_2 = 2$ can be obtained by utilizing the Petersen graph. Moreover, there exists a strongly regular graph with parameters $\tilde{v} = q = 4t + 1$ (for prime power q), $\tilde{k} = 2t$, $p_{11}^1 = t - 1$, $p_{11}^2 = t$, which is called the *Paley graph* (see [10, pp.668–683]). Paley graphs satisfy the condition of Theorem 4.2. By utilizing Paley graphs, we obtain the following corollary.

Corollary 4.2. *For a prime power* $q = 4t + 1$, *there exists a regular symmetric GD design with parameters* $v = b = 2q$, $r = k = 4t + 1 = q$, $\lambda_1 = 4t$, $\lambda_2 = 2t$.

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