On Weierstrass 7-semigroups

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§1. Introduction.
Let $\mathbb{N}$ be the additive semigroup of non-negative integers. A subsemigroup $H$ of $\mathbb{N}$ is called a numerical semigroup if the complement $\mathbb{N}\setminus H$ of $H$ in $\mathbb{N}$ is a finite set. For any positive integer $n$ a numerical semigroup $H$ is called an $n$-semigroup if $H$ starts with $n$, i.e., the minimum positive integer in $H$ is $n$. For a non-singular complete irreducible curve $C$ over an algebraically closed field $k$ of characteristic 0 (which is called a curve in this paper) and its point $P$ we set

$$H(P) = \{ n \in \mathbb{N} | \exists \text{ a rational function } f \text{ on } C \text{ with } (f)_\infty = nP \}.$$ 

A numerical semigroup is Weierstrass if there exists a curve $C$ with its point $P$ such that $H(P) = H$. We are interested in the following problem:

**Problem 1.** Is every $n$-semigroup Weierstrass?

We have the following positive results:

**Fact 2.** For $n \leq 5$ every $n$-semigroup is Weierstrass. (For $n = 2$, classical, for $n = 3$, see [8] and for $n = 4, 5$, see [4], [5] respectively.)

But we know the negative result as follows:

**Fact 3.** For any $n \geq 13$, there exists a non-Weierstrass $n$-semigroup. (For $n = 13$, [1] and for $n \geq 14$ see, for example, [7].)

Thus, we have the following problem:

**Problem 4.** For $6 \leq n \leq 12$, is every $n$-semigroup Weierstrass or is there a non-Weierstrass $n$-semigroup?

In this paper we are devoted to the study of 7-semigroups. In Section 2 we determine the 7-semigroups which are the semigroups $H(P)$ of ramification points $P$ on cyclic coverings of the projective line $\mathbb{P}^1$ with degree 7. In Section 3 we divide the Weierstrass 7-semigroups generated by 4 elements into 31 cases and investigate whether such a 7-semigroup is of toric type in each case where a numerical semigroup is said to be of toric type if roughly speaking, the monomial curve associated to the numerical semigroup is a specialization of some affine toric variety, because we know that a numerical semigroup of toric type is Weierstrass ([4]).
§2. Cyclic 7-semigroups.

An \( n \)-semigroup is said to be cyclic if it is the semigroup \( H(P) \) for some totally ramification point \( P \) on a cyclic covering of the projective line \( \mathbb{P}^1 \) with degree \( n \). In this section we describe a necessary and sufficient condition on a 7-semigroup to be cyclic. Moreover, some non-cyclic Weierstrass 7-semigroups are given. We use the following notation: For an \( n \)-semigroup \( H \) we set

\[
S(H) = \{n, s_1, \ldots, s_{n-1}\}
\]

where \( s_i = \text{Min}\{h \in H|h \equiv i \mod n\} \). We have the following necessary condition on an \( n \)-semigroup to be cyclic if \( n \) is prime.

**Fact 5 ([9]).** Let \( p \) be a prime number. If \( H \) is a cyclic \( p \)-semigroup with

\[
S(H) = \{p, s_1, \ldots, s_{p-1}\},
\]

then

\[
s_i + s_{p-i} = s_j + s_{p-j}, \text{ all } i, j.
\]

We had already obtained an answer to the converse problem of the above statement.

**Fact 6.** i) For a prime number \( p \leq 7 \), the converse of Fact 5 is true (See [9]).

ii) For any prime number \( p \geq 11 \), the converse of Fact 5 is false (See [3]).

By Fact 6 i) we get the following:

**Proposition 7.** Let \( H \) be a 7-semigroup with

\[
S(H) = \{7, s_1, \ldots, s_6\}.
\]

Assume that

\[
s_1 + s_6 = s_2 + s_5 = s_3 + s_4.
\]

Then \( H \) is cyclic, hence Weierstrass.

For any positive integers \( b_0, \ldots, b_m, <b_0, \ldots, b_m> \) denotes the semigroup generated by \( b_0, \ldots, b_m \). We give examples of cyclic 7-semigroups.

**Example 8.** (1) Let \( H = <7, 8, 10, 12> \). Then \( S(H) = \{7, 8, 10, 12, 16, 18, 20\} \). Since \( 8 + 20 = 16 + 12 = 10 + 18 \), \( H \) is cyclic, hence Weierstrass.

(2) Let \( H = <7, 15, 16, 17, 25, 26, 27> \). Then \( S(H) = \{7, 15, 16, 17, 25, 26, 27\} \). Since \( 15 + 27 = 16 + 26 = 17 + 25 \), \( H \) is cyclic, hence Weierstrass.

We also have non-cyclic Weierstrass 7-semigroups.
Fact 9. For integers $g$ and $s$ with $7 \leq g \leq s \leq 12$, let $H_{s,g}$ be a 7-semigroup with $N_0 \setminus H_{s,g} = \{1, \ldots, 6, 8 + s - g, \ldots, s + 1\}$. Then we have the following:

i) There exists a covering $C \to \mathbb{P}^1$ of degree 3 with non-ramification point $P \in C$ such that $H(P) = H_{s,g}$. Hence, $H_{s,g}$ is a Weierstrass 7-semigroup (See [2]).

ii) If $(s,g) \neq (9,9), (12,9), (12,12)$, then $H_{s,g}$ is non-cyclic. For example, $H_{11,9} = \langle 7,8,9,13,19 \rangle$ and $H_{12,10} = \langle 7,8,9,19,20 \rangle$ are non-cyclic Weierstrass 7-semigroups.

Fact 10. Let $H$ be the 7-semigroup $\langle 7,9,10,11,12,13 \rangle$. Then there is a cyclic covering of an elliptic curve of degree 8 with only two ramification points $P_1$ and $P_2$, which are totally ramified, such that $H(P_1) = H(P_2) = H$. Hence $\langle 7,9,10,11,12,13 \rangle$ is a non-cyclic Weierstrass 7-semigroup (See [6]).

§3. 7-semigroups of toric type.

For a numerical semigroup $H$ we denote by $M(H)$ the minimal set of generators for $H$. In this section we are interested in 7-semigroups $H$ with $M(H) = \{7, a_1, a_2, a_3\}$ which satisfy the following condition:

Definition 11. Let $H$ be a numerical semigroup with $|M(H)| = m + 1$. The semigroup $H$ is said to be of toric type if

\[ \exists l: \text{a positive integer,} \]
\[ \exists S: \text{a saturated subsemigroup of } \mathbb{Z}^l \text{ generated by } b_1, \ldots, b_{l+m} \text{ which generates } \mathbb{Z}^l \]
\[ \text{as a group and} \]
\[ \exists g_j's (j = 1, \ldots, l + m): \text{monomials in } k[X_0, X_1, \ldots, X_m] \text{ such that} \]
\[ \begin{array}{c}
\text{Spec } k[H] \\
\downarrow \\
\text{Spec } k \\
(0) \\
\end{array} \quad \quad \begin{array}{c}
\text{Spec } k[S][X_0, X_1, \ldots, X_m] \\
\downarrow \\
\text{Spec } k[Y_1, \ldots, Y_{l+m}] \\
\text{the origin} \\
\end{array} \]

where the right vertical map is induced by the $k$-algebra homomorphism

\[ \eta_S : k[Y_1, \ldots, Y_{l+m}] \to k[S][X_0, X_1, \ldots, X_m] \]

which sends $Y_j$ to $T^{b_j} - g_j$, that is to say,

\[ \begin{array}{c}
\text{Spec } k[H] \\
\downarrow \\
\text{Spec } k \\
(0) \\
\end{array} \quad \quad \begin{array}{c}
\text{Spec } k[X_0, X_1, \ldots, X_m] \\
\downarrow \\
\text{Spec } k[Y_1, \ldots, Y_{l+m}] \\
\end{array} \]

where the horizontal maps are the embeddings through the generators and the right vertical map is induced by the $k$-algebra morphism from $k[Y_1, \ldots, Y_{l+m}]$ to $k[X_0, X_1, \ldots, X_m]$ sending $Y_j$ to $g_j$. 
We explain how to find a subsemigroup $S$ of $\mathbb{Z}^l$ as in Definition 11 below.

**Remark 12.** Let $H$ be a numerical semigroup with $M(H) = \{a_0, a_1, \ldots, a_m\}$.
i) Determine a generating system of relations among $a_0, a_1, \ldots, a_m$, i.e., a set of generators for the ideal of the monomial curve $\text{Spec } k[H]$.
ii) Determine a fundamental system of relations among $a_0, a_1, \ldots, a_m$, i.e., a basis of the relation $\mathbb{Z}$-module among $a_0, a_1, \ldots, a_m$.
iii) We construct a subsemigroup $S$ of $\mathbb{Z}^l$ from the fundamental system. In this case, $S$ is generated by $l + m$ elements $b_j$'s and generates $\mathbb{Z}^l$ as a group naturally. Moreover, we associate the generators $b_j$'s for $S$ to monomials $g_j$'s in $k[X_0, \ldots, X_m]$ such that we have the fiber products in Definition 11.
iv) The remaining problem is whether the semigroup $S$ is saturated or not. We note that $S$ is saturated if and only if the semigroup ring $k[S]$ is normal, i.e., $\text{Spec } k[S]$ is an affine toric variety. If $S$ is saturated, the numerical semigroup $H$ become of toric type.

From now on we treat only 7-semigroups generated by 4 elements.

**Lemma 13.** Let $H$ be a 7-semigroup generated by 4 elements, i.e., $M(H) = \{7, a_1, a_2, a_3\}$. Renumbering $a_1, a_2$ and $a_3$ it satisfies one of the following:

(I) $a_1 + a_2 + a_3 \equiv 0 (7)$,

(II) $a_1 + a_2 \equiv 0 (7)$,

(III) $2a_1 + a_2 \equiv 0 (7)$ and $2a_2 + a_3 \equiv 0 (7)$.

We give the construction of a saturated subsemigroup $S$ of $\mathbb{Z}^l$ as in Definition 11 in (I) and some cases of (II).

Case (I) $a_1 + a_2 + a_3 \equiv 0 (7)$. A fundamental system of relations consists of

$$a_1 + a_2 + a_3/7 \equiv a_0, \
2a_1 = 2a_1 - a_2/7 \equiv a_0 + a_2, \ 
2a_2 = 2a_2 - a_3/7 \equiv a_0 + a_3.$$

For example, the relation

$$2a_3 = 2a_3 - a_1/7 \equiv a_0 + a_1$$

is derived from the addition of the three relations. The determinant of the matrix consisting of the coefficients of the three relations is

$$\begin{vmatrix}
(a_1 + a_2 + a_3)/7 & -1 & -1 \\
-(2a_1 - a_2)/7 & 2 & -1 \\
-(2a_2 - a_3)/7 & 0 & 2
\end{vmatrix} = a_3.$$

A numerical semigroup $H$ with $M(H) = \{a_0, a_1, a_2, a_3\}$ satisfying the above condition is said to be 1-neat. Under the above condition we get a saturated subsemigroup $S$ of $\mathbb{Z}^6$ as in Definition 11 from the fundamental system.
Case (II-1) \( a_1 + a_2 \equiv 0 \) (7) and \( 2a_1 \equiv a_3 \) (7).

Case (II-1-i) \( 2a_2 < a_1 + 2a_3 \) and \( 2a_3 < 3a_2 \). A generating system for relations consists of

\[
\frac{a_1 + a_2}{7}a_0 = a_1 + a_2, 2a_1 = \frac{2a_1 - a_3}{7}a_0 + a_3, 3a_2 = \frac{3a_2 - 2a_3}{7}a_0 + 2a_3,
\]

\[
3a_3 = \frac{3a_3 - a_2}{7}a_0 + a_2, \quad \frac{a_2 + a_3 - a_1}{7}a_0 + a_1 = a_2 + a_3,
\]

\[
\frac{a_1 + 2a_3 - 2a_2}{7}a_0 + 2a_2 = a_1 + 2a_3.
\]

i.e., the kernel of

\[
\varphi_H : k[X_0, X_1, X_2, X_3] \rightarrow k[t^{a_0}, t^{a_1}, t^{a_2}, t^{a_3}]
\]

is generated by

\[
\begin{align*}
X_0^{a_1+a_2} - X_1X_2, & \quad X_1^2 - X_0^{2a_1-a_3}X_3, & \quad X_2^3 - X_0^{3a_3-2a_2}X_3, \\
X_3 - X_0^{3a_3-a_2}X_2, & \quad X_0^{a_1+a_3-a_1}X_1 - X_2X_3, & \quad X_0 a_1^{a_1+2a_3-2a_2}X_2^2 - X_1X_3^2.
\end{align*}
\]

A fundamental system of relations is the following:

\[
\frac{a_1 + a_2}{7}a_0 = a_1 + a_2, 2a_1 = \frac{2a_1 - a_3}{7}a_0 + a_3, 3a_2 = \frac{3a_2 - 2a_3}{7}a_0 + 2a_3.
\]

For example, the addition of the first and second relations

\[
\frac{a_1 + a_2}{7}a_0 + 2a_1 = \left( a_1 + a_2 \right) + \left( \frac{2a_1 - a_3}{7}a_0 + a_3 \right)
\]

induces the fifth relation. To get a subsemigroup \( S \) of \( \mathbb{Z}^l \) we divide this case into three cases again.

Case (II-1-i-A) \( a_1 + 2a_2 > 3a_3 \). We divide the coefficients in the fundamental system of relations into the following:

\[
(a_0' + a_0'' + a_0^m)a_0 = a_01a_1 + a_02a_2, 2a_01a_1 = (a_0' + a_0^m)a_0 + a_13a_3,
\]

\[
(2a_02 + a_0')a_2 = (a_0' + a_0^m)a_0 + a_23a_3.
\]

We associate elements of \( \mathbb{Z}^5 \) to the components of the above system as follows:

\[
\begin{align*}
\alpha_0'a_0 & \mapsto b_1 = e_1, \quad \alpha_0''a_0 \mapsto b_2 = e_2, \quad \alpha_0^m a_0 \mapsto b_3 = e_3, \quad \alpha_01a_1 \mapsto b_4 = e_4, \\
\alpha_0'a_2 & \mapsto b_5 = e_5, \quad \alpha_02a_2 \mapsto b_6 = (1, 1, 1, -1, 0),
\end{align*}
\]
$\alpha_{13}a_3 \mapsto b_7 = (-1, -1, 0, 2, 0)$, $\alpha_{23}a_3 \mapsto b_8 = (1, 2, 1, -2, 1)$.

where $e_i$ denotes the vector whose $i$-th component is 1 and $j$-th component is 0 if $j \neq i$. Let $S$ be the subsemigroup of $\mathbb{Z}^5$ generated by $b_1, \ldots, b_8$. We can show that

$$\sum_{i=1}^{8} \mathbb{R}_+ b_i \cap \mathbb{Z}^5 \subseteq S$$

where $\mathbb{R}_+$ denotes the set of non-negative real numbers. Hence, $S$ is saturated.

Case (II-1-i-B) $a_1 + 2a_2 < 3a_3$. We divide the coefficients in the fundamental system of relations into the following:

$$(\alpha'_0 + \alpha_{10} + \alpha_{20})a_0 = \alpha_{01}a_1 + \alpha_{02}a_2, 2\alpha_{01}a_1 = \alpha_{10}a_0 + \alpha_{13}a_3,$$

$$(2\alpha_{02} + \alpha'_2)a_2 = \alpha_{20}a_0 + \alpha_{23}a_3.$$}

We associate elements of $\mathbb{Z}^5$ to the components of the above system as follows:

$\alpha'_0 a_0 \mapsto b_1 = e_1$, $\alpha_{10}a_0 \mapsto b_2 = e_2$, $\alpha_{20}a_0 \mapsto b_3 = e_3$, $\alpha_{01}a_1 \mapsto b_4 = e_4$,

$\alpha'_2 a_2 \mapsto b_5 = e_5, \alpha_{02}a_2 \mapsto b_6 = (1, 1, 1, -1, 0)$,

$\alpha_{13}a_3 \mapsto b_7 = (0, -1, 0, 2, 0)$, $\alpha_{23}a_3 \mapsto b_8 = (2, 2, 1, -2, 1)$.

Let $S$ be the subsemigroup of $\mathbb{Z}^5$ generated by $b_1, \ldots, b_8$. Then $S$ is saturated.

Case (II-1-i-C) $a_1 + 2a_2 = 3a_3$. In the Case (II-1-i-A) let $\alpha'_0 = 0$. We get a subsemigroup $S$ of $\mathbb{Z}^4$ generated by 7 elements. Then $S$ is saturated.

But our method does not work well in the following case.

Case (III-2-i) $2a_1 + a_2 \equiv 0$, $2a_2 + a_3 \equiv 0$, $2a_1 \leq a_2 + a_3$, $2a_2 > 3a_1$. We have the following generating system of relations

$$\frac{2a_1 + a_2}{7}a_0 = 2a_1 + a_2,$$

$$4a_1 = \frac{4a_1 + a_3}{7}a_0 + a_3,$$

$$2a_2 = \frac{2a_2 + 3a_1}{7}a_0 + 3a_1,$$

$$2a_3 = \frac{2a_3 - a_1}{7}a_0 + a_1,$$

$$\frac{a_2 + a_3 - 2a_1}{7}a_0 + 2a_1 = a_2 + a_3,$$

$$\frac{a_1 + a_3 - a_2}{7}a_0 + a_2 = a_1 + a_3.$$
The three equations (1), (2) and (6) in the generating system of relations form a fundamental system. In fact,

$$(1) + (2) = (5), \quad t(1) + t(2) + (6) = (3) \quad \text{and} \quad t(1) + t(2) + t(6) = (4).$$

We divide the coefficients in the fundamental system of relations into the following:

$$\alpha_{10} + \alpha_{20} + \alpha'_{0}a_{0} = \alpha_{01}a_{1} + \alpha'_{2}a_{2}, \quad (\alpha_{01} + \alpha'_{1} + \alpha_{31})a_{1} = \alpha_{10}a_{0} + \alpha_{13}a_{3},$$

$$\alpha'_{0}a_{0} + \alpha'_{2}a_{2} = \alpha'_{1}a_{1} + \alpha_{13}a_{3}.$$

We associate elements of $\mathbb{Z}^{5}$ to the components of the above system as follows:

$$\alpha_{10}a_{0} \mapsto b_{1} = e_{1}, \quad \alpha_{20}a_{0} \mapsto b_{2} = e_{2}, \quad \alpha'_{0}a_{0} \mapsto b_{3} = e_{3}, \quad \alpha_{01}a_{1} \mapsto b_{4} = e_{4},$$

$$\alpha'_{1}a_{1} \mapsto b_{5} = e_{5}, \quad \alpha'_{2}a_{2} \mapsto b_{6} = (1, 1, 1, -1, 0),$$

$$\alpha_{31}a_{1} \mapsto b_{7} = (2, 1, 2, -2, -2), \quad \alpha_{13}a_{3} \mapsto b_{8} = (1, 1, 2, -1, -1).$$

Let $S$ be the subsemigroup of $\mathbb{Z}^{5}$ generated by $b_{1}, \ldots, b_{8}$. Then $S$ is not saturated. In fact,

$$2(1, 1, 1, -1, -1) = (2, 2, 2, -2, -2) = b_{2} + b_{7} \in S,$$

but

$$(1, 1, 1, -1, -1) \notin S \quad \text{and} \quad (1, 1, 1, -1, -1) \in \mathbb{Z}^{5}.$$ 

Hence, $\text{Spec } k[S]$ is not a toric variety.

To check whether a 7-semigroup generated by 4 elements is of toric type we divide them into the 31 cases in the following table. But this problem is still open in the last three cases. The right-hand side of column in the table means the dimension of the affine toric variety which is constructed from a numerical semigroup of given type in our way.
<table>
<thead>
<tr>
<th>Condition</th>
<th>Toric</th>
<th>dim</th>
</tr>
</thead>
<tbody>
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<td>$a_1 + a_2 + a_3 \equiv 0$</td>
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<td>6</td>
</tr>
<tr>
<td>$a_1 + a_2 \equiv 0, 2a_1 + a_3, 2a_2 &lt; a_1 + 2a_3, 2a_3 &lt; 3a_2, a_1 + 2a_2 &gt; 3a_3$</td>
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<tr>
<td>$a_1 + a_2 \equiv 0, 2a_1 + a_3, 2a_3 &gt; 3a_2, 4a_2 &gt; a_1 + a_3$</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>$a_1 + a_2 \equiv 0, 2a_1 + a_3, 2a_3 &gt; 3a_2, 4a_2 = a_1 + a_3$</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
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</tr>
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<td>4</td>
</tr>
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<td>$a_1 + a_2 \equiv 0, 3a_1 + a_3, 2a_2 &lt; 2a_1 + a_3, 2a_1 &lt; a_2 + a_3, 3a_2 &lt; a_1 + a_3, 1 + 2a_2 = 2a_3$</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
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<td>0</td>
<td>5</td>
</tr>
<tr>
<td>$a_1 + a_2 \equiv 0, 3a_1 + a_3, 2a_2 &lt; 2a_1 + a_3, 2a_1 &lt; a_2 + a_3, 3a_2 &lt; a_1 + a_3, 2a_1 + 2a_2 = 2a_3$</td>
<td>0</td>
<td>5</td>
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<td>$2a_1 + a_2 \equiv 0, 2a_2 + a_3 \equiv 0, 2a_1 &gt; a_2 + a_3$</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>$2a_1 + a_2 \equiv 0, 2a_2 + a_3 \equiv 0, 2a_1 \leq a_2 + a_3, 2a_2 &gt; 3a_1$</td>
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<tr>
<td>$2a_1 + a_2 \equiv 0, 2a_2 + a_3 \equiv 0, 2a_1 \leq a_2 + a_3, 2a_2 &lt; 3a_1$</td>
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<tr>
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<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

References


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