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<th>Title</th>
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</thead>
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<td>Ito, Shunji</td>
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Kyoto University
Tilings and Fractals from Pisot substitutions

Shunji ITO
(Kanazawa University)

This is the note for the lecture at RIMS (Kyoto University).

1 Definition of Pisot Unit Substitutions

\[ A := \{1, 2, \cdots, d\} \quad \text{(alphabet)} \]
\[ A^* := \bigcup_{n=0}^{\infty} A^n \quad \text{(free monoid, i.e., the set of finite words)} \]

\(G \{1, \cdots, d\} : \) a free group of rank \(d\)

\textbf{Definition 1.1} \(\sigma : A^* \rightarrow A^*\) is a substitution if

1. \(\sigma(i) = W^{(i)} \in A^*, \ W^{(i)} \neq \emptyset\)
   \[= w_1^{(i)} \cdots w_k^{(i)} \cdots w_l^{(i)}, \ w_k^{(i)} \in A\]
   \[= P_k^{(i)} w_k^{(i)} S_k^{(i)};\]

2. \(\sigma(w_1 \cdots w_k) := \sigma(w_1) \cdots \sigma(w_k) \) for \(w_1 \cdots w_k \in A^*\).

(A substitution \(\sigma\) is invertible if \(\sigma\) is an automorphism on \(G \{1, 2, \cdots, d\}\).)

Let \(L_\sigma\) be a matrix of \(\sigma\), that is,

\(L_\sigma (i,j) := \) the number of the letter \(i\) contained in \(\sigma(j)\).

\textbf{Example 1.1} (Fibonacci substitution)

\[ \sigma_F : 1 \rightarrow \begin{array}{c} 12 \\ 1 \end{array}, \quad L_{\sigma_F} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \]

\textbf{Example 1.2} (Rauzy substitution)

\[ \sigma_R : 1 \rightarrow \begin{array}{c} 12 \\ 1 \end{array}, \quad L_{\sigma_R} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \]
Assumption  For the substitution $\sigma$,

1. $L_\sigma$ is primitive, that is, $\exists N : L_\sigma^N > 0$;
2. $L_\sigma$ is unimodular, that is, $\det L_\sigma = \pm 1$;
3. $L_\sigma$ is Pisot type, that is, for eigenvalues $\lambda = \lambda_1, \lambda_2, \cdots, \lambda_d$ of $L_\sigma$,
   \[ \lambda = \lambda_2 > 1 > |\lambda_i|, \quad i = 2, \cdots, d; \]
4. the characteristic polynomial $\Phi_\sigma(x)$ of $L_\sigma$ is irreducible;
5. $\sigma(1) = 1\omega'$.

We say the substitution $\sigma$ satisfying Assumption the Pisot Unit substitution.

On Assumption (5),
\[ w_\sigma := \lim_{n \to \infty} \sigma^n(1) = s_1 s_2 \cdots s_m \cdots \]
is a fixed point of $\sigma$, that is, $\sigma(w_\sigma) = w_\sigma$.

$f : A^* \to Z^d$ is a homomorphism given by
\[ f(i) := e_i \quad \text{and} \quad f(w_1 \cdots w_k) := \sum_{j=1}^{k} f(w_j). \]

Lemma 1.1  The following relation holds:
\[
\begin{array}{ccc}
A^* & \xrightarrow{\sigma} & A^* \\
\downarrow f & & \downarrow f \\
Z^d & \xrightarrow{L_\sigma} & Z^d
\end{array}
\]

Let $v > 0$ be a positive eigenvector of $\lambda$ and $P$ be $L_\sigma$-invariant contractive plain, that is,
\[ R^d := \mathcal{L}(v) \oplus P. \]
The projection $\pi$ is given by
\[ \pi : Z^d \to P \quad \text{along} \quad v. \]

Lemma 1.2  The following relation holds:
\[
\begin{array}{ccc}
Z^d & \xrightarrow{L_\sigma} & Z^d \\
\downarrow \pi & & \downarrow \pi \\
P & \xrightarrow{L_\sigma} & P
\end{array}
\]
For the fixed point $w_{\sigma} = s_{1}s_{2}\cdots s_{n}\cdots$,

\begin{align*}
y & := \pi \{ f(s_{0}s_{1}\cdots s_{k}) \mid k = 0, 1, \ldots \}, \\
y_{i} & := \pi \{ f(s_{0}s_{1}\cdots s_{k-1}) \mid \exists k : s_{k} = i, \ k = 1, 2, \cdots \}, \\
y'_{i} & := \pi \{ f(s_{0}s_{1}\cdots s_{k}) \mid \exists k : s_{k} = i, \ k = 0, 1, \cdots \}
\end{align*}

where $s_{0} = \varepsilon$ (the empty word).

**Definition 1.2** $X_{i} : = \text{the closure of } \pi Y_{i}$ is called atomic surfaces
($X'_{i} : = \text{the closure of } \pi Y'_{i}$) and $X := \bigcup_{i=1}^{d} X_{i} (= \bigcup_{i=1}^{d} X'_{i})$ is called atomic surface of the substitution $\sigma$.

On Example 1.1 and Example 1.2, we have the figures (see Figure 2 and Figure 3 respectively).

**Remark** There is the theorem. Theorem ([E-I]): On $d = 2$, $X_{i}$ is interval is the interval iff $\sigma$ is invertible.

Figure 2: The figure of the atomic surface on $\sigma_{F} : 1 \mapsto 12, \ 2 \mapsto 1$. 
Theorem 1 ([A-I], [I-R]) Atomic surfaces satisfy

(1) $X$, $X_i$, $X_i'$ are compact sets;
(2) $\text{int} \times = X$;
(3) $L_{\sigma}^{-1}X_i = \bigcup_{j=1}^{d} \bigcup_{k:W_k^{(j)}=i} \{ X_j + L_{\sigma}^{-1}(f(P_k^{(j)})) \}$ (non-overlapping);
(4) $|X_1|, \ldots, |X_d|$ is the eigenvector of $L_{\sigma}$ with respect to $\lambda = \lambda_1 > 1$ where $|B|$ is the volume of the set $B$.

Sketch of proof. On the notation,

$$(x, i) := \{ x + \lambda e_i \mid 0 \leq \lambda \leq 1 \},$$

$$\sigma(0, i) := \sum_{k=1}^{l_{i}} \{ f(P_k^{(i)}), W_k^{(i)} \}, \quad 1 \leq i \leq d,$$

$$\overline{w_{\sigma}} := \lim_{n \to \infty} \sigma^n(0, 1) \quad \text{(the broken line starting from 0)},$$

we define

$$(Y_i, i) := \{ (y, i) \mid (y, i) \in \overline{w} \}, \quad \overline{w} = \bigcup_{i=1}^{d} (Y_i, i)$$

$$= \left\{ (y, i) \middle| (y, i) \in \sigma \left( \bigcup_{j=1}^{d} (Y_j, j) \right) \right\}$$

$$= \bigcup_{j=1}^{d} \{ (y, i) \mid (y, i) \in \sigma(Y_j, j) \}$$

$$= \bigcup_{j=1}^{d} \bigcup_{k:W_k^{(j)}=i} \{ (L_{\sigma}y + f(P_k^{(j)}), i) \mid y \in Y_j \}.$$
Taking only the starting points of line segments, we get Theorem 1 (3).

**Question** Are $X_i$, $i = 1, 2, \ldots, d$ non-overlapping?

**Definition 1.3** Substitution $\sigma$ satisfies the coincidence condition if $\exists n, k$:

1. $f \left( P_k^{(n,1)} \right) = f \left( P_k^{(n,2)} \right) = \cdots = f \left( P_k^{(n,d)} \right)$

2. $w_k^{(n,1)} = w_k^{(n,2)} = \cdots = w_k^{(n,d)}$

where

$\sigma^n(1) = w_1^{(n,1)} \cdots w_k^{(n,1)} \cdots w_{l(n,1)}^{(n,1)}$  

$\sigma^n(d) = w_1^{(n,d)} \cdots w_k^{(n,d)} \cdots w_{l(n,d)}^{(n,d)}$.

**Proposition 1.1** If $\sigma$ satisfies the coincidence condition, then $X_i$, $i = 1, 2, \ldots, d$ are non-overlapping.

**Conjecture** Any Pisot unimodular substitutions satisfy the coincidence condition.

**Remark** On $d = 2$, the conjecture is proved by Barge and Diamond ([B-D]).

If $X_i$, $i = 1, 2, \ldots, d$ are non-overlapping, then we have two dynamical systems on $X$:

1. $T : X \rightarrow X$,  

   $Tx = L^{-1}_\sigma x - L^{-1}_\sigma \pi f \left( P_k^{(j)} \right)$ if $x \in X_i$, $\exists j, k : L^{-1}_\sigma x \in X_j + L^{-1}_\sigma \pi f \left( P_k^{(j)} \right)$.

   Therefore, $T$ is Markov endomorphism with the structure matrix $L^t_\sigma$.

2. $W : X \rightarrow X$  

   $x \mapsto x - \pi e_i$ if $x \in X_i$  

   is well-defined  

   and $W$ is called the domain exchange transformation (later, we will see $W \simeq$ the rotation on $T^{d-1}$).

From Markov endomorphism (1), we have the following numerical expression.

**Corollary 1.1** Using Markov endomorphism, $X_i$ is presented by

$X_i = \left\{ \pi_\varepsilon f \left( P_{k_0}^{(j_0)} \right) + \pi_\varepsilon L_\sigma f \left( P_{k_1}^{(j_1)} \right) + \pi_\varepsilon L_\sigma^2 f \left( P_{k_2}^{(j_2)} \right) + \cdots + \pi_\varepsilon L_\sigma^n f \left( P_{k_n}^{(j_n)} \right) + \cdots \mid (*) \right\}$

where $(*)$ is defined by $\left( j_0 \ k_0 \ j_1 \ k_1 \ \cdots \ j_n \ k_n \ \cdots \right)$ is given by $w_{k_0}^{(j_0)} = j_{n-1}$ and $w_{k_0}^{(j_0)} = i$.  

(see Figre 4).
Figure 4: Figures of Markov endomorphism $T$ and the domain exchange transformation $W$ on Example 1.2.
2 Stepped Surfaces and Tiling Substitutions

For

$$(x, i^*) \in \mathbb{Z}^d \times \{1^*, \ldots, d^*\},$$

we give a geometrical meaning such that

$$(x, i^*) := \left\{ \sum_{j=1, \ldots, d^*} x + \mu_j e_j \mid 0 \leq \mu_j \leq 1, j \neq i \right\}$$

(see Figure 5).

Figure 5: The figures of $(0, i^*)$.

Definition 2.1 $S := \{(x, i^*) \mid (x, u) \geq 0, (x - e_i, u) < 0\}$ is called the stepped surface of the contract plain $P$ where the row vector $u \geq 0$ is the eigenvector such that $u L_\sigma = \lambda u$ and the contract plain $P$ is given by $P = \{x \mid (x, u) = 0\}$.

Definition 2.2 $\pi S := \{\pi (x, i^*), (x, i^*) \in S\}$ is called a tiling of $P$ from the stepped surface $S$ (see Figure 6).

3 Dual Substitution $\sigma^*$ (Tiling Substitution)

$S^* := \{\text{the finite sum of elements of } S\} \simeq \{\text{the patches of tiles of the tiling } \pi S\}$.

Let us define the dual (tiling) substitution $\sigma^*$ by

$$\sigma^* \pi (x, i^*) = L_\sigma^{-1} \pi x + \sum_{j=1}^{d} \sum_{W_k^{(j)} = i} \pi_c \left( L_\sigma^{-1} f(P_k^{(j)}), j^* \right)$$

(see Figure 7 and Figure 8).
Figure 6: The figure of the tiling $\pi S$ of $P$ from the stepped surface $S$ on $\sigma_R$ in Example 1.2.

Figure 7: The figure of $\sigma_R^* \pi (0, i^*)$. 
Theorem 2 ([A-I]) \( \sigma \): An Pisot Unit substition, then

1. \( \sigma^* : S^* \rightarrow S^* \) is well-defined;
2. \( \forall (x, i^*) \in S, \exists (y, j^*) \in S: (x, i^*) \in \sigma^*(y, j^*) \);
3. \( (x, i^*) \neq (y, j^*) \in S \Rightarrow \sigma^*(x, i^*) \cap \sigma^*(y, j^*) = \emptyset \).

Theorem 3 ([I-R]) Let \( \mathcal{U} = \{ \pi(e_i, i^*) | i = 1, 2, \ldots, d \} \), then,

1. \( \sigma^* \mathcal{U} \succ \mathcal{U} \);
2. if \( d(\partial(\sigma^{*n} \mathcal{U}), 0) \rightarrow \infty \ (n \rightarrow \infty) \), then
   \[ \tau' = \{ \pi(x, j^*) | \pi(x, j^*) \in \sigma^{*n} \pi(e_i, i^*) \text{ for some } n \text{ and } j^* \} \]
   coincides with \( \pi S \) and a quasi-periodic tiling;
3. \( -X_i = \lim_{n \rightarrow \infty} L_0^0 \sigma^{*n} \pi(e_i, i^*) \);
4. \( \tau = \{ \pi x - X_j | \pi(x, j^*) \in \tau' \} \) is also a quasi-periodic tiling of \( P \).
Figure 9: The figure of the quasi-periodic tiling $\tau$.

References


