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Endomorphisms of a Module over a Valuation Domain

by

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Unless specified, $R$ is a valuation ring, that is, an integral domain in which either $a$ divides $b$, or $b$ divides $a$ for any nonzero $a$, $b$ in $R$. This shows that $R$ is a local ring with the unique maximal ideal $\mathfrak{m}$ consisting of all nonunits of $R$. Clearly its unit group is $R^* = R - \mathfrak{m}$.

Let $M$ be a left free module over $R$ of rank $n$, and $\text{End}_R(M)$ or $\text{End}(M)$ the right $R$-algebra of $R$-endomorphisms of $M$. The unit group of $\text{End}_R(M)$ is $\text{Aut}_R(M)$ or simply $\text{Aut}(M)$. We write an endomorphism $\sigma$ on the right side of a module element $x \in M$.

The special elements in $\text{End}_R(M)$ used here are (a) to (f) following, where $E = \{e_1, e_2, \cdots, e_n\}$ is a fixed basis for $M$ over $R$, and $X = \{x_1, x_2, \cdots, x_n\}$ is an arbitrarily chosen basis for $M$ over $R$.

(a) For $x, y \in M$ and $L \subseteq M$, let $M = Rx \oplus Ry \oplus L$. A transposition $\Delta = \Delta_{x,y,L} \in \text{Aut}(M)$ is defined by

$$x\Delta = y, \ y\Delta = x \text{ and } \Delta = 1 \text{ on } L.$$

(b) For $a \in R$, $x, y \in M$ and $U \subseteq M$, let $M = Rx \oplus Ry \oplus U$ and $L = Ry \oplus U$. A transvection $\tau = \tau_{x,ay,u} \in \text{Aut}(M)$ is defined by

$$x\tau = x + ay \text{ and } \tau = 1 \text{ on } L.$$

(c) For $a \in R$, $x, y \in M$ and $U \subseteq M$, let $M = Rx \oplus Ry \oplus U$.

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We define a left transposed transvection or a left skew transvection \( \varphi = \varphi_{x, ay} \in \text{Aut}(M) \) by 
\[
\varphi = \Delta_{x, y, U} \tau_{x, ay, U},
\]
i.e., 
\[
 x \varphi = y, \ y \varphi = x + ay \text{ and } \varphi = 1 \text{ on } U.
\]

Similarly, a right skew transvection \( \tau_{x, ay, U} \Delta_{x, y, U} \) is possible to define. However, as we will see, left skew is right skew and right skew is left skew. Therefore, we will often call them just skew transvections.

\[(d)\] For any elements \( a_1, a_2, \cdots, a_n \) in \( R \) and for \( X \) a basis for \( M \), we define \( \delta = \delta_X(a_1, a_2, \cdots, a_n) \in \text{End}(M) \) by 
\[
x_i \delta = a_i x_i, \quad i = 1, 2, \cdots, n.
\]

\[(e)\] An element \( \eta = \eta_X \in \text{Aut}(M) \) is defined by 
\[
x_1 \eta = x_1 \text{ and } x_i \eta = x_1 + x_i, \quad 2 \leq i \leq n.
\]
If \( n = 1 \), i.e., \( |X| = 1 \), we define \( \eta_X = 1 \), i.e., the identity map on \( M \).

\[(f)\] For \( \pi \in S_n \) we define a permutation automorphism \( \pi_X \in \text{End}_R(M) \) by 
\[
x_i \pi_X = x_{\pi i}
\]
The set of such \( \pi_X \) is denoted by \( S_X \). Clearly \( S_X \) is a subgroup of \( \text{Aut}(M) \) isomorphic to \( S_n \).

If \( X = \mathcal{E} \), i.e., the canonical basis, then for \( \pi \in S_n \) the permutation automorphism \( \pi_{\mathcal{E}} \) is said to be a canonical permutation. For simplicity we may write \( \pi \) and \( S_n \) instead of \( \pi_X \) and \( S_X \), respectively.

Moreover, providing these particular elements in \( \text{End}_R(M) \), we define the following three subsets of \( \text{End}_R(M) \), where \( X = \{x_1, x_2, \cdots, x_n\} \) is again an arbitrary chosen basis for \( M \):

The set of transvections relative to \( X \) is

\[(5.1)\]
\[
\tau_{R, X} = \{ \tau_{x_i, ax_j, U} | a \in R, \ U = \bigoplus_{h \neq i, j}^{n-1} Rx_h, \ 1 \leq i \neq j \leq n \},
\]

the set of skew transvections relative to \( X \) is

\[(5.2)\]
\[
\varphi_{R, X} = \{ \varphi_{x_i, ax_j, U} | a \in R, \ U = \bigoplus_{h \neq i, j}^{n-1} Rx_h, \ 1 \leq i \neq j \leq n \},
\]
and for subsets \( S_1, S_2, \cdots, S_n \) of \( R \) we write

\[(5.3)\]
\[
\delta_X(S_1, S_2, \cdots, S_n) = \{ \delta_X(a_1, a_2, \cdots, a_n) | a_i \in S_i \}.
\]
For any $\sigma \in \text{End}_R M$ we define the fixed submodule $M_{\sigma}$ of $\sigma$ by

$$M_{\sigma} = \{ x \in M \mid x\sigma = x \}.$$

**Definition.** For $i = 0, 1, \cdots, n$ we define

$$S^{(i)} = \{ \sigma \in \text{End}(M) \mid \text{rank } M_{\sigma} = n - i \}.$$

An element $\sigma$ in $S^{(1)}$ is called a simple element, i.e., $\sigma$ is simple if and only if $\sigma$ fixes a hyper plane of $M$.

By definition, $\Delta$ in (a) and $\tau$ in (b) are in $S^{(1)}$, and $\varphi$ in (c) is in $S^{(2)}$. Also $\delta$ in (d) is in $S^{(n-i)}$ if exactly $i$ of $\{a_1, a_2, \cdots, a_n\}$ is 1. Further $\eta$ in (e) belongs to $S^{(n-1)}$.

**Main Theorem.** Let $0 \neq \sigma \in \text{End}_R M$. Then there exist

(i) $\eta_{Z'}$ with $Z' \subseteq Z$ for some basis $Z$ for $M$
and

(ii) skew transvections $\psi_1, \psi_2, \cdots, \psi_l$ with $0 \leq l \leq n - 1$

such that

$$\psi_l \cdots \psi_2 \psi_1 \eta_{Z'} \sigma = \delta_X (a_1, a_2, \cdots, a_l, a_{l+1}, \cdots, a_n)$$

for some basis $X$ for $M$ with $a_1 \mid a_2 \mid \cdots \mid a_l$ and $a_l \mid a_i$ for $l \leq i \leq n$. 
References


