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An experiment in computer of the Banach-Tarski paradox on the lattice points of the plane

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The purpose of this paper is to realize the Banach-Tarski paradox on $\mathbb{Z}^2$. Ordinary Banach-Tarski paradoxes are the followings:

Banach-Tarski paradox for Euclidean spaces.

$n \geqq 3$: an integer,

$U, V \subseteqq \mathbb{R}^n$: bdd, $\mathrm{int}\, U \neq \emptyset$, $\mathrm{int}\, V \neq \emptyset$

$\Rightarrow$ *$^\exists \ell$: a positive integer such that*

U and V are $SG_n(\mathbb{R})$-equidecomposable using ℓ pieces (denoted by $U \underset{SG_n(\mathbb{R})}{\overset{\ell}{\sim}} V$), i.e.,

$^\exists\{U_0, U_1, \ldots, U_{\ell-1}\}$: a partition of U

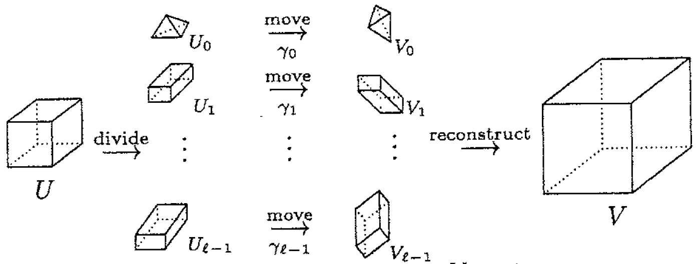
(that is, $U = \bigcup_{i=0}^{\ell-1} U_i$ and $U_i \cap U_j = \emptyset$ for $i \neq j$),

$^\exists\{V_0, V_1, \ldots, V_{\ell-1}\}$: a partition of V such that

$$U_i \underset{SG_n(\mathbb{R})}{\approx} V_i \quad \text{for} \quad i = 0, 1, \ldots, \ell-1$$

(that is, $^\exists \gamma_i \in SG_n(\mathbb{R})$ such that $\gamma_i(U_i) = V_i$),

where $SG_n(\mathbb{R}) = \{\gamma : \mathbb{R}^n \to \mathbb{R}^n$: an orientation-preserving isometry$\}$ (by Banach & Tarski).



non-measurable sets non-measurable sets

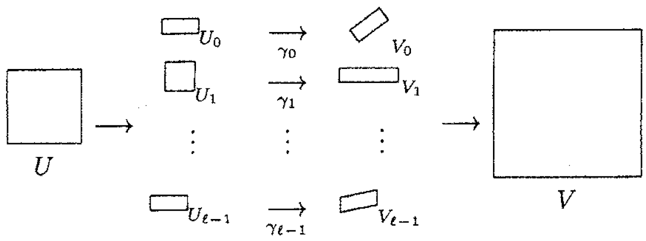
Banach-Tarski paradox for plane.

$U, V \subseteqq \mathbb{R}^2$: bdd, $\mathrm{int}\, U \neq \emptyset$, $\mathrm{int}\, V \neq \emptyset$

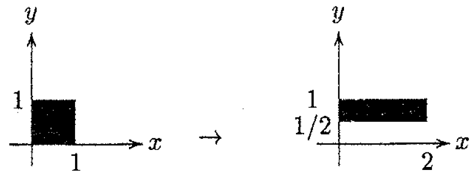
$\Rightarrow$ *$^\exists \ell$: a positive integer such that $U \underset{SA_2(\mathbb{R})}{\overset{\ell}{\sim}} V$,*

This is an abstract and the details will be published elsewhere. The title in Japanese is "平面内の格子点集合上の Banach-Tarski の逆理の計算機実験".

where $SA_2(\mathbb{R}) = \{\gamma : \mathbb{R}^2 \to \mathbb{R}^2$: *an affine transformation with determinant* $+1\}$ (by von Neumann).



Example of an element of $SA_2(\mathbb{R})$.



$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

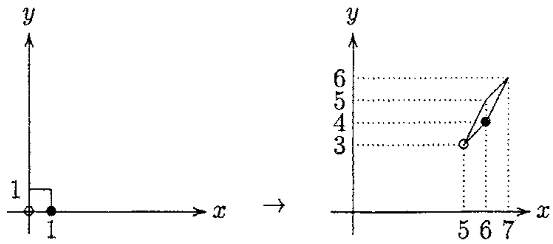
The author is considering the Banach-Tarski paradoxes on denumerable sets of Euclidean spaces or denumerable sets of the sphere of the Euclidean spaces, because we do not have to use the axiom of choice to prove them. The following is one of them:

Hausdorff decomposition for lattice points in plane (Main theorem).
$\mathbb{Z}^2$ has a $SA_2(\mathbb{Z})$-Hausdorff decomposition, i.e, there exists a partition $\{P, Q, R\}$ of $\mathbb{Z}^2$ such that

$$P \underset{SA_2(\mathbb{Z})}{\approx} Q \underset{SA_2(\mathbb{Z})}{\approx} R \underset{SA_2(\mathbb{Z})}{\approx} P \cup Q \underset{SA_2(\mathbb{Z})}{\approx} Q \cup R \underset{SA_2(\mathbb{Z})}{\approx} R \cup P,$$

where $SA_2(\mathbb{Z}) = \{\gamma : \mathbb{Z}^2 \to \mathbb{Z}^2$: *an affine transformation with determinant* $+1\}$ (by S.).

Example of an element of $SA_2(\mathbb{Z})$.



$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Sketch of proof.
Let F_3 be the group generated by three transformations of $SA_2(\mathbb{Z})$:

$$\alpha : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 4 \\ 4 & 17 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \end{pmatrix},$$

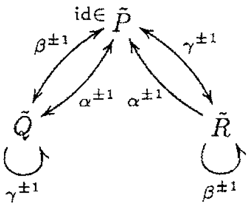
$$\beta : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 9 & 20 \\ 4 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 7 \\ 3 \end{pmatrix},$$

$$\gamma : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 17 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

Then, F_3 is *a free group of rank* 3, i.e., any non-trivial words in $\{\alpha^{-1}, \beta^{-1}, \gamma^{-1}, \alpha, \beta, \gamma\}$ without the form $\cdots \lambda^{-1}\lambda \cdots$ is not equal to the identity

$$\mathrm{id} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and each non-identical element of F_3 has no fixed point in $\mathbb{Z}^2$. We can make a partition $\{\tilde{P}, \tilde{Q}, \tilde{R}\}$ of F_3 according to the figure below:



(e.g., id $\in \tilde{P}$, $\gamma \in \tilde{R}$, $\beta\gamma \in \tilde{R}$, $\gamma^{-1}\beta\gamma \in \tilde{P}$, $\alpha\gamma^{-1}\beta\gamma \in \tilde{Q}$, ...). Then, $\alpha(\tilde{P}) = \tilde{Q} \cup \tilde{R}$, $\beta(\tilde{P}) = \tilde{Q}$, and $\gamma(\tilde{P}) = \tilde{R}$. So, for a choice set M of $\mathbb{Z}^2/F_3$, if we set

$$P = \bigcup_{w \in \tilde{P}} w(M), \quad Q = \bigcup_{w \in \tilde{Q}} w(M), \quad R = \bigcup_{w \in \tilde{R}} w(M),$$

then we have $\alpha(P) = Q \cup R$, $\beta(P) = Q$, and $\gamma(P) = R$. □

The Hausdorff decomposition on $\mathbb{Z}^2$ is realized as follows:

Computer program.
We can make three sets of Hausdorff decomposition P, Q, and R step by step, by well-ordering of F_3 and $\mathbb{Z}^2$:

$$w \lneqq w' \Leftrightarrow \sharp w \lneqq \sharp w' \text{ or } (\sharp w = \sharp w' \text{ and } {}^{\exists} w_0, \lambda, \lambda', w_1, w_1' \text{ s.t. } w = w_0 \lambda w_1,\ w' = w_0 \lambda' w_1', \text{ and } \lambda \lneqq \lambda'),$$

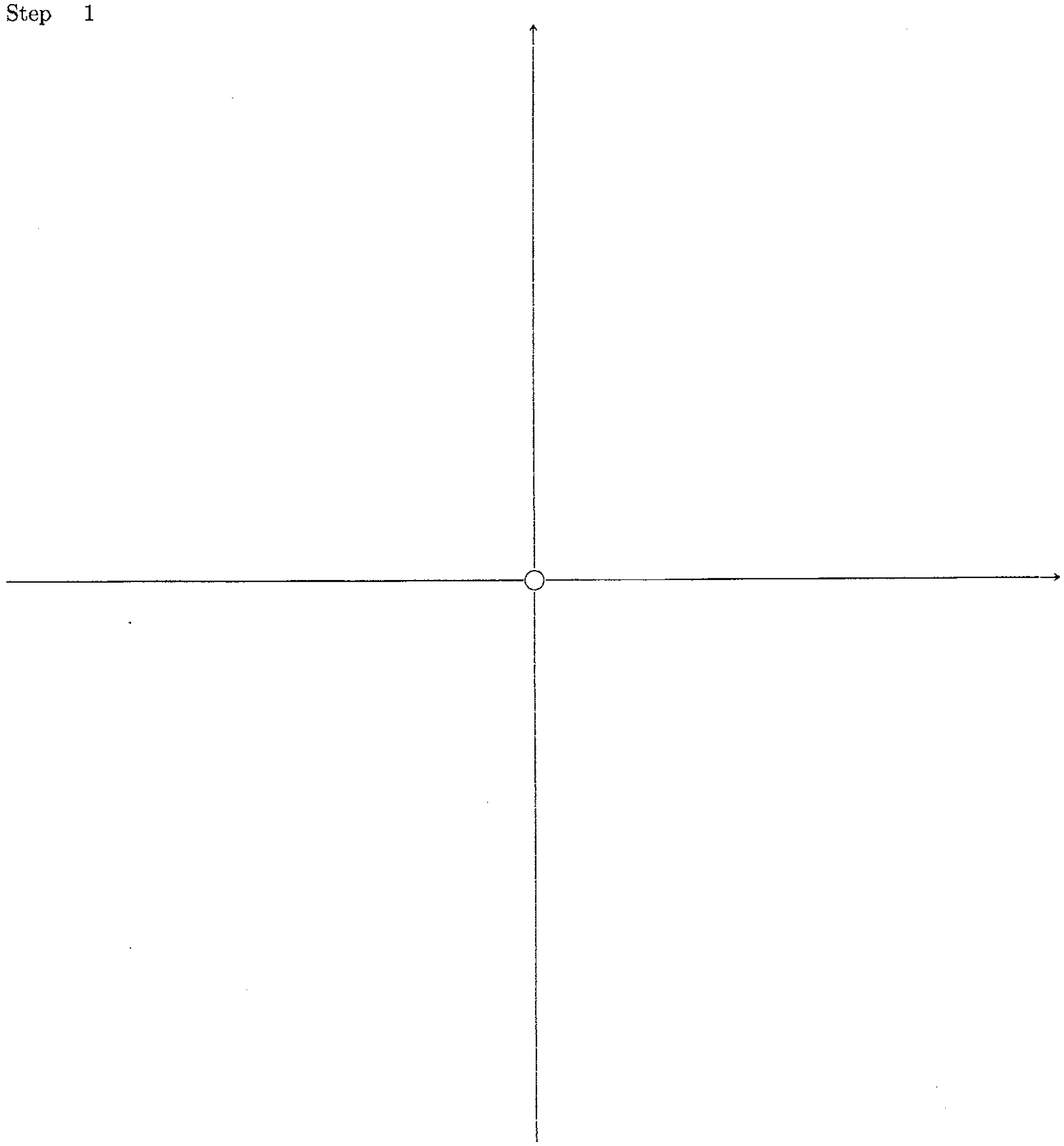
with $\alpha^{-1} \lneqq \beta^{-1} \lneqq \gamma^{-1} \lneqq \alpha \lneqq \beta \lneqq \gamma$,

$$(x, y) \lneqq (x', y') \Leftrightarrow |x| + |y| \lneqq |x'| + |y'| \text{ or } (|x| + |y| = |x'| + |y'| \text{ and } \theta \lneqq \theta'),$$

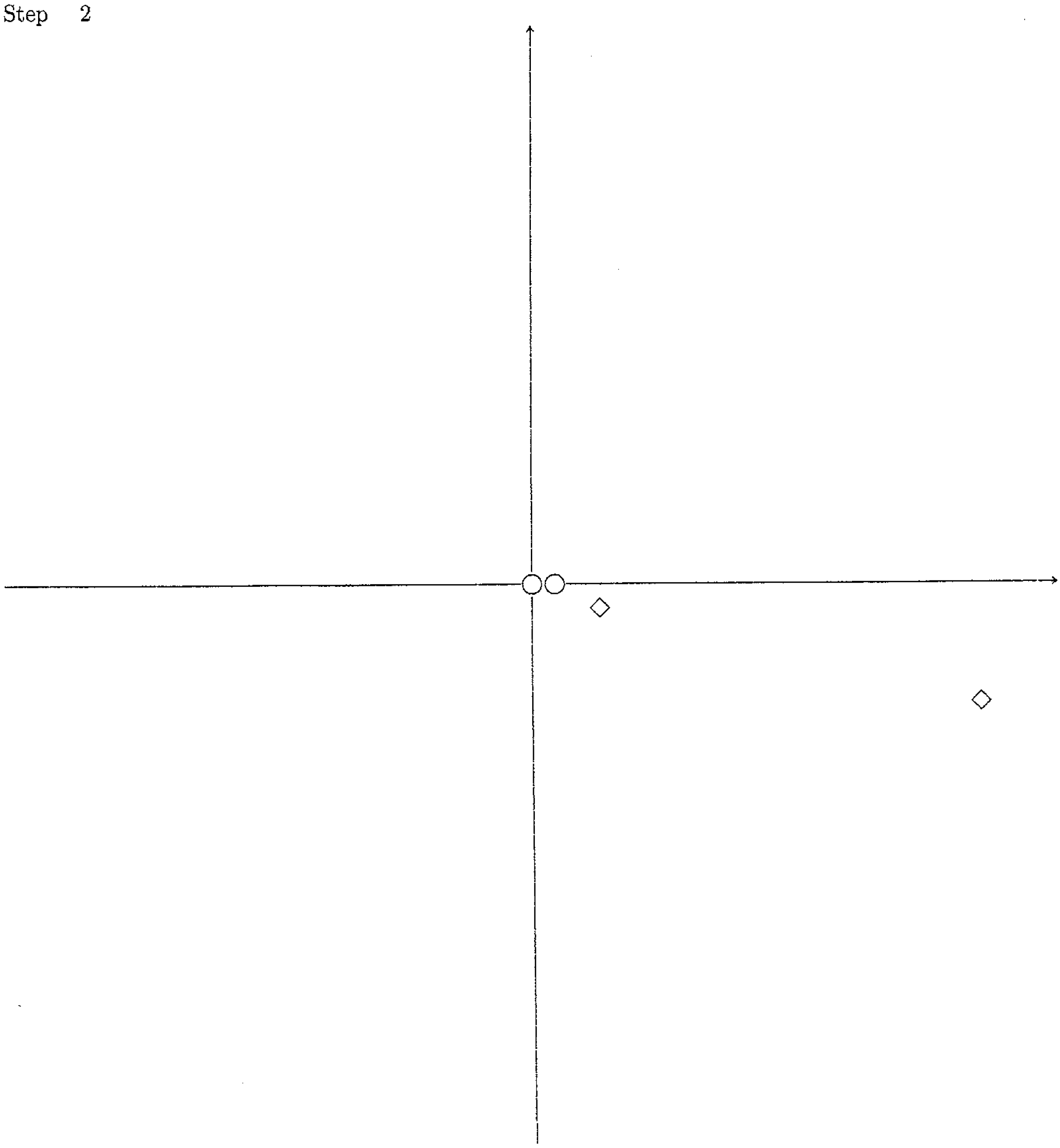
where θ, $\theta' \in [0, 2\pi[$ are arguments of complex numbers $x + \sqrt{-1}y$ and $x' + \sqrt{-1}y'$, respectively.

	$P(\bigcirc)$			$Q(\diamond)$			$R(\triangle)$
Step 0							
Step 1	ε $(0,0)$						
Step 2	ε $(0,0)$ $(1,0)$			α^{-1} $(3,-1)$ $(20,-5)$			
Step 3	ε $(0,0)$ $(1,0)$ $(0,1)$			α^{-1} $(3,-1)$ $(20,-5)$ $(-1,0)$	β^{-1} $(-3,1)$ $(6,-3)$ $(-23,10)$		
Step 4	ε $(0,0)$ $(1,0)$ $(0,1)$ $(-1,0)$			α^{-1} $(3,-1)$ $(20,-5)$ $(-1,0)$ $(-14,3)$	β^{-1} $(-3,1)$ $(6,-3)$ $(-23,10)$ $(-12,5)$		γ^{-1} $(-1,3)$ $(0,-1)$ $(-5,20)$ $(-2,7)$
	ε $(0,0)$ $(1,0)$ $(0,1)$	α^{-2} $(-14,3)$	$\beta^{-1}\alpha^{-1}$ $(-12,5)$	α^{-1} $(3,-1)$ $(20,-5)$ $(-1,0)$	β^{-1} $(-3,1)$ $(6,-3)$ $(-23,10)$	$\gamma^{-1}\alpha^{-1}$ $(-2,7)$	γ^{-1} $(-1,3)$ $(0,-1)$ $(-5,20)$

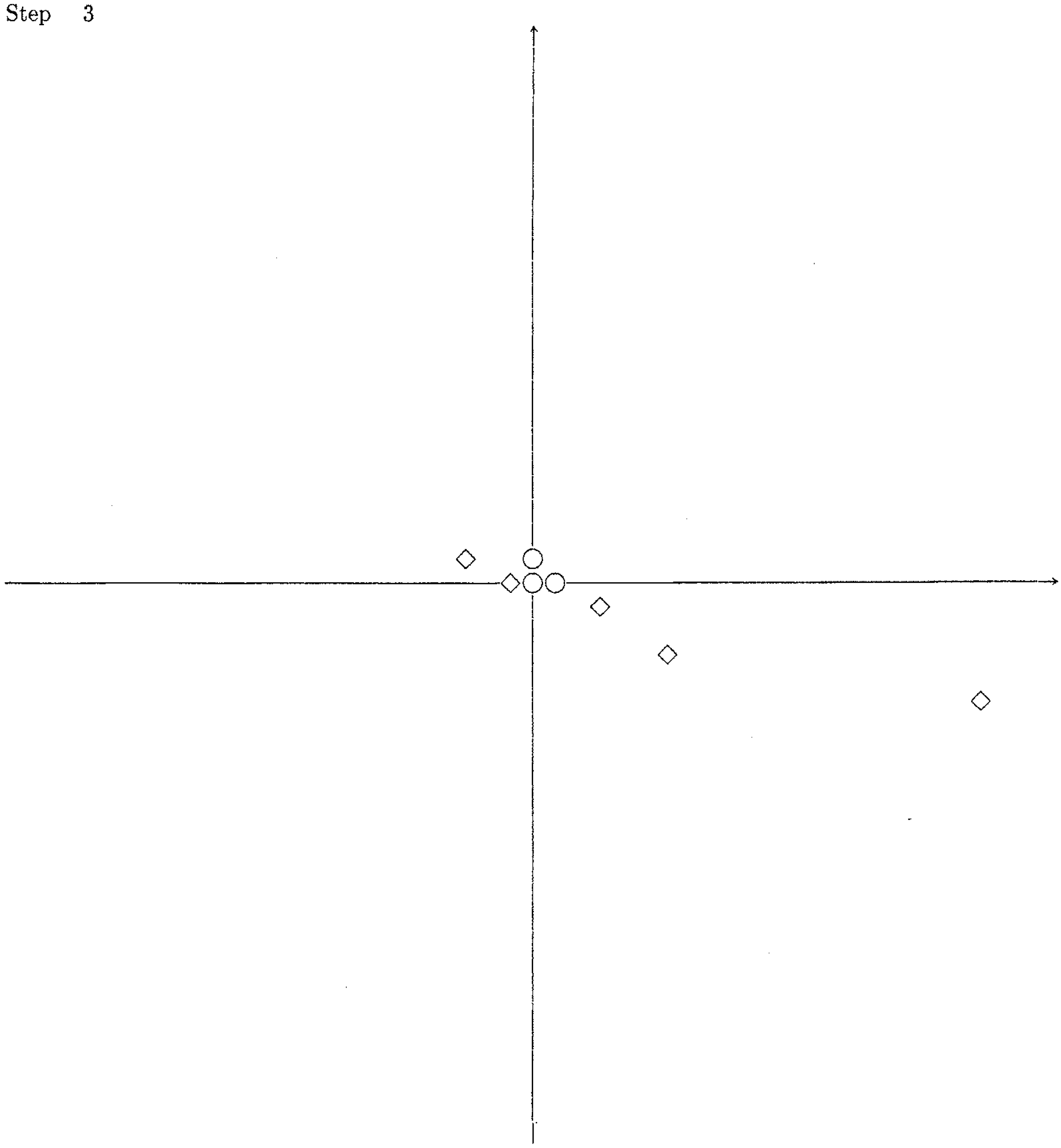
Step 1



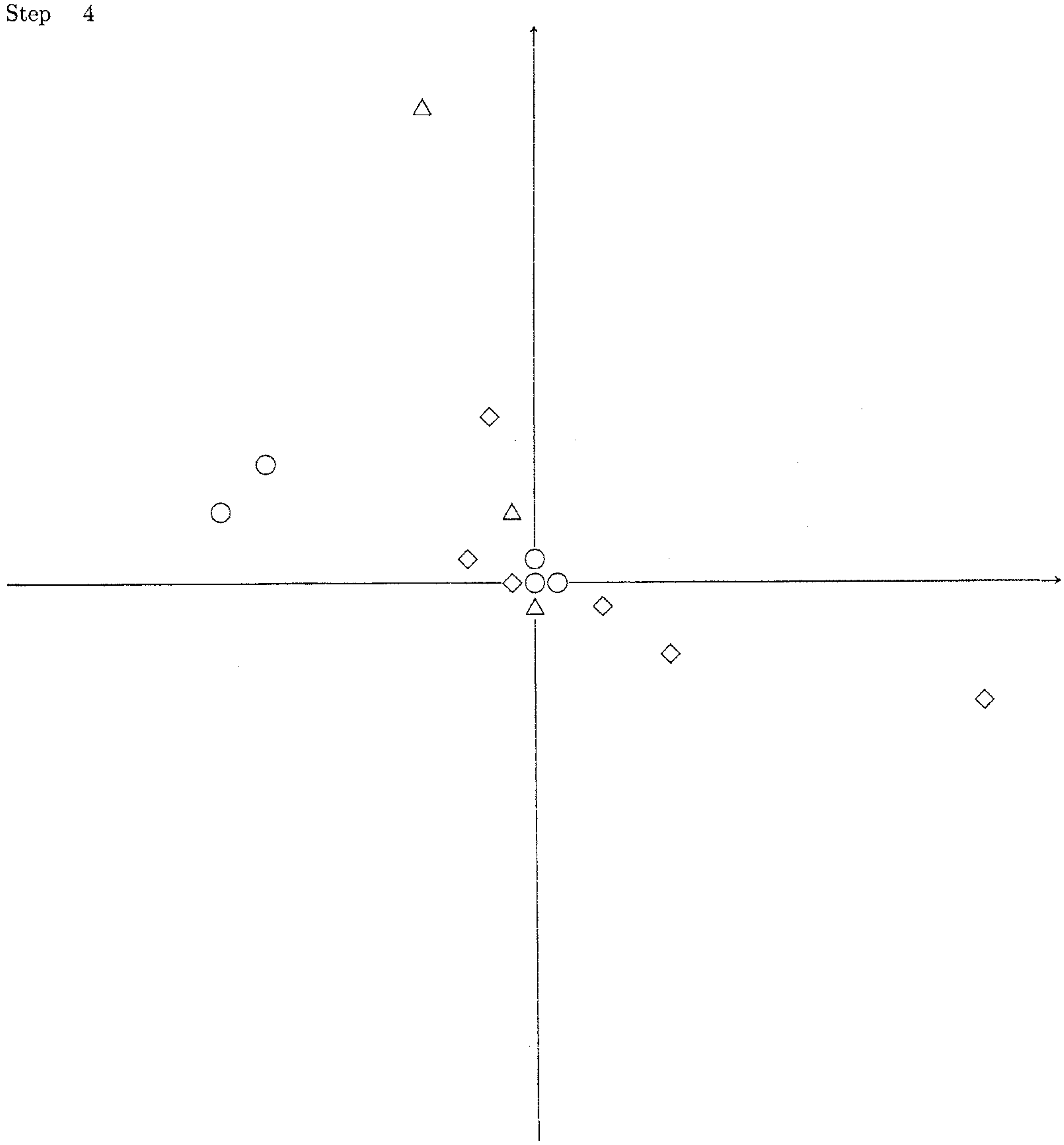
Step 2



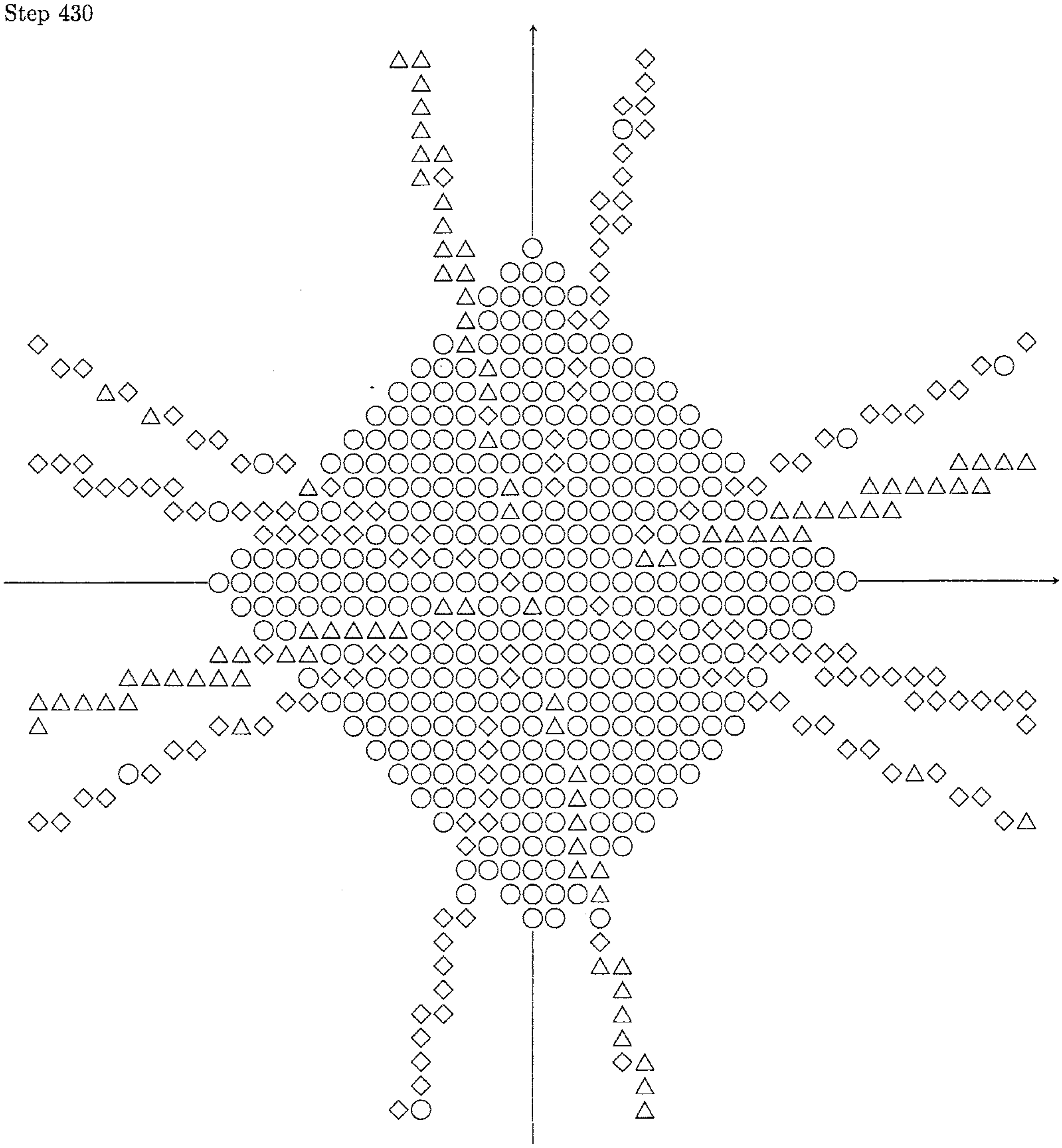
Step 3



Step 4



Step 430



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