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<th>Tilings and Fractals from Pisot substitutions (Algebra, Languages and Computation)</th>
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<tr>
<td>Author(s)</td>
<td>Ito, Shunji</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2005), 1437: 145-154</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2005-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/47484">http://hdl.handle.net/2433/47484</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Tilings and Fractals from Pisot substitutions

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This is the note for the lecture at RIMS (Kyoto University).

1 Definition of Pisot Unit Substitutions

\[ \mathcal{A} := \{1, 2, \ldots, d\} \quad \text{(alphabet)} \]
\[ \mathcal{A}^* := \bigcup_{n=0}^{\infty} \mathcal{A}^n \quad \text{(free monoid, i.e., the set of finite words)} \]

\[ (G \{1, \ldots, d\}: \text{a free group of rank } d) \]

Definition 1.1 \( \sigma : \mathcal{A}^* \rightarrow \mathcal{A}^* \) is a substitution if

1. \( \sigma(i) = W^{(i)} \in \mathcal{A}^* \), \( W^{(i)} \neq \emptyset \)
   \[ = w_1^{(i)} \ldots w_k^{(i)} \ldots w_l^{(i)}, \quad w_k^{(i)} \in \mathcal{A} \]
   \[ = P_k^{(i)} w_k^{(i)} S_k^{(i)} \];

2. \( \sigma(w_1 \ldots w_k) := \sigma(w_1) \cdots \sigma(w_k) \) for \( w_1 \ldots w_k \in \mathcal{A}^* \).

(A substitution \( \sigma \) is invertible if \( \sigma \) is an automorphism on \( G \{1, 2, \ldots, d\} \).)

Let \( L_\sigma \) be a matrix of \( \sigma \), that is,

\[ L_\sigma(i,j) := \text{the number of the letter } i \text{ contained in } \sigma(j). \]

Example 1.1 (Fibonacci substitution)

\[ \sigma_F : \begin{array}{c|c}
1 & 12 \\
1 & 1 \\
\end{array} \]
\[ L_{\sigma_F} = \begin{bmatrix}
1 & 1 \\
1 & 0 \\
\end{bmatrix}. \]

Example 1.2 (Rauzy substitution)

\[ \sigma_R : \begin{array}{c|c}
1 & 12 \\
1 & 13 \\
1 & 1 \\
\end{array} \]
\[ L_{\sigma_R} = \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}. \]
Assumption  For the substitution $\sigma$,

1. $L_\sigma$ is primitive, that is, $\exists N : L_\sigma^N > 0$;
2. $L_\sigma$ is unimodular, that is, $\det L_\sigma = \pm 1$;
3. $L_\sigma$ is Pisot type, that is, for eigenvalues $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_d$ of $L_\sigma$,
   $$\lambda = \lambda_1 > 1 > |\lambda_i|, \ i = 2, \ldots, d;$$
4. the characteristic polynomial $\Phi_\sigma(x)$ of $L_\sigma$ is irreducible;
5. $\sigma(1) = 1W'$.

We say the substitution $\sigma$ satisfying Assumption the Pisot Unit substitution.

On Assumption (5),

$$w_\sigma := \lim_{n \to \infty} \sigma^n(1) = s_1s_2\cdots s_m \cdots$$

is a fixed point of $\sigma$, that is, $\sigma(w_\sigma) = w_\sigma$.

$f : A^* \to \mathbb{Z}^d$ is a homomorphism given by

$$f(i) := e_i \quad \text{and} \quad f(w_1 \cdots w_k) := \sum_{j=1}^k f(w_j).$$

Lemma 1.1 The following relation holds:

$$A^* \xrightarrow{\sigma} A^*$$

$$\downarrow f \quad \downarrow f$$

$$\mathbb{Z}^d \xrightarrow{L_\sigma} \mathbb{Z}^d.$$ 

Let $v > 0$ be a positive eigenvector of $\lambda$ and $P$ be $L_\sigma$-invariant contractive plain, that is,

$$R^d := L(v) \oplus P.$$

The projection $\pi$ is given by

$$\pi : \mathbb{Z}^d \to P \text{ along } v.$$

Lemma 1.2 The following relation holds:

$$\mathbb{Z}^d \xrightarrow{L_\sigma} \mathbb{Z}^d$$

$$\downarrow \pi \quad \downarrow \pi$$

$$\mathbb{Z}^d \xrightarrow{L_\sigma} \mathbb{Z}^d.$$
For the fixed point $w_{\sigma} = s_1s_2\cdots s_n\cdots$,

\begin{align*}
Y & := \pi \{ f(s_0s_1\cdots s_k) \mid k = 0, 1, \cdots \}, \\
Y_i & := \pi \{ f(s_0s_1\cdots s_{k-1}) \mid \exists k : s_k = i, k = 1, 2, \cdots \}, \\
Y_i' & := \pi \{ f(s_0s_1\cdots s_k) \mid \exists k : s_k = i, k = 0, 1, \cdots \}
\end{align*}

where $s_0 = \varepsilon$ (the empty word).

**Definition 1.2** $X_i :=$ the closure of $\pi Y_i$ is called atomic surfaces

($X_i' :=$ the closure of $\pi Y_i'$) and $X := \bigcup_{i=1}^{d} X_i (= \bigcup_{i=1}^{d} X_i')$ is called atomic surface of the substitution $\sigma$.

On Example 1.1 and Example 1.2, we have the figures (see Figure 2 and Figure 3 respectively).

**Remark** There is the theorem. Theorem ([E-I]): On $d = 2$, $X_i$ is interval is the interval iff $\sigma$ is invertible.

![Figure 2: The figure of the atomic surface on $\sigma_F : 1 \mapsto 12$, 2 $\mapsto$ 1.](image-url)
Figure 3: The figure of the atomic surface on $\sigma_R: 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$.

**Theorem 1** ([A-I], [I-R]) Atomic surfaces satisfy

1. $X, X_i, X_i'$ are compact sets;
2. $\text{int}.X = X$;
3. $L_{\sigma}^{-1}X_i = \bigcup_{j=1}^{d} \bigcup_{k:W_k^{(j)}=i} \left( X_j + L_{\sigma}^{-1}\left( f\left(P_k^{(j)}\right)\right) \right)$ (non-overlapping);
4. $\left( |X_1|, \ldots, |X_d| \right)$ is the eigenvector of $L_{\sigma}$ with respect to $\lambda = \lambda_1 > 1$ where $|B|$ is the volume of the set $B$.

**Sketch of proof.** On the notation,

$(x, i) := \{ x + \lambda e_i | 0 \leq \lambda \leq 1 \}$,

$\sigma(0, i) := \Sigma_{k=1}^{l_i} \left( f\left(P_k^{(i)}\right), W_k^{(i)} \right), 1 \leq i \leq d$,

$\overline{w}_{\sigma} := \lim_{n \rightarrow \infty} \sigma^n(0, 1)$ (the broken line starting from 0),

we define

$$(Y_i, i) := \{(y, i) | (y, i) \in \overline{w}\}, \overline{w} = \bigcup_{i=1}^{d} (Y_i, i)$$

$$= \left\{(y, i) | (y, i) \in \sigma\left( \bigcup_{j=1}^{d} (Y_j, j) \right) \right\}$$

$$= \bigcup_{j=1}^{d} \{ (y, i) | (y, i) \in \sigma(Y_j, j) \}$$

$$= \bigcup_{j=1}^{d} \bigcup_{k:W_k^{(j)}=i} \left\{ \left( L_{\sigma}y + f\left(P_k^{(j)}\right), i \right) \right\}.$$
Taking only the starting points of line segments, we get Theorem 1 (3).

**Question** Are \(X_i, i = 1, 2, \ldots, d\) non-overlapping?

**Definition 1.3** Substitution \(\sigma\) satisfies the coincidence condition if \(\exists n, k:\)

1. \(f(P_k^{(n,1)}) = f(P_k^{(n,2)}) = \cdots = f(P_k^{(n,d)})\)
2. \(w_k^{(n,1)} = w_k^{(n,2)} = \cdots = w_k^{(n,d)}\)

where

\[\sigma^n (1) = w_1^{(n,1)} \cdots w_k^{(n,1)} \cdots w_l^{(n,1)}\]

\[\sigma^n (d) = w_1^{(n,d)} \cdots w_k^{(n,d)} \cdots w_l^{(n,d)}\]

**Proposition 1.1** If \(\sigma\) satisfies the coincidence condition, then \(X_i, i = 1, 2, \ldots, d\) are non-overlapping.

**Conjecture** Any Pisot unimodular substitutions satisfy the coincidence condition.

**Remark** On \(d = 2\), the conjecture is proved by Barge and Diamond ([B-D]).

If \(X_i, i = 1, 2, \ldots, d\) are non-overlapping, then we have two dynamical systems on \(X:\)

1. \(T : X \to X,\)

\[Tx = L_{\sigma}^{-1}x - L_{\sigma}^{-1}\pi f(P_k^{(j)})\quad \text{if } x \in X_i, \exists j, k : L_{\sigma}^{-1}x \in X_j + L_{\sigma}^{-1}\pi f(P_k^{(j)})\]

Therefore, \(T\) is Markov endomorphism with the structure matrix \(L_{\sigma}\).

2. \(W : X \to X\)

\[\omega_x \mapsto x - \pi e_i \quad \text{is well-defined}\]

and \(W\) is called the domain exchange transformation (later, we will see \(W \simeq\) the rotation on \(T^{d-1}\)).

From Markov endomorphism (1), we have the following numerical expression.

**Corollary 1.1** Using Markov endomorphism, \(X_i\) is presented by

\[X_i = \left\{ \pi c f(P_{k_0}^{(j_0)}) + \pi c L_{\sigma} f(P_{k_1}^{(j_1)}) + \pi c L_{\sigma}^2 f(P_{k_2}^{(j_2)}) + \cdots + \pi c L_{\sigma}^n f(P_{k_n}^{(j_n)}) + \cdots \mid (*) \right\}\]

where \((*)\) is defined by \((j_0, k_0) \cdots (j_n, k_n)\) is given by \(w_{k_0}^{(j_0)} = j_{n-1}\) and \(w_{k_0}^{(j_0)} = i\).

(see Figure 4).
$\downarrow L_{\sigma}^{-1}$

$T X_1 = X_1 \cup X_2 \cup X_3$
$T X_2 = X_1$
$T X_3 = X_2$

$L_{\sigma}^{-1} X_1 = X_1 \cup X_2 \cup X_3$
$L_{\sigma}^{-1} X_2 = \pi e_3 + X_1$
$L_{\sigma}^{-1} X_3 = \pi e_3 + X_2$

$\downarrow W$

Figure 4: Figures of Markov endomorphism $T$ and the domain exchange transformation $W$ on Example 1.2.
2 Stepped Surfaces and Tiling Substitutions

For 
\[(x, i^*) \in \mathbb{Z}^d \times \{1^*, \ldots, d^*\},\]
we give a geometrical meaning such that
\[(x, i^*) := \left\{ \sum_{j=1, j \neq i}^{d} x + \mu_j e_j \right\} 0 \leq \mu_j \leq 1 \]
(see Figure 5).

(0,1*)
\ochrome{0,2*}
(0,2*)
\ochrome{0,3*}

Figure 5: The figures of \((0, i^*)\).

Definition 2.1 \(S := \{(x, i^*) \mid (x, u) \geq 0, (x - e_i, u) < 0\}\) is called the stepped surface of the contract plain \(P\) where the row vector \(u \geq 0\) is the eigenvector such that \(uL_\sigma = \lambda u\) and the contract plain \(P\) is given by \(P = \{x \mid (x, u) = 0\}\).

Definition 2.2 \(\pi S := \{\pi(x, i^*) \mid (x, i^*) \in S\}\) is called a tiling of \(P\) from the stepped surface \(S\) (see Figure 6).

3 Dual Substitution \(\sigma^*\) (Tiling Substitution)

\(S^* := \{\text{the finite sum of elements of } S\} \approx \{\text{the patches of tiles of the tiling } \pi S\}\).

Let us define the dual (tiling) substitution \(\sigma^*\) by
\[\sigma^*\pi(x, i^*) = L_\sigma^{-1}\pi x + \sum_{j=1}^{d} \sum_{W_{k}^{(j)} = i} \pi_c \left(L_\sigma^{-1} f \left(P_k^{(j)}\right), j^*\right)\]
(see Figure 7 and Figure 8).
Figure 6: The figure of the tiling $\pi S$ of $P$ from the stepped surface $S$ on $\sigma_R$ in Example 1.2.

Figure 7: The figure of $\sigma_R^* \pi(0, i^*)$. 
Figure 8: The figure of $\sigma_R^* \bigcup_{i=1,2,3} \pi(e_i, i^*)$ on $\sigma_R$.

**Theorem 2** ([A-I]) $\sigma$: An Pisot Unit substitution, then

1. $\sigma^* : S^* \rightarrow S^*$ is well-defined;
2. $\forall (x, i^*) \in S, \exists (y, j^*) \in S: (x, i^*) \in \sigma^*(y, j^*)$;
3. $\forall (x, i^*) \in S \Rightarrow \sigma^*(x, i^*) \cap \sigma^*(y, j^*) = \emptyset$.

**Theorem 3** ([I-R]) Let $\mathcal{U} = \{\pi(e_i, i^*) | i = 1, 2, \ldots, d\}$, then,

1. $\sigma^* \mathcal{U} \succ \mathcal{U}$;
2. if $d(\partial(\sigma^* \mathcal{U}), 0) \rightarrow \infty (n \rightarrow \infty)$, then
   $$\tau' = \{\pi(x, j^*) | \pi(x, j^*) \in \sigma^* \pi(e_i, i^*) \text{ for some } n \text{ and } j^*\}$$
   coincides with $\pi S$ and a quasi-periodic tiling;
3. $-X_i = \lim_{n \rightarrow \infty} L_n^o \sigma^n \pi(e_i, i^*)$;
4. $\tau = \{\pi x - X_j | \pi(x, j^*) \in \tau'\}$ is also a quasi-periodic tiling of $P$. 


Figure 9: The figure of the quasi-periodic tiling $\tau$.

References


