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<th>On Weierstrass 7-semigroups (Algebra, Languages and Computation)</th>
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<tr>
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On Weierstrass 7-semigroups

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§1. Introduction.
Let \( \mathbb{N} \) be the additive semigroup of non-negative integers. A subsemigroup \( H \) of \( \mathbb{N} \) is called a numerical semigroup if the complement \( \mathbb{N} \setminus H \) of \( H \) in \( \mathbb{N} \) is a finite set. For any positive integer \( n \) a numerical semigroup \( H \) is called an \( n \)-semigroup if \( H \) starts with \( n \), i.e., the minimum positive integer in \( H \) is \( n \). For a non-singular complete irreducible curve \( C \) over an algebraically closed field \( k \) of characteristic 0 (which is called a curve in this paper) and its point \( P \) we set

\[
H(P) = \{ n \in \mathbb{N} | \exists \text{ a rational function } f \text{ on } C \text{ with } (f)_{\infty} = nP \}.
\]

A numerical semigroup is Weierstrass if there exists a curve \( C \) with its point \( P \) such that \( H(P) = H \). We are interested in the following problem:

**Problem 1.** Is every \( n \)-semigroup Weierstrass?

We have the following positive results:

**Fact 2.** For \( n \leq 5 \) every \( n \)-semigroup is Weierstrass. (For \( n = 2 \), classical, for \( n = 3 \), see [8] and for \( n = 4, 5 \), see [4], [5] respectively.)

But we know the negative result as follows:

**Fact 3.** For any \( n \geq 13 \), there exists a non-Weierstrass \( n \)-semigroup. (For \( n = 13 \), [1] and for \( n \geq 14 \) see, for example, [7].)

Thus, we have the following problem:

**Problem 4.** For \( 6 \leq n \leq 12 \), is every \( n \)-semigroup Weierstrass or is there a non-Weierstrass \( n \)-semigroup?

In this paper we are devoted to the study of 7-semigroups. In Section 2 we determine the 7-semigroups which are the semigroups \( H(P) \) of ramification points \( P \) on cyclic coverings of the projective line \( \mathbb{P}^1 \) with degree 7. In Section 3 we divide the Weierstrass 7-semigroups generated by 4 elements into 31 cases and investigate whether such a 7-semigroup is of toric type in each case where a numerical semigroup is said to be of toric type if roughly speaking, the monomial curve associated to the numerical semigroup is a specialization of some affine toric variety, because we know that a numerical semigroup of toric type is Weierstrass ([4]).
§2. Cyclic 7-semigroups.

An \( n \)-semigroup is said to be cyclic if it is the semigroup \( H(P) \) for some totally ramification point \( P \) on a cyclic covering of the projective line \( \mathbb{P}^1 \) with degree \( n \). In this section we describe a necessary and sufficient condition on a 7-semigroup to be cyclic. Moreover, some non-cyclic Weierstrass 7-semigroups are given. We use the following notation: For an \( n \)-semigroup \( H \) we set

\[
S(H) = \{n, s_1, \ldots, s_{n-1}\}
\]

where \( s_i = \text{Min}\{h \in H | h \equiv i \mod n\} \). We have the following necessary condition on an \( n \)-semigroup to be cyclic if \( n \) is prime.

**Fact 5** ([9]). Let \( p \) be a prime number. If \( H \) is a cyclic \( p \)-semigroup with

\[
S(H) = \{p, s_1, \ldots, s_{p-1}\},
\]

then

\[
s_i + s_{p-i} = s_j + s_{p-j}, \text{ all } i, j.
\]

We had already obtained an answer to the converse problem of the above statement.

**Fact 6.** i) For a prime number \( p \leq 7 \), the converse of Fact 5 is true (See [9]).

ii) For any prime number \( p \geq 11 \), the converse of Fact 5 is false (See [3]).

By Fact 6 i) we get the following:

**Proposition 7.** Let \( H \) be a 7-semigroup with

\[
S(H) = \{7, s_1, \ldots, s_6\}.
\]

Assume that

\[
s_1 + s_6 = s_2 + s_5 = s_3 + s_4.
\]

Then \( H \) is cyclic, hence Weierstrass.

For any positive integers \( b_0, \ldots, b_m, < b_0, \ldots, b_m > \) denotes the semigroup generated by \( b_0, \ldots, b_m \). We give examples of cyclic 7-semigroups.

**Example 8.** (1) Let \( H = < 7, 8, 10, 12 > \). Then \( S(H) = \{7, 8, 10, 12, 16, 18, 20\} \). Since \( 8 + 20 = 16 + 12 = 10 + 18 \), \( H \) is cyclic, hence Weierstrass.

(2) Let \( H = < 7, 15, 16, 17, 25, 26, 27 > \). Then \( S(H) = \{7, 15, 16, 17, 25, 26, 27\} \). Since \( 15 + 27 = 16 + 26 = 17 + 25 \), \( H \) is cyclic, hence Weierstrass.

We also have non-cyclic Weierstrass 7-semigroups.
Fact 9. For integers $g$ and $s$ with $7 \leq g \leq s \leq 12$, let $H_{s,g}$ be a 7-semigroup with
\[ \mathbb{N}_0 \backslash H_{s,g} = \{1, \ldots, 6, 8 + s - g, \ldots, s + 1\}. \]
Then we have the following:
i) There exists a covering $C \to \mathbb{P}^1$ of degree 3 with non-ramification point $P \in C$ such that $H(P) = H_{s,g}$. Hence, $H_{s,g}$ is a Weierstrass 7-semigroup (See [2]).
ii) If $(s, g) \neq (9, 9), (12, 9), (12, 12)$, then $H_{s,g}$ is non-cyclic. For example, $H_{11,9} = <7,8,9,13>$ and $H_{12,10} = <7,8,9,19,20>$ are non-cyclic Weierstrass 7-semigroups.

Fact 10. Let $H$ be the 7-semigroup $<7,9,10,11,12>$. Then there is a cyclic covering of an elliptic curve of degree 8 with only two ramification points $P_1$ and $P_2$, which are totally ramified, such that $H(P_1) = H(P_2) = H$. Hence $<7,9,10,11,12>$ is a non-cyclic Weierstrass 7-semigroup (See [6]).

§3. 7-semigroups of toric type.

For a numerical semigroup $H$ we denote by $M(H)$ the minimal set of generators for $H$. In this section we are interested in 7-semigroups $H$ with $M(H) = \{7, a_1, a_2, a_3\}$ which satisfy the following condition:

Definition 11. Let $H$ be a numerical semigroup with $\# M(H) = m + 1$. The semigroup $H$ is said to be of toric type if
\[ \exists l: \text{a positive integer}, \]
\[ \exists S: \text{a saturated subsemigroup of } \mathbb{Z}^l \text{ generated by } b_1, \ldots, b_{l+m} \text{ which generates } \mathbb{Z}^l \text{ as a group and} \]
\[ \exists g_j' \text{ s} (j = 1, \ldots, l + m) : \text{monomials in } k[X_0, X_1, \ldots, X_m] \text{ such that} \]
\[ \text{Spec } k[H] \rightarrow \text{Spec } k[S][X_0, X_1, \ldots, X_m] \]
\[ \downarrow \quad \square \quad \downarrow \]
\[ \text{Spec } k \rightarrow \text{Spec } k[Y_1, \ldots, Y_{l+m}] \]
\[ (0) \quad \rightarrow \quad \text{the origin} \]

where the right vertical map is induced by the $k$-algebra homomorphism
\[ \eta_S : k[Y_1, \ldots, Y_{l+m}] \rightarrow k[S][X_0, X_1, \ldots, X_m] \]
which sends $Y_j$ to $T^{|g_j|} - g_j$, that is to say,
\[ \text{Spec } k[H] \rightarrow \text{Spec } k[X_0, X_1, \ldots, X_m] \]
\[ \downarrow \quad \square \quad \downarrow \]
\[ \text{Spec } k[S] \rightarrow \text{Spec } k[Y_1, \ldots, Y_{l+m}] \]

where the horizontal maps are the embeddings through the generators and the right vertical map is induced by the $k$-algebra morphism from $k[Y_1, \ldots, Y_{l+m}]$ to $k[X_0, X_1, \ldots, X_m]$ sending $Y_j$ to $g_j$. 
We explain how to find a subsemigroup $S$ of $\mathbb{Z}^l$ as in Definition 11 below.

**Remark 12.** Let $H$ be a numerical semigroup with $M(H) = \{a_0, a_1, \ldots, a_m\}$. 

i) Determine a generating system of relations among $a_0, a_1, \ldots, a_m$, i.e., a set of generators for the ideal of the monomial curve $\text{Spec } k[H]$. 

ii) Determine a fundamental system of relations among $a_0, a_1, \ldots, a_m$, i.e., a basis of the relation $\mathbb{Z}$-module among $a_0, a_1, \ldots, a_m$.

iii) We construct a subsemigroup $S$ of $\mathbb{Z}^l$ from the fundamental system. In this case, $S$ is generated by $l + m$ elements $b_j$'s and generates $\mathbb{Z}^l$ as a group naturally. Moreover, we associate the generators $b_j$'s for $S$ to monomials $g_j$'s in $k[X_0, \ldots, X_m]$ such that we have the fiber products in Definition 11.

iv) The remaining problem is whether the semigroup $S$ is saturated or not. We note that $S$ is saturated if and only if the semigroup ring $k[S]$ is normal, i.e., $\text{Spec } k[S]$ is an affine toric variety. If $S$ is saturated, the numerical semigroup $H$ become of toric type.

From now on we treat only 7-semigroups generated by 4 elements.

**Lemma 13.** Let $H$ be a 7-semigroup generated by 4 elements, i.e., $M(H) = \{7, a_1, a_2, a_3\}$. Renumbering $a_1, a_2$ and $a_3$ it satisfies one of the following:

(I) $a_1 + a_2 + a_3 \equiv 0 \ (7)$,

(II) $a_1 + a_2 \equiv 0 \ (7)$,

(III) $2a_1 + a_2 \equiv 0 \ (7)$ and $2a_2 + a_3 \equiv 0 \ (7)$.

We give the construction of a saturated subsemigroup $S$ of $\mathbb{Z}^l$ as in Definition 11 in (I) and some cases of (II).

Case (I) $a_1 + a_2 + a_3 \equiv 0 \ (7)$. A fundamental system of relations consists of

\[
\frac{a_1 + a_2 + a_3}{7} = a_0 = a_1 + a_2 + a_3, \quad 2a_1 = \frac{2a_1 - a_2}{7}a_0 + a_2, \quad 2a_2 = \frac{2a_2 - a_3}{7}a_0 + a_3.
\]

For example, the relation

\[
2a_3 = \frac{2a_3 - a_1}{7}a_0 + a_1
\]

is derived from the addition of the three relations. The determinant of the matrix consisting of the coefficients of the three relations is

\[
\begin{vmatrix}
(a_1 + a_2 + a_3)/7 & -1 & -1 \\
-(2a_1 - a_2)/7 & 2 & -1 \\
-(2a_2 - a_3)/7 & 0 & 2
\end{vmatrix} = a_3.
\]

A numerical semigroup $H$ with $M(H) = \{a_0, a_1, a_2, a_3\}$ satisfying the above condition is said to be 1-neat. Under the above condition we get a saturated subsemigroup $S$ of $\mathbb{Z}^6$ as in Definition 11 from the fundamental system.
Case (II-1) $a_1 + a_2 \equiv 0 \pmod{7}$ and $2a_1 \equiv a_3 \pmod{7}$.

Case (II-1-i) $2a_2 < a_1 + 2a_3$ and $2a_3 < 3a_2$. A generating system for relations consists of

\[
\frac{a_1 + a_2}{7}a_0 = a_1 + a_2, \quad 2a_1 = \frac{2a_1 - a_3}{7}a_0 + a_3, \quad 3a_2 = \frac{3a_2 - 2a_3}{7}a_0 + 2a_3,
\]

\[
3a_3 = \frac{3a_3 - a_2}{7}a_0 + a_2, \quad \frac{a_2 + a_3 - a_1}{7}a_0 + a_1 = a_2 + a_3,
\]

\[
\frac{a_1 + 2a_3 - 2a_2}{7}a_0 + 2a_2 = a_1 + 2a_3.
\]

i.e., the kernel of

\[
\varphi_H : k[X_0, X_1, X_2, X_3] \rightarrow k[t^{a_0}, t^{a_1}, t^{a_2}, t^{a_3}]
\]

is generated by

\[
X_0^{a_1 + a_2} - X_1X_2, \quad X_1^2 - X_0^{2a_1 - a_3}X_3, \quad X_2^3 - X_0^{3a_3 - 2a_3}X_3,
\]

\[
X_3 - X_0^{3a_3 - 2a_2}X_2, \quad X_0^{a_2 + a_3 - a_1}X_1 - X_2X_3, \quad X_0^{a_1 + 2a_3 - 2a_2}X_0^2 - X_1X_3^2.
\]

A fundamental system of relations is the following:

\[
\frac{a_1 + a_2}{7}a_0 = a_1 + a_2, \quad 2a_1 = \frac{2a_1 - a_3}{7}a_0 + a_3, \quad 3a_2 = \frac{3a_2 - 2a_3}{7}a_0 + 2a_3.
\]

For example, the addition of the first and second relations

\[
\frac{a_1 + a_2}{7}a_0 + 2a_1 = \left( a_1 + a_2 \right) + \left( \frac{2a_1 - a_3}{7}a_0 + a_3 \right)
\]

induces the fifth relation. To get a subsemigroup $S$ of $\mathbb{Z}^l$ we divide this case into three cases again.

Case (II-1-i-A) $a_1 + 2a_2 > 3a_3$. We divide the coefficients in the fundamental system of relations into the following:

\[
(a_0' + a_0'' + a_0^m)a_0 = a_0a_1 + a_0a_2, \quad 2a_0a_1 = (a_0' + a_0''a_0 + a_1a_3,
\]

\[
(2a_0 + a_0')a_2 = (a_0' + a_0''a_0 + a_2a_3.
\]

We associate elements of $\mathbb{Z}^5$ to the components of the above system as follows:

\[
\alpha_0'0_0 \mapsto b_1 = e_1, \quad \alpha_0''a_0 \mapsto b_2 = e_2, \quad \alpha_0'a_0 \mapsto b_3 = e_3, \quad \alpha_0a_1 \mapsto b_4 = e_4,
\]

\[
\alpha_0'a_2 \mapsto b_5 = e_5, \quad \alpha_0a_2 \mapsto b_6 = (1, 1, 1, -1, 0),
\]
\[ \alpha_{13}a_{3} \mapsto b_{7} = (-1, -1, 0, 2, 0), \quad \alpha_{23}a_{3} \mapsto b_{8} = (1, 2, 1, -2, 1). \]

where \( e_{i} \) denotes the vector whose \( i \)-th component is 1 and \( j \)-th component is 0 if \( j \neq i \). Let \( S \) be the subsemigroup of \( \mathbb{Z}^{5} \) generated by \( b_{1}, \ldots, b_{8} \). We can show that

\[ \sum_{i=1}^{8} \mathbb{R}_{+}b_{i} \cap \mathbb{Z}^{5} \subseteq S \]

where \( \mathbb{R}_{+} \) denotes the set of non-negative real numbers. Hence, \( S \) is saturated.

Case (II-1-i-B) \( a_{1} + 2a_{2} < 3a_{3} \). We divide the coefficients in the fundamental system of relations into the following:

\[ (\alpha'_{0} + \alpha_{10} + \alpha_{20})a_{0} = \alpha_{01}a_{1} + \alpha_{02}a_{2}, \quad 2\alpha_{01}a_{1} = \alpha_{10}a_{0} + \alpha_{13}a_{3}, \]

\[ (2\alpha_{02} + \alpha'_{2})a_{2} = \alpha_{20}a_{0} + \alpha_{23}a_{3}. \]

We associate elements of \( \mathbb{Z}^{5} \) to the components of the above system as follows:

\[ \alpha'_{0}a_{0} \mapsto b_{1} = e_{1}, \quad \alpha_{10}a_{0} \mapsto b_{2} = e_{2}, \quad \alpha_{20}a_{0} \mapsto b_{3} = e_{3}, \quad \alpha_{01}a_{1} \mapsto b_{4} = e_{4}, \]

\[ \alpha'_{2}a_{2} \mapsto b_{5} = e_{5}, \quad \alpha_{02}a_{2} \mapsto b_{6} = (1, 1, 1, -1, 0), \]

\[ \alpha_{13}a_{3} \mapsto b_{7} = (0, -1, 0, 2, 0), \quad \alpha_{23}a_{3} \mapsto b_{8} = (2, 2, 1, -2, 1). \]

Let \( S \) be the subsemigroup of \( \mathbb{Z}^{5} \) generated by \( b_{1}, \ldots, b_{8} \). Then \( S \) is saturated.

Case (II-1-i-C) \( a_{1} + 2a_{2} = 3a_{3} \). In the Case (II-1-i-A) let \( \alpha'_{0} = 0 \). We get a subsemigroup \( S \) of \( \mathbb{Z}^{4} \) generated by 7 elements. Then \( S \) is saturated.

But our method does not work well in the following case.

Case (III-2-i) \( 2a_{1} + a_{2} \equiv 0, \quad 2a_{2} + a_{3} \equiv 0, \quad 2a_{1} \leq a_{2} + a_{3}, \quad 2a_{2} > 3a_{1} \). We have the following generating system of relations

\[ \frac{2a_{1} + a_{2}}{7}a_{0} = 2a_{1} + a_{2}, \quad (1) \]

\[ 4a_{1} = \frac{4a_{1} - a_{3}}{7}a_{0} + a_{3}, \quad (2) \]

\[ 2a_{2} = \frac{2a_{2} - 3a_{1}}{7}a_{0} + 3a_{1}, \quad (3) \]

\[ 2a_{3} = \frac{2a_{3} - a_{1}}{7}a_{0} + a_{1}, \quad (4) \]

\[ \frac{a_{2} + a_{3} - 2a_{1}}{7}a_{0} + 2a_{1} = a_{2} + a_{3}, \quad (5) \]

\[ \frac{a_{1} + a_{3} - a_{2}}{7}a_{0} + a_{2} = a_{1} + a_{3}. \quad (6) \]
The three equations (1), (2) and (6) in the generating system of relations form a fundamental system. In fact,

\[(1) + (2) = (5), \quad {}^{t}(1) + {}^{t}(2) + (6) = (3) \text{ and } {}^{t}(1) + {}^{t}(2) + {}^{t}(6) = (4).\]

We divide the coefficients in the fundamental system of relations into the following:

\[ (\alpha_{10} + \alpha_{20} + \alpha_{0}')a_{0} = \alpha_{01}a_{1} + \alpha_{2}'a_{2}, \quad (\alpha_{01} + \alpha_{1}' + \alpha_{31})a_{1} = \alpha_{10}a_{0} + \alpha_{13}a_{3}, \]

\[ \alpha_{0}'a_{0} + \alpha_{2}'a_{2} = \alpha_{1}'a_{1} + \alpha_{13}a_{3}. \]

We associate elements of \( \mathbb{Z}^{5} \) to the components of the above system as follows:

\[ \alpha_{10}a_{0} \mapsto b_{1} = e_{1}, \quad \alpha_{20}a_{0} \mapsto b_{2} = e_{2}, \quad \alpha_{0}'a_{0} \mapsto b_{3} = e_{3}, \quad \alpha_{01}a_{1} \mapsto b_{4} = e_{4}, \]

\[ \alpha_{1}'a_{1} \mapsto b_{5} = e_{5}, \quad \alpha_{2}'a_{2} \mapsto b_{6} = (1, 1, 1, -1, 0), \]

\[ \alpha_{31}a_{1} \mapsto b_{7} = (2, 1, 2, -2, -2), \quad \alpha_{13}a_{3} \mapsto b_{8} = (1, 1, 2, -1, -1). \]

Let \( S \) be the subsemigroup of \( \mathbb{Z}^{5} \) generated by \( b_{1}, \ldots, b_{8} \). Then \( S \) is not saturated. In fact,

\[ 2(1, 1, 1, -1, -1) = (2, 2, 2, -2, -2) = b_{2} + b_{7} \in S, \]

but

\[ (1, 1, 1, -1, -1) \not\in S \text{ and } (1, 1, 1, -1, -1) \in \mathbb{Z}^{5}. \]

Hence, \( \text{Spec } k[S] \) is not a toric variety.

To check whether a 7-semigroup generated by 4 elements is of toric type we divide them into the 31 cases in the following table. But this problem is still open in the last three cases. The right-hand side of column in the table means the dimension of the affine toric variety which is constructed from a numerical semigroup of given type in our way.
<table>
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<th>Condition</th>
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<th>dim</th>
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<tbody>
<tr>
<td>I</td>
<td>(a_1 + a_2 + a_3 \equiv 0)</td>
<td>○</td>
<td>6</td>
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<tr>
<td>II-1-i-A</td>
<td>(a_1 + a_2 \equiv 0, 2a_1 \equiv a_3, 2a_2 &lt; a_1 + 2a_3, 2a_3 &lt; 3a_2, a_1 + 2a_2 &gt; 3a_3)</td>
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<td>○</td>
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<td>II-1-i-C</td>
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<td>○</td>
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<tr>
<td>II-1-ii-A</td>
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<td>II-1-ii-B</td>
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<tr>
<td>II-2-ii-C</td>
<td>(a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 &lt; 2a_1 + a_3, a_1 &gt; 2a_2, 3a_2 &lt; a_1 + a_3, 4a_2 &gt; a_1 + a_3)</td>
<td>○</td>
<td>6</td>
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<tr>
<td>II-2-ii-D</td>
<td>(a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 &lt; 2a_1 + a_3, a_1 &gt; 2a_2, 3a_2 &lt; a_1 + a_3, 4a_2 &lt; a_1 + a_3, 4a_2 = a_1 + a_3)</td>
<td>○</td>
<td>5</td>
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<tr>
<td>II-2-ii-E</td>
<td>(a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 &lt; 2a_1 + a_3, a_1 &gt; 2a_2, 3a_2 &lt; a_1 + a_3, a_1 + 2a_2 &lt; a_3)</td>
<td>○</td>
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<tr>
<td>II-2-iii-A</td>
<td>(a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 &lt; 2a_1 + a_3, 2a_1 &lt; 2a_2, 3a_2 \geq a_1 + a_3, a_1 + 2a_2 \geq a_3)</td>
<td>○</td>
<td>6</td>
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<tr>
<td>II-2-iii-B</td>
<td>(a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 &lt; 2a_1 + a_3, 2a_1 &lt; 2a_2, 3a_2 \geq a_1 + a_3, a_1 + 2a_2 = a_3)</td>
<td>○</td>
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<tr>
<td>II-2-iii-C</td>
<td>(a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 &lt; 2a_1 + a_3, 2a_1 &lt; 2a_2, 3a_2 &lt; a_1 + a_3, a_1 + 2a_2 &lt; a_3)</td>
<td>○</td>
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<tr>
<td>II-2-iii-D</td>
<td>(a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 &lt; 2a_1 + a_3, 2a_1 &lt; 2a_2, 3a_2 = a_1 + a_3, a_1 + 2a_2 = a_3)</td>
<td>○</td>
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<tr>
<td>II-2-iii-E</td>
<td>(a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 &lt; 2a_1 + a_3, 2a_1 &lt; 2a_2, 3a_2 &lt; a_1 + a_3, a_1 + 2a_2 &lt; a_3)</td>
<td>○</td>
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<tr>
<td>II-2-iii-F</td>
<td>(a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 &lt; 2a_1 + a_3, 2a_1 &lt; 2a_2, 3a_2 &lt; a_1 + a_3, a_1 + 2a_2 = a_3)</td>
<td>○</td>
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<td>II-2-iii-G</td>
<td>(a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 &lt; 2a_1 + a_3, 2a_1 &lt; 2a_2, 3a_2 &lt; a_1 + a_3, 2a_1 + 3a_2 &gt; a_3)</td>
<td>○</td>
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<td>II-2-iv-A</td>
<td>(a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 &lt; 2a_1 + a_3, 2a_1 = a_2 + a_3, 3a_2 &gt; a_1 + a_3)</td>
<td>○</td>
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<tr>
<td>II-2-iv-B</td>
<td>(a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 &lt; 2a_1 + a_3, 2a_1 = a_2 + a_3, 3a_2 &lt; a_1 + a_3)</td>
<td>○</td>
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<tr>
<td>II-2-iv-C</td>
<td>(a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 &lt; 2a_1 + a_3, 2a_1 = a_2 + a_3, 3a_2 &lt; a_1 + a_3)</td>
<td>○</td>
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<td>III-1</td>
<td>(2a_1 + a_2 \equiv 0, 2a_2 + a_3 \equiv 0, 2a_1 \geq a_2 + a_3)</td>
<td>○</td>
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<td>III-2-i</td>
<td>(2a_1 + a_2 \equiv 0, 2a_2 + a_3 \equiv 0, 2a_1 \leq a_2 + a_3, 2a_2 &gt; 3a_1)</td>
<td>?</td>
<td>(5)</td>
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<td>III-2-ii</td>
<td>(2a_1 + a_2 \equiv 0, 2a_2 + a_3 \equiv 0, 2a_1 \leq a_2 + a_3, 2a_2 &lt; 3a_1)</td>
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<td>(5)</td>
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<tr>
<td>III-2-iii</td>
<td>(2a_1 + a_2 \equiv 0, 2a_2 + a_3 \equiv 0, 2a_1 \leq a_2 + a_3, 2a_2 = 3a_1)</td>
<td>?</td>
<td>(4)</td>
</tr>
</tbody>
</table>

References


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