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The Simple Graphs Associated with Rings and Semigroups

Mitsuo Kanemitsu

Let $R$ be a commutative ring, and let $Z(R)$ denote its set of zero-divisors. We associate a simple graph $\Gamma(R)$ to $R$ with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero-divisors of $R$. Two distinct vertices $x$ and $y$ are adjacent if $xy = 0$. This graphs are called the zero-divisor graphs of rings $R$.

We also associate a simple graph $\Delta(Z_n)$ to $Z_n$ with vertices $Z_n$ and for distinct elements $x, y \in Z_n$, the vertices $x$ and $y$ are adjacent if and only if $y = x^2 (x \neq y)$. This graphs are called the parabola graphs.

For a commutative multiplicative semigroup $S$ with $0 (0x = 0$ for all $x \in S)$, we can defined the zero-divisor graph $\Gamma(S)$ as above ([DMS]).

We denote an edge such that $a$ and $b$ are adjacent by $a - b$. We also denote a path by $a - b - c - d$ etc. Also, let $\chi(G)$ denote the chromatic number of the graph $G$ and let $\chi'(G)$ denote the edge chromatic number of the graph $G$.

The notion of a zero-divisor graph was first introduced by I. Beck in [B1] and further investigated in [A1], though their vertices set included the zero element. Let $G$ be a graph.

The diameter of $G$ is
\[
diam(G) = \sup \{d(x, y) \mid x \text{ and } y \text{ are distinct vertices of } G\},
\]
where $d(x, y)$ denotes the length of the shortest path from $x$ to $y$. The girth of $G$, denoted by $g(G)$, is defined as the length of the shortest cycle in $G$.

A complete subgraph of $G$ is called clique. $\omega(G)$, the clique number of $G$, is the greatest integer $r \geq 1$ such that $K^r \subset G$. Also, $c(G)$, the circumference of $G$, is the length of the longest cycle in $G$. Let $n = p_1^{2n_1} \cdots p_k^{2n_k} q_1^{2m_1+1} \cdots q_r^{2m_r+1}$ for distinct primes $p_i$, $q_j$ and integers $n_i, m_j \geq 0$. Then
\[
\omega(\Gamma(Z_n)) = p_1^{n_1} \cdots p_k^{n_k} q_1^{m_1} \cdots q_r^{m_r} + r - 1
\]

1 This is a part of an abstract and details will be published elsewhere.
by [B, Proposition 2.3].

For a graph $G$ and an integer $n \geq 1$, we define $\lambda(G, n)$ to be the number of complete subgraphs (cliques) of $G$ of order $n$. Note that $\lambda(G, 1)$ is the number of vertices of $G$, $\lambda(G, 2)$ is the number of edges of $G$, $\lambda(G, 3)$ is the number of triangles in $G$ and $\lambda(G, n) = 0$ for all $n \geq \omega(G) + 1$ ([AFLL]).

A graph $G$ is planar if it can be drawn in such a way that no two edges intersect.

We examine the zero-divisor graph $\Gamma(Z_n)$, of the ring $Z_n$ where the residue class ring modulo $n$ where $n$ is a positive integer. Many parts of this note are contained in [DS] and [AF].

We also investigate the parabora graphs.

For any commutative semigroup $S$, let $C(S) = \text{the core of } \Gamma(S)$ be the union of the cycles in $\Gamma(S)$. A vertex $x$ of $\Gamma(S)$ is called an end point in case there is at most one edge in $\Gamma(S)$ with vertex $x$.

§1. Some examples of zero-divisor graphs and parabora graphs.

Example 1. Let $S$ be the commutative nilsemigroup

$$S = \langle a, b \mid a^3 = a^2 b = ab^2 = b^3 = 0 \rangle = \{a, b, a^2, ab, b^2, 0\}.$$  

For this zero-divisor graph $\Gamma(S) = (V(\Gamma(S)), E(\Gamma(S)))$, we have that 

$$V(\Gamma(S)) = \{a, b, a^2, ab, b^2\}$$

is the set of all vertices and $E(\Gamma(S)) = \{a-a^2, a-b^2, a-ab, b^2-ab, b^2-a^2, b^2-b, \}$. This graph is connected, $d(a, b^2) = 1, d(a, b) = 2$ and $a - a^2 - b^2 - a$ is a cycle of length 3. And so $diam(\Gamma(S)) = 2$ and $g(\Gamma(S)) = 3$. There is not an end point in $\Gamma(S)$ and $C(S) = \Gamma(S)$. Also, we have that $\omega(\Gamma(S)) = 4$. Also we have that $\lambda(S, 1) = 5, \lambda(S, 2) = 9$ and $\lambda(S, 2) = 6$.

Example 2. Let $S$ be the commutative nilsemigroup

$$S = \langle a, b \mid a^3 = a^2 b = ab^2 = b^3 = 0, \ a^2 = b^2 \rangle = \{a, b, a^2, ab, 0\}.$$
\[ \Gamma(S) \text{ is connected, } d(a, a^2) = 1, d(a, b) = 2 \text{ and } a - a^2 - ab - a \text{ is a cycle of length 3. And so } diam(\Gamma(S)) = 2 \text{ and } g(\Gamma(S)) = 3. \text{ There is not an end point in } \Gamma(S) \text{ and } C(S) = \Gamma(S). \text{ Also, we have that } \omega(\Gamma(S)) = 3 \text{ and } c(\Gamma(S)) = 3. \text{ Also we have that } \lambda(S, 1) = 4, \lambda(S, 2) = 5 \text{ and } \lambda(S, 2) = 2. \]

This graph \( \Gamma(S) \) is planar.

**Example 3.** \( V(\Gamma(Z_{12})) \) of the zero-divisor graph \( \Gamma(Z_{12}) \) is the set

\[
\{2, 3, 4, 6, 8, 9, 10\}
\]

and

\[
E(\Gamma(Z_{12})) = \{2 - 6, 3 - 4, 3 - 8, 4 - 6, 4 - 9, 6 - 8, 6 - 10, 8 - 9\}.
\]

The elements 4, 9 are idempotent elements and 6 is a nilpotent element. Also, \{0, 6\} is an ideal of \( Z_{12} \). \( \omega(\Gamma(Z_{12})) = 4 \). Also we have that \( \lambda(Z_{12}, 1) = 7, \lambda(Z_{12}, 2) = 8 \) and \( \lambda(Z_{12}, 3) = 0 \). Also \( diam(\Gamma(Z_{12})) = 3, g(\Gamma(Z_{12})) = 4 \) and \( c(\Gamma(Z_{12})) = 4 \). We have that \( \chi(\Gamma(Z_{12})) = 2 \) abd \( \chi'(\Gamma(Z_{12})) = 4 \). \( \Gamma(Z_{12}) \) is planar.

**Example 4.** \( V(\Gamma(Z_{15})) \) of the zero-divisor graph \( \Gamma(Z_{15}) \) is the set

\[
\{3, 5, 6, 9, 10, 12\}
\]

and

\[
E(\Gamma(Z_{15})) = \{3 - 5, 3 - 10, 5 - 6, 5 - 9, 5 - 12, 6 - 10, 9 - 10, 10 - 12\}.
\]

The elements 6, 10 are idempotent elements. The ring \( Z_{15} \) has no ideals contained only two elements. We have that \( \omega(Z_{15}) = 4, c(Z_{15}) = 4 \) and also we have that \( \lambda(Z_{15}, 1) = 6, \lambda(Z_{15}, 2) = 8 \) and \( \lambda(Z_{15}, 2) = 0 \). Also \( diam(\Gamma(Z_{15})) = 2 \) and \( g(\Gamma(Z_{15})) = 4 \). We have that \( \chi(\Gamma(Z_{15})) = 2 \) abd \( \chi'(\Gamma(Z_{15})) = 4 \). \( \Gamma(Z_{15}) \) is planar.

**Example 5.** \( V(\Gamma(Z_{16})) \) of the zero-divisor graph \( \Gamma(Z_{16}) \) is the set

\[
\{2, 4, 6, 8, 10, 12, 14\}
\]

and

\[
E(\Gamma(Z_{16})) = \{2 - 8, 4 - 8, 4 - 12, 6 - 8, 8 - 10, 8 - 12, 8 - 14\}.
\]

The element 8 is an idempotent element and 4, 12 are nilpotent elements. The set \{0, 8\} is an ideal of \( Z_{16} \). We have that \( \omega(Z_{16}) = 3 \). Also we have that \( \lambda(Z_{16}, 1) = 7, \lambda(Z_{16}, 2) = 7 \) and \( \lambda(Z_{16}, 3) = 1 \). Also \( diam(\Gamma(Z_{16})) = \)
2, \( g(\Gamma(Z_{16})) = 3 \) and \( c(\Gamma(Z_{16})) = 3 \). We have that \( \chi(\Gamma(Z_{16})) = 3 \) and \( \chi'(\Gamma(Z_{16})) = 6 \). \( \Gamma(Z_{16}) \) is planar.

**Example 6.** Let \( \Gamma(Z_4[X]/(X^2)) = \Gamma(Z_4[x]) \) \((x^2 = 0)\) be a zero-divisor graph associated with the ring \( Z_4[X]/(X^2) \). Set \( R_4[x] = Z_4[x] \). Then \( V(\Gamma(R_4[x])) = \{2, x, 2+x, 2x, 2+2x, 3x, 2+3x\} \) and \( E(R_4[x]) = \{2-2x, 2-2+2x, x-2x, x-3x, 2+x-2x, 2x-2+2x, 2+3x-2x\} \).

We have that \( \omega(R_4[x]) = 3 \), \( c(R_4[x]) = 3 \) and also we have that \( \lambda(R_4[x], 1) = 7 \), \( \lambda(R_4[x], 2) = 8 \), \( \lambda(R_4[x], 3) = 2 \) and \( \lambda(R_4[x], 4) = 0 \). Also \( diam(\Gamma(R_4[x])) = 2 \) and \( g(\Gamma(R_4[x])) = 3 \). We have that \( \chi(\Gamma(R_4[x])) = 3 \) and \( \chi'(\Gamma(R_4[x])) = 6 \). \( \Gamma(R_4[x]) \) is planar.

**Lemma 1.** \( \Gamma(Z_n) \) is connected and \( diam(Z_n) \leq 3 \).

**Lemma 2.** If \( \Gamma(Z_n) \) contains a cycle, then \( g(Z_n) \leq 4 \).

**Conjecture.** The number of end points of \( \Gamma(Z_n) \) is an even number.

This conjecture is valid for \( n \leq 30 \).

**Lemma 3.** If \( n \geq 9 \) and \( \Gamma(Z_n) \) is a zero-divisor graph of \( Z_n \), then there exists an element \( x \) such that \( \{0, x\} \) is an ideal of \( Z_n \). \( Ann(x) \) is a maximal ideal of \( Z_n \) and \( Z_n/Ann(x) \cong Z_2 \).

If \( p \) is a prime number \((\neq 2)\), then \( \{0, p\} \) is an ideal of \( Z_{2p} \). So \( \Gamma(Z_{2p}) \) is a star graph.

**Lemma 4.** If any vertex in \( \Gamma(Z_n) \) is either a vertex of the core \( C(\Gamma(Z_n)) \) or else is an end point of \( \Gamma(Z_n) \).

**Lemma 5 [[B] and [AN]].** Let \( p, q \) and \( r \) be all distinct prime numbers. The following statements are satisfied.

(1) \( \omega(\Gamma(Z_n)) = 1 \) if and only if \( n = 4 \).

(2) \( \omega(\Gamma(Z_n)) = 2 \) if and only if \( n = 8, 9, pq \), \( 4p \) \((p \neq 2)\).

(3) \( \omega(\Gamma(Z_n)) = 3 \) if and only if \( n = pqr \), \( 4pq \) \((p \neq 2, q \neq 2)\), \( 8p \) \((p \neq 2)\), \( 9p \) \((p \neq 3)\), 16, 27.
§2. The parabola graph $y = x^2$

We associate a simple graph $\Delta(Z_n)$ to $Z_n$ with vertices $Z_n$ and for distinct elements $x, y \in Z_n$, the vertices $x$ and $y$ are adjacent if and only if $y = x^2$ ($x \neq y$).

Example 1. The vertices set $V(\Delta(Z_7))$ of the simple graph $\Delta(Z_7)$ is the set $Z_7$ and $E(\Delta(Z_7)) = \{1 - 6, 2 - 3, 2 - 4, 4 - 5\}$. $\lambda(\Delta(Z_7), 1) = 7, \lambda(\Delta(Z_7), 2) = 4$ and $\lambda(\Delta(Z_7), 3) = 0$. We have that $\text{diam}(\Delta(Z_7)) = 3, \omega(\Delta(Z_7)) = 2, c(\Delta(Z_7)) = g(\Delta(Z_7)) = 0$.

Example 2. Let $\Delta(Z_{11}) = (V(\Delta(Z_{11})), E(\Delta(Z_{11})))$ be a simple graph associated with $Z_{11}$. We have that $V(\Delta(Z_{11})) = Z_{11}$ and $E(\Delta(Z_{11})) = \{2 - 10, 3 - 5, 3 - 6, 9 - 4, 5 - 9, 5 - 7, 8 - 9\}$. $\lambda(\Delta(Z_{11}), 1) = 11, \lambda(\Delta(Z_{11}), 2) = 8$ and $\lambda(\Delta(Z_{11}), 3) = 0, \lambda(\Delta(Z_{11}), 4) = 1$. We have that $\text{diam}(\Delta(Z_{11})) = 4, \omega(\Delta(Z_{11})) = 2, c(\Delta(Z_{11})) = 4, g(\Delta(Z_{11})) = 4$. This graph is not a forest.

Theorem. (a) Let $\Delta(Z_7)$ be a parabola graph and let $A_7$ be an adjacent matrix of a parabola graph $\Delta(Z_7)$. Also, $f_n(X) = f_{A_n}(X)$ be a minimal polynomial of $A_7$ Then the following statements hold.

1. If $n$ is an even natural number, then $f_n(X)$ has a divisor $X^2 - L_nX + 1$.
2. If $n$ is an odd natural number, then $f_n(X)$ has a divisor $X^2 - L_nX - 1$, where $L_n$ is a Lucas number.

(b) (1) $\Delta(Z_p)$ has no triangles, that is, $\lambda(\Delta(Z_p), 3) = 0$ for a prime number $p$.
(2) $\lambda(\Delta(Z_n)) = 0$ for $1 \leq n \leq 10$ and $12 \leq n \leq 20$.
(3) For parabola graphs $\Delta(Z_n)$ ($2 \leq n \leq 20$), their graphs are forest except $n = 11$.

References


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