On Weierstrass 7-semigroups

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§1. Introduction.

Let \( \mathbb{N} \) be the additive semigroup of non-negative integers. A subsemigroup \( H \) of \( \mathbb{N} \) is called a numerical semigroup if the complement \( \mathbb{N}\setminus H \) of \( H \) in \( \mathbb{N} \) is a finite set. For any positive integer \( n \) a numerical semigroup \( H \) is called an \( n \)-semigroup if \( H \) starts with \( n \), i.e., the minimum positive integer in \( H \) is \( n \). For a non-singular complete irreducible curve \( C \) over an algebraically closed field \( k \) of characteristic 0 (which is called a curve in this paper) and its point \( P \) we set

\[
H(P) = \{ n \in \mathbb{N} | \exists \text{ a rational function } f \text{ on } C \text{ with } (f)_\infty = nP \}.
\]

A numerical semigroup is Weierstrass if there exists a curve \( C \) with its point \( P \) such that \( H(P) = H \). We are interested in the following problem:

**Problem 1.** Is every \( n \)-semigroup Weierstrass?

We have the following positive results:

**Fact 2.** For \( n \leq 5 \) every \( n \)-semigroup is Weierstrass. (For \( n = 2 \), classical, for \( n = 3 \), see [8] and for \( n = 4,5 \), see [4],[5] respectively.)

But we know the negative result as follows:

**Fact 3.** For any \( n \geq 13 \), there exists a non-Weierstrass \( n \)-semigroup. (For \( n = 13 \), [1] and for \( n \geq 14 \) see, for example, [7].)

Thus, we have the following problem:

**Problem 4.** For \( 6 \leq n \leq 12 \), is every \( n \)-semigroup Weierstrass or is there a non-Weierstrass \( n \)-semigroup?

In this paper we are devoted to the study of 7-semigroups. In Section 2 we determine the 7-semigroups which are the semigroups \( H(P) \) of ramification points \( P \) on cyclic coverings of the projective line \( \mathbb{P}^1 \) with degree 7. In Section 3 we divide the Weierstrass 7-semigroups generated by 4 elements into 31 cases and investigate whether such a 7-semigroup is of toric type in each case where a numerical semigroup is said to be of toric type if roughly speaking, the monomial curve associated to the numerical semigroup is a specialization of some affine toric variety, because we know that a numerical semigroup of toric type is Weierstrass ([4]).
§2. Cyclic 7-semigroups.

An $n$-semigroup is said to be cyclic if it is the semigroup $H(P)$ for some totally ramification point $P$ on a cyclic covering of the projective line $\mathbb{P}^1$ with degree $n$. In this section we describe a necessary and sufficient condition on a 7-semigroup to be cyclic. Moreover, some non-cyclic Weierstrass 7-semigroups are given. We use the following notation: For an $n$-semigroup $H$ we set

$$S(H) = \{n, s_1, \ldots, s_{n-1}\}$$

where $s_i = \text{Min}\{h \in H|h \equiv i \mod n\}$. We have the following necessary condition on an $n$-semigroup to be cyclic if $n$ is prime.

**Fact 5** ([9]). Let $p$ be a prime number. If $H$ is a cyclic $p$-semigroup with

$$S(H) = \{p, s_1, \ldots, s_{p-1}\},$$

then

$$s_i + s_{p-i} = s_j + s_{p-j}, \text{ all } i, j.$$

We had already obtained an answer to the converse problem of the above statement.

**Fact 6.** i) For a prime number $p \leq 7$, the converse of Fact 5 is true (See [9]).

ii) For any prime number $p \geq 11$, the converse of Fact 5 is false (See [3]).

By Fact 6 i) we get the following:

**Proposition 7.** Let $H$ be a 7-semigroup with

$$S(H) = \{7, s_1, \ldots, s_6\}.$$

Assume that

$$s_1 + s_6 = s_2 + s_5 = s_3 + s_4.$$

Then $H$ is cyclic, hence Weierstrass.

For any positive integers $b_0, \ldots, b_m$, $<b_0, \ldots, b_m>$ denotes the semigroup generated by $b_0, \ldots, b_m$. We give examples of cyclic 7-semigroups.

**Example 8.** (1) Let $H = <7, 8, 10, 12>$. Then $S(H) = \{7, 8, 10, 12, 16, 18, 20\}$. Since $8 + 20 = 16 + 12 = 10 + 18$, $H$ is cyclic, hence Weierstrass.

(2) Let $H = <7, 15, 16, 17, 25, 26, 27>$. Then $S(H) = \{7, 15, 16, 17, 25, 26, 27\}$. Since $15 + 27 = 16 + 26 = 17 + 25$, $H$ is cyclic, hence Weierstrass.

We also have non-cyclic Weierstrass 7-semigroups.
Fact 9. For integers $g$ and $s$ with $7 \leq g \leq s \leq 12$, let $H_{s,g}$ be a 7-semigroup with
\[ \mathbb{N}_0 \setminus H_{s,g} = \{1, \ldots, 6, 8 + s - g, \ldots, s + 1\}. \]
Then we have the following:
i) There exists a covering $C \rightarrow \mathbb{P}^1$ of degree 3 with non-ramification point $P \in C$ such that $H(P) = H_{s,g}$. Hence, $H_{s,g}$ is a Weierstrass 7-semigroup (See [2]).
ii) If $(s, g) \neq (9, 9), (12, 9), (12, 12)$, then $H_{s,g}$ is non-cyclic. For example, $H_{11,9} = \langle 7, 8, 9, 13, 19 \rangle$ and $H_{12,10} = \langle 7, 8, 9, 19, 20 \rangle$ are non-cyclic Weierstrass 7-semigroups.

Fact 10. Let $H$ be the 7-semigroup $\langle 7, 9, 10, 11, 12, 13 \rangle$. Then there is a cyclic covering of an elliptic curve of degree 8 with only two ramification points $P_1$ and $P_2$, which are totally ramified, such that $H(P_1) = H(P_2) = H$. Hence $\langle 7, 9, 10, 11, 12, 13 \rangle$ is a non-cyclic Weierstrass 7-semigroup (See [6]).

§3. 7-semigroups of toric type.

For a numerical semigroup $H$ we denote by $M(H)$ the minimal set of generators for $H$. In this section we are interested in 7-semigroups $H$ with $M(H) = \{7, a_1, a_2, a_3\}$ which satisfy the following condition:

Definition 11. Let $H$ be a numerical semigroup with $\# M(H) = m + 1$. The semigroup $H$ is said to be of toric type if
\begin{itemize}
  \item $\exists l$: a positive integer,
  \item $\exists S$: a saturated subsemigroup of $\mathbb{Z}^l$ generated by $b_1, \ldots, b_{l+m}$ which generates $\mathbb{Z}^l$ as a group and
  \item $\exists g_j's$ ($j = 1, \ldots, l + m$): monomials in $k[X_0, X_1, \ldots, X_m]$ such that
\end{itemize}
where the right vertical map is induced by the $k$-algebra homomorphism
\[ \eta_S : k[Y_1, \ldots, Y_{l+m}] \rightarrow k[S][X_0, X_1, \ldots, X_m] \]
which sends $Y_j$ to $T^{b_j} - g_j$, that is to say,
\begin{itemize}
  \item Spec $k[H] \rightarrow$ Spec $k[S][X_0, X_1, \ldots, X_m]$
  \item Spec $k$ \arrow $\Rightarrow$ Spec $k[Y_1, \ldots, Y_{l+m}]$
  \item (0) \arrow $\Rightarrow$ the origin
\end{itemize}
where the horizontal maps are the embeddings through the generators and the right vertical map is induced by the $k$-algebra morphism from $k[Y_1, \ldots, Y_{l+m}]$ to $k[X_0, X_1, \ldots, X_m]$ sending $Y_j$ to $g_j$. 

We explain how to find a subsemigroup $S$ of $\mathbb{Z}^l$ as in Definition 11 below.

**Remark 12.** Let $H$ be a numerical semigroup with $M(H) = \{a_0, a_1, \ldots, a_m\}$.

i) Determine a generating system of relations among $a_0, a_1, \ldots, a_m$, i.e., a set of generators for the ideal of the monomial curve $\text{Spec} \ k[H]$.

ii) Determine a fundamental system of relations among $a_0, a_1, \ldots, a_m$, i.e., a basis of the relation $\mathbb{Z}$-module among $a_0, a_1, \ldots, a_m$.

iii) We construct a subsemigroup $S$ of $\mathbb{Z}^l$ from the fundamental system. In this case, $S$ is generated by $l + m$ elements $b_j$'s and generates $\mathbb{Z}^l$ as a group naturally. Moreover, we associate the generators $b_j$'s for $S$ to monomials $g_j$'s in $k[X_0, \ldots, X_m]$ such that we have the fiber products in Definition 11.

iv) The remaining problem is whether the semigroup $S$ is saturated or not. We note that $S$ is saturated if and only if the semigroup ring $k[S]$ is normal, i.e., $\text{Spec} \ k[S]$ is an affine toric variety. If $S$ is saturated, the numerical semigroup $H$ become of toric type.

From now on we treat only 7-semigroups generated by 4 elements.

**Lemma 13.** Let $H$ be a 7-semigroup generated by 4 elements, i.e., $M(H) = \{7, a_1, a_2, a_3\}$. Renumbering $a_1, a_2$ and $a_3$ it satisfies one of the following:

(I) $a_1 + a_2 + a_3 \equiv 0 \ (7)$,

(II) $a_1 + a_2 \equiv 0 \ (7)$,

(III) $2a_1 + a_2 \equiv 0 \ (7)$ and $2a_2 + a_3 \equiv 0 \ (7)$.

We give the construction of a saturated subsemigroup $S$ of $\mathbb{Z}^l$ as in Definition 11 in (I) and some cases of (II).

Case (I) $a_1 + a_2 + a_3 \equiv 0 \ (7)$. A fundamental system of relations consists of

$$\frac{a_1 + a_2 + a_3}{7}a_0 = a_1 + a_2 + a_3, \ 2a_1 = \frac{2a_1 - a_2}{7}a_0 + a_2, \ 2a_2 = \frac{2a_2 - a_3}{7}a_0 + a_3.$$ 

For example, the relation

$$2a_3 = \frac{2a_3 - a_1}{7}a_0 + a_1$$

is derived from the addition of the three relations. The determinant of the matrix consisting of the coefficients of the three relations is

$$\begin{vmatrix}
(a_1 + a_2 + a_3)/7 & -1 & -1 \\
-(2a_1 - a_2)/7 & 2 & -1 \\
-(2a_2 - a_3)/7 & 0 & 2
\end{vmatrix} = a_3.$$

A numerical semigroup $H$ with $M(H) = \{a_0, a_1, a_2, a_3\}$ satisfying the above condition is said to be 1-neat. Under the above condition we get a saturated subsemigroup $S$ of $\mathbb{Z}^6$ as in Definition 11 from the fundamental system.
Case (II-1) $a_1 + a_2 = 0$ (7) and $2a_1 = a_3$ (7).

Case (II-1-i) $2a_2 < a_1 + 2a_3$ and $2a_3 < 3a_2$. A generating system for relations consists of

$$\frac{a_1 + a_2}{7} a_0 = a_1 + a_2, \quad 2a_1 = \frac{2a_1 - a_3}{7} a_0 + a_3, \quad 3a_2 = \frac{3a_2 - 2a_3}{7} a_0 + 2a_3,$$

$$3a_3 = \frac{3a_3 - a_2}{7} a_0 + a_2, \quad \frac{a_2 + a_3 - a_1}{7} a_0 + a_1 = a_2 + a_3,$$

$$\frac{a_1 + 2a_3 - 2a_2}{7} a_0 + 2a_2 = a_1 + 2a_3.$$

i.e., the kernel of

$$\varphi_H : k[X_0, X_1, X_2, X_3] \rightarrow k[t^{a_0}, t^{a_1}, t^{a_2}, t^{a_3}]$$

is generated by

$$X_0^{a_1 + a_2} - X_1 X_2, \quad X_1^2 - X_0^{2a_1 - a_3} X_3, \quad X_2^3 - X_0^{3a_3 - 2a_3} X_3,$$

$$X_3 - X_0^{3a_3 - a_2} X_2, \quad X_0^{a_2 + a_3 - a_1} - X_1 - X_2 X_3, \quad X_0^{a_1 + 2a_3 - 2a_2} X_2^2 - X_1 X_3^2.$$

A fundamental system of relations is the following:

$$\frac{a_1 + a_2}{7} a_0 = a_1 + a_2, \quad 2a_1 = \frac{2a_1 - a_3}{7} a_0 + a_3, \quad 3a_2 = \frac{3a_2 - 2a_3}{7} a_0 + 2a_3.$$

For example, the addition of the first and second relations

$$\frac{a_1 + a_2}{7} a_0 + 2a_1 = \left( a_1 + a_2 \right) + \left( \frac{2a_1 - a_3}{7} a_0 + a_3 \right)$$

induces the fifth relation. To get a subsemigroup $S$ of $\mathbb{Z}^3$ we divide this case into three cases again.

Case (II-1-i-A) $a_1 + 2a_2 > 3a_3$. We divide the coefficients in the fundamental system of relations into the following:

$$(a_0' + a_0'' + a_0''') a_0 = a_0 a_1 + a_0' a_2, \quad 2a_0 a_1 = (a_0' + a_0'') a_0 + a_1 a_3,$$

$$(2a_0 + a_2') a_2 = (a_0' + a_0''') a_0 + a_2 a_3.$$

We associate elements of $\mathbb{Z}^5$ to the components of the above system as follows:

$$a_0' a_0 \mapsto b_1 = e_1, \quad a_0'' a_0 \mapsto b_2 = e_2, \quad a_0''' a_0 \mapsto b_3 = e_3, \quad a_0 a_1 \mapsto b_4 = e_4,$$

$$a_0' a_2 \mapsto b_5 = e_5, \quad a_0 a_2 \mapsto b_6 = (1, 1, 1, -1, 0),$$
\[ \alpha_{13}a_3 \mapsto b_7 = (-1, -1, 0, 2, 0), \alpha_{23}a_3 \mapsto b_8 = (1, 2, 1, -2, 0). \]

where \( e_i \) denotes the vector whose \( i \)-th component is 1 and \( j \)-th component is 0 if \( j \neq i \). Let \( S \) be the subsemigroup of \( \mathbb{Z}^5 \) generated by \( b_1, \ldots, b_8 \). We can show that

\[
\sum_{i=1}^{8} \mathbb{R}_+ b_i \cap \mathbb{Z}^5 \subseteq S
\]

where \( \mathbb{R}_+ \) denotes the set of non-negative real numbers. Hence, \( S \) is saturated.

Case (II-i-B) \( a_1 + 2a_2 < 3a_3 \). We divide the coefficients in the fundamental system of relations into the following:

\[
\begin{align*}
(\alpha'_0 + \alpha_{10} + \alpha_{20})a_0 &= \alpha_{01}a_1 + \alpha_{02}a_2, \\
2\alpha_{01}a_1 &= \alpha_{10}a_0 + \alpha_{13}a_3, \\
(2\alpha_{02} + \alpha'_2)a_2 &= \alpha_{20}a_0 + \alpha_{23}a_3.
\end{align*}
\]

We associate elements of \( \mathbb{Z}^5 \) to the components of the above system as follows:

\[
\begin{align*}
\alpha'_0 a_0 &\mapsto b_1 = e_1, \alpha_{10}a_0 &\mapsto b_2 = e_2, \alpha_{20}a_0 &\mapsto b_3 = e_3, \alpha_{01}a_1 &\mapsto b_4 = e_4, \\
\alpha'_2 a_2 &\mapsto b_5 = e_5, \alpha_{02}a_2 &\mapsto b_6 = (1, 1, 1, -1, 0), \\
\alpha_{13}a_3 &\mapsto b_7 = (0, -1, 0, 2, 0), \alpha_{23}a_3 &\mapsto b_8 = (2, 2, 1, -2, 1).
\end{align*}
\]

Let \( S \) be the subsemigroup of \( \mathbb{Z}^5 \) generated by \( b_1, \ldots, b_8 \). Then \( S \) is saturated.

Case (II-i-C) \( a_1 + 2a_2 = 3a_3 \). In the Case (II-i-A) let \( \alpha'_0 = 0 \). We get a subsemigroup \( S \) of \( \mathbb{Z}^4 \) generated by 7 elements. Then \( S \) is saturated.

But our method does not work well in the following case.

Case (III-2-i) \( 2a_1 + a_2 \equiv 0, 2a_2 + a_3 \equiv 0, 2a_1 \leq a_2 + a_3, 2a_2 > 3a_1 \). We have the following generating system of relations

\[
\begin{align*}
\frac{2a_1 + a_2}{7}a_0 &= 2a_1 + a_2, \\
4a_1 &= \frac{4a_1 - a_3}{7}a_0 + a_3, \\
2a_2 &= \frac{2a_2 - 3a_1}{7}a_0 + 3a_1, \\
2a_3 &= \frac{2a_3 - a_1}{7}a_0 + a_1, \\
\frac{a_2 + a_3 - 2a_1}{7}a_0 + 2a_1 &= a_2 + a_3, \\
\frac{a_1 + a_3 - a_2}{7}a_0 + a_2 &= a_1 + a_3.
\end{align*}
\]
The three equations (1), (2) and (6) in the generating system of relations form a fundamental system. In fact,

\[(1) + (2) = (5), \quad t(1) + t(2) + (6) = (3)\] and \[t(1) + t(2) + t(6) = (4).\]

We divide the coefficients in the fundamental system of relations into the following:

\[(\alpha_{10} + \alpha_{20} + \alpha_{0}')a_{0} = \alpha_{01}a_{1} + \alpha_{1}'a_{1}, \quad (\alpha_{01} + \alpha_{1} + \alpha_{31})a_{1} = \alpha_{10}a_{0} + \alpha_{31}a_{3},\]

\[\alpha_{0}'a_{0} + \alpha_{2}'a_{2} = \alpha_{1}'a_{1} + \alpha_{13}a_{3}.\]

We associate elements of \(\mathbb{Z}^5\) to the components of the above system as follows:

\[\alpha_{10}a_{0} \mapsto b_{1} = e_{1}, \quad \alpha_{20}a_{0} \mapsto b_{2} = e_{2}, \quad \alpha_{0}'a_{0} \mapsto b_{3} = e_{3}, \quad \alpha_{01}a_{1} \mapsto b_{4} = e_{4},\]

\[\alpha_{1}'a_{1} \mapsto b_{5} = e_{5}, \quad \alpha_{2}'a_{2} \mapsto b_{6} = (1, 1, 1, -1, 0),\]

\[\alpha_{31}a_{1} \mapsto b_{7} = (2, 1, 2, -2, -2), \quad \alpha_{13}a_{3} \mapsto b_{8} = (1, 1, 2, -1, -1).\]

Let \(S\) be the subsemigroup of \(\mathbb{Z}^5\) generated by \(b_{1}, \ldots, b_{8}\). Then \(S\) is not saturated. In fact,

\[2(1, 1, 1, -1, -1) = (2, 2, 2, -2, -2) = b_{2} + b_{7} \in S,\]

but

\[(1, 1, 1, -1, -1) \notin S \text{ and } (1, 1, 1, -1, -1) \in \mathbb{Z}^5.\]

Hence, \(\text{Spec } k[S]\) is not a toric variety.

To check whether a 7-semigroup generated by 4 elements is of toric type we divide them into the 31 cases in the following table. But this problem is still open in the last three cases. The right-hand side of column in the table means the dimension of the affine toric variety which is constructed from a numerical semigroup of given type in our way.
<table>
<thead>
<tr>
<th>Condition</th>
<th>Toric</th>
<th>dim</th>
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</thead>
<tbody>
<tr>
<td>$a_1 + a_2 + a_3 = 0$</td>
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<td>6</td>
</tr>
<tr>
<td>II-1-i-A</td>
<td>$a_1 + a_2 = 0$, $2a_1 \equiv a_3$, $2a_2 &lt; a_1 + 2a_3$, $2a_3 &lt; 3a_2$, $a_1 + 2a_2 &gt; 3a_3$</td>
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<td>II-1-i-B</td>
<td>$a_1 + a_2 = 0$, $2a_1 \equiv a_3$, $2a_2 &lt; a_1 + 2a_3$, $2a_3 &lt; 3a_2$, $a_1 + 2a_2 &lt; 3a_3$</td>
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<tr>
<td>II-1-ii-A</td>
<td>$a_1 + a_2 = 0$, $2a_1 \equiv a_3$, $2a_3 &gt; 3a_2$, $4a_2 &gt; a_1 + a_3$</td>
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<td>II-1-ii-B</td>
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<td>II-1-iii-A</td>
<td>$a_1 + a_2 = 0$, $2a_1 \equiv a_3$, $2a_2 &gt; a_1 + 2a_3$</td>
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<tr>
<td>II-1-iii-B</td>
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<tr>
<td>II-1-iv</td>
<td>$a_1 + a_2 = 0$, $2a_1 \equiv a_3$, $2a_3 = 3a_2$</td>
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<td>II-2-i-A</td>
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<td>II-2-i-B</td>
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<tr>
<td>II-2-iii-C</td>
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<td>II-2-iii-F</td>
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<td>II-2-iii-G</td>
<td>$a_1 + a_2 = 0$, $3a_1 \equiv a_3$, $2a_2 &lt; 2a_1 + a_3$, $2a_2 &gt; a_1 + 2a_3$, $3a_2 &lt; a_1 + a_3$, $3a_2 &gt; 2a_2 &gt; 2a_3$</td>
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<td>II-2-iv-A</td>
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<td>II-2-iv-B</td>
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<td>II-2-iv-C</td>
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<td>III-1</td>
<td>$2a_1 + a_2 = 0$, $2a_2 + a_3 = 0$, $2a_1 &gt; a_2 + a_3$</td>
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<td>III-2-i</td>
<td>$2a_1 + a_2 = 0$, $2a_2 + a_3 = 0$, $2a_1 &lt; a_2 + a_3$, $2a_2 &gt; 3a_1$</td>
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<td>III-2-ii</td>
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<td>III-2-iii</td>
<td>$2a_1 + a_2 = 0$, $2a_2 + a_3 = 0$, $2a_1 &lt; a_2 + a_3$, $2a_2 = 3a_1$</td>
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</table>

References


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