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Schutzenberger graphs and embedding theorems

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Abstract

The embedding of an inverse semigroup into a variant of HNN extension is obtained. We charcterize HNN extensions where the set of idempotents do not get larger by the construction. We clarify the relationship among the definitions of HNN extensions of inverse semigroups; the embedding theorem for HNN extensions is obtained by the automata theoretical technique based on the combinatorial and geometrical properties of Schützenberger graphs. This paper is an extended abstract and the detailed version will be published elsewhere.

1 Introduction

The concept of an HNN extension of groups is introduced by Higman, Neumann and Neumann [5]. Several generalizations are possible for inverse semigroups depending upon the interpretations of the concept of *conjugacy*. The concept is generalized to the class of semigroups and inverse semigroups in [8]. HNN extensions of groupoids are considered in Higgins [4] and transfered into inverse semigroups by Gilbert [2]. The constructions in [8] and [2] are defined under different assumptions and sêem to stand apart away from one another. The definition in [8] has the features of abstraction of free objects in inverese semigroups such as free inverse semigroups. On the other hand, the construction in [2] has a strong connection to groupoid theory. In this paper, we clarify the relationship between these two constructions. First, we show that HNN extension in the sense of Gilbert can be extended to more general context and embeddability property still holds. Second, we show that every HNN extension in the sense of [2] can be naturally embedded into an HNN extension of the original inverse semigroup adjoined two extra idempotents in the sense of [8]. Therefore, twese two constructions share almost identical algebraic structures. To prove the embedability, we use the iterative construction of automata based on Schützenberger graphs by Stephen [7].

1.1 Concepts of HNN extensions

We recall the concept of HNN extensions in [2, 8]. Let S be an inverse semigroup, and let A and B be inverse subsemigroups of S. Suppose that $e \in A \subset eSe$, $f \in B \subset fSf$ for some idempotents e, f of S and that $\phi : A \to B$ is an isomorphism. Then the inverse semigroup presented by

$$\mathrm{Inv}(S, t \mid t^{-1}at = \phi(a) \text{ for } \forall a \in A,\ t^{-1}t = f,\ tt^{-1} = e), \tag{1.1}$$

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or equivalently,

$$\mathrm{Inv}(S, t \mid t^{-1}at = \phi(a) \text{ for } \forall a \in A',\ t^{-1}t = f,\ tt^{-1} = e), \tag{1.2}$$

where A' is a set of generators of A, is called the *HNN extension of S associated with $\phi : A \to B$*. We here denote the inverse semigroup presented by (1.1) (or (1.2)) by $S(\phi : A \to B)$. The element t in $S(\phi : A \to B)$ is called the *stable letter*. Higgins [4] introduces a concept of an HNN extension of a groupoid. Interpreting into inverse semigroups context, Gilbert [2] defines the HNN extension $S_{*U,\varphi}$ of an inverse semigroup S with associated order ideals U and V to be presented by

$$\mathrm{Inv}(S,\ t_e\ (e \in E(U)) \mid t_e t_f^{-1} = ef,\ t_e^{-1} t_f = \varphi(ef),\ t_{uu^{-1}}^{-1} u t_{u^{-1}u} = \varphi(u), u \in U). \tag{1.3}$$

2 Embedding theorems

Suppose that S is an inverse semigroup, A and B are isomorphic inverse subsemigroups of S. We do not assume A and B are order ideals. Let ϕ be an isomorphism of A onto B. We consider the inverse semigroup $S[\phi : A \to B]$ presented by

$$\mathrm{Inv}(S,\ t_e\ (e \in E(A)) \mid t_e t_f^{-1} = ef,\ t_e^{-1} t_f = \phi(ef),\ t_{aa^{-1}}^{-1} a t_{a^{-1}a} = \phi(a), a \in A). \tag{2.1}$$

When A and B are order ideals, $S[\phi : A \to B]$ is exactly the construction of Gilbert. Thus, this is a generalization of the construction by Gilbert. Because of the normal form of the Gilbert's construction, the original semigroup S is embedded into $S[\phi : U \to V]$ provided U is an ordere ideal.

First of all, we shall show that the embeddability holds even though we do not assume that A and B are order ideals. Next, we discuss the relationship between the constructions by [2] and [8]. We shall show that $S[\phi : A \to B]$ is embedded into $S_{1_A,1_B}(\phi : A \to B)$, where $S_{1_A,1_B}$ is the inverse semigroup obtained from S by adjoining new idempotents 1_A and 1_B, in such a way that $A \cup \{1_A\}$ and $B \cup \{1_B\}$ are inverse monoids of $S_{1_A,1_B}$. As a matter of fact, we shall obtain the following commutative diagram

$$\begin{tikzcd}
S \arrow[r, "\mu"] \arrow[d, "\nu"'] & S[\phi : A \to B] \arrow[d, "\eta"] \\
S_{1_A,1_B} \arrow[r, "\xi"'] & S_{1_A,1_B}(\phi : A \cup \{1_A\} \to B \cup \{1_B\}),
\end{tikzcd}$$

where each homomorphism is an embeddings. In fact, the mapping η is defined by

$$\eta(x) = \begin{cases} x & \text{if } x \in S, \\ et\phi(e) & \text{if } x = t_e. \end{cases} \tag{2.2}$$

Because η is an embedding, $S[\phi : A \to B]$ can be considered as a subsemigroup of $S_{1_A,1_B}(\phi : A \cup \{1_A\} \to B \cup \{1_B\})$.

Theorem 2.1 *S is naturally embedded into $S[\phi : A \to B]$.*

Theorem 2.2 (Main theorem) *The homomorphism η given by (2.2) is an embedding of $S[\phi : A \to B]$ into $S_{1_A,1_B}(\phi : A \to B)$.*

It seems hard to prove this theorem by a linear method and so we employ automaton theoretic methods using geometry of Schützenberger graphs.

3 Schützenberger graphs and automata

3.1 Schützenberger graphs

Let S be an inverse semigroup and X the set of generators of S. The Schützenberger graph $\mathrm{S}\Gamma(S, \mathrm{X}, u)$ for the word u is given by

$$\begin{aligned}\mathrm{Vert}(\mathrm{S}\Gamma(S,\mathrm{X},u)) &= \{s \mid s \in S,\ s\mathcal{R}u\},\\ \mathrm{Edge}(\mathrm{S}\Gamma(S,\mathrm{X},u)) &= \{(s_1,x,s_2) \mid s_1x = s_2,\ s_1\ \mathcal{R}\ s_2\ \mathcal{R}\ u,\ s_1,s_2 \in S,\ x \in \mathrm{X}\cup\mathrm{X}^{-1}\}.\end{aligned}$$

The initial and terminal vertex are given by

$$\alpha(s_1,x,s_2) = s_1, \quad \omega(s_1,x,s_2) = s_2.$$

Even in the case that $x = x^{-1}$ in S, we distinguish the edges labeled by x and x^{-1}. The Schützenberger graph $\mathrm{S}\Gamma(S,\mathrm{X},u)$ for the word u is denoted by $\mathrm{S}\Gamma(u)$ if the presentation is understood.

Suppose that $\mathcal{A}$ is an automaton with input alphabet $X \cup X^{-1}$. We say that $\mathcal{A}$ is an *inverse word automaton* if the tansition is consistent with the involution $x \to x^{-1}$, that is, if $\overset{q_1}{\circ}\xrightarrow{x}\overset{q_2}{\circ}$ is an edge in $\mathcal{A}$, then so is $\overset{q_1}{\circ}\xleftarrow{x^{-1}}\overset{q_2}{\circ}$. We can regard $\mathrm{S}\Gamma(u)$ as an inverse word automaton. The initial and terminal state of the automaton $\mathrm{S}\Gamma(u)$ are ss^{-1} and s, respectively. We call $\mathrm{S}\Gamma(u)$ the Schützenberger automaton.

We define a morphism of inverse word automata. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are inverse word automata with input alphabets $X_1 \cup X_1^{-1}$ and $X_2 \cup X_2^{-1}$, respectively. We identify automata and their transition graphs, respectively. Suppose that τ is a mapping of X_1 into X_2. A graph morphism ξ of $\mathcal{A}$ and $\mathcal{B}$ is called a *morphism of inverse word automata with* τ if ξ maps an edge $\overset{q_1}{\circ}\xrightarrow{x}\overset{q_2}{\circ}$ in $\mathcal{A}$ to $\overset{\xi(q_1)}{\circ}\xrightarrow{\tau(x)}\overset{\xi(q_2)}{\circ}$ in $\mathcal{B}$ and ξ maps the initial and final states of $\mathcal{A}$ to those of $\mathcal{B}$. If τ is a mapping of X_1 into X_2 and w is a word $x_1x_2\cdots x_n$ on $X_1 \cup X_1^{-1}$, then we denote the word $\tau(x_1)\tau(x_2)\cdots\tau(x_n)$ simply by $\tau(w)$.

Lemma 3.1 ([7]) *For each word u the language accepted by the Schützenberger automaton $S\Gamma(u)$ is given by*

$$L(\mathrm{S}\Gamma(u)) = u\uparrow = \{w \mid u \le w \text{ in } S\}.$$

The words u and w reprent the same element in S if and only if $u\uparrow = w\uparrow$ if and only if the Schützenberger automata of u and w accept the same languages.

3.2 Automata production

In [7] the construction of Schützenberger graph from Munn tree [6] is given. There are two operations for the automata production process; *expansions* and *reductions*. Suppose that we are given an inverese semigroup presenation $\mathrm{Inv}(X \mid R)$, where R is a set of defining relations. We consider inverse word automata whose input alphabets are $X \cup X^{-1}$.

Expansions. Given an automaton A, an expansion B of A is obtained as follows. Suppose that there is a path form the state q_1 to q_2 labeled by the word r_1, where $r_1 = r_2$ is a defining relation belonging to R, and there is no path from q_1 to q_2 labeled by r_2. Now B is obtained from A by adding a new path from q_1 to q_2 labeled by r_2. The automaton B is called an expansion of A.

Reductions. Given an automaton A, an expansion B of A is obtained as follows. Suppose that in A there are two edges $q \to q_1$ and $q \to q_2$ labeled by the same letter x. The new automaton is obtained from A by identifying these edges. Note that the states q_1 and q_2 are identified in B. The automaton B is called a reduction of A.

Schützenberger automaton We review the construction of automata $\mathcal{A}_i(u)$ $(i = 1, 2, 3, \ldots)$ and $\mathcal{B}_i(u)$ $(i = 0, 1, 2, \ldots)$ in general. The automaton $\mathcal{B}_0(u)$ is the linear automaton of the word u, that is, $\mathcal{B}_0(u)$ is the linear automaton reading the word u. Suppose u is the word $s_1 s_2 \cdots s_k$. Then the linear automaton $\mathcal{B}_0(u)$ is represented by

$$\downarrow\circ \xrightarrow{s_1} \circ \xrightarrow{s_2} \circ \xrightarrow{s_3} \circ \xrightarrow{s_4} \cdots \circ \xrightarrow{s_k} \circ\uparrow . \tag{3.1}$$

The automaton $\mathcal{A}_1(u)$ is obtained from $\mathcal{B}_0(u)$ applying finite sequence of reductions until no more reductions can be applied. We note that $\mathcal{A}_1(u)$ is graph isomorphic to the Munn tree for u.

Suppose now that $\mathcal{A}_k(u)$ has been constructed. We construct an automaton $\mathcal{B}_k(u)$ by applying all possible expansions to $\mathcal{A}_k(u)$. Clearly only finite number of expansions can be applicable. Then $\mathcal{A}_{k+1}(u)$ is obtained from $\mathcal{B}_k(u)$ by applying reductions to $\mathcal{B}_k(u)$ until no more reductions can be applied. It is shown in [7] that the Schützenberger automaton $S\Gamma(u)$ for u is given by the join $\bigvee_{k=1}^{\infty} \mathcal{A}_k(u)$ and $L(\mathcal{A}_k(u)) \subset L(\mathcal{A}_{k+1}(u))$ for every $k = 0, 1, 2, \ldots$ and

$$\bigcup_{k=1}^{\infty} L(\mathcal{A}_k(u)) = L(S\Gamma(u)) = u\uparrow = \{w \mid u \leq w \text{ in } S\}. \tag{3.2}$$

4 Synchronous production of automata

To prove Theorem 2.2, we synchronously construct automata $\mathcal{C}_i(\eta(u), S_{1_A,1_B}(\phi : A \to B)), \mathcal{N}_i(u, S[\phi : A \to B])$ and $\mathcal{N}_i(\eta(u), S_{1_A,1_B}(\phi : A \to B))$ so that these satisfy some desired properties. We start from the linear automata and use expansions and reductions like the construction in the previous section.

Let us suppose that $\eta(u) = \eta(w)$, where u and w are words on the generators of $S[\phi : A \to B]$. Then we shall show that $u = w$. Recall that the set of generators of $S[\phi : A \to B]$ consists of that of S and the set of stable letters $\{t_e \mid e \in E(A)\}$. We introduce the following convension. Let w be a word on the set of generators of $S[\phi : A \to B]$. We denote the word obtained from w by substituting $et_e\phi(e)$ for every stable letter t_e by

$$w(t_e \to et_e\phi(e)). \tag{4.1}$$

Likewise, we denote the word obtained from w by substtituting $et_d\phi(e)$ for some stable letters by

$$w[t_e \to et_d\phi(e)], \tag{4.2}$$

where d may be any idempotent $d \in E(A)$. The replacement in $w[t_e \to et_d\phi(e)]$ is not necessarily compatible for each occurence of the stable letter in w. Therefore, $w(t_e \to et_e\phi(e))$ is uniquely determined, but there are many candidates for $w[t_e \to et_d\phi(e)]$.

The automaton $\mathcal{N}_0(u, S[\phi : A \to B])$ is obtained from the automaton $\mathcal{B}_0(u(t_e \to et_e\phi(e)), S_{*A,\phi})$ by making every edge labeled by an idempotent letter a loop. The automaton $\mathcal{N}_0(u, S[\phi : A \to B])$ is uniquely determined (and well-defined). By the construction, for any edge y labeled by t_e, there exist loops y_1 and y_2 labeled by e and $\phi(e)$ rooted at $\alpha(y)$ and $\omega(y)$, respectively. The automaton $\mathcal{N}_0(\eta(u), S_{1_A,1_B}(\phi : A \to B))$ is obtained from $\mathcal{B}_0(\eta(u), S_{1_A,1_B}(\phi : A \to B))$ by making every edge labeled by an idempotent letter a loop. The automaton $\mathcal{N}_0(\eta(u), S_{1_A,1_B}(\phi : A \to B))$ is uniquely determined (and well-defined). The automata $\mathcal{C}_i(\eta(u), S_{1_A,1_B}(\phi : A \to B))$, $\mathcal{N}_i(u, S[\phi : A \to B])$ and $\mathcal{N}_i(\eta(u), S_{1_A,1_B}(\phi : A \to B))$ and the morphisms ρ_i and λ_i satisfy the following.

1. $L(\mathcal{N}_i(u, S[\phi : A \to B])) \subset L(S\Gamma(u, S[\phi : A \to B]))$ for every $i = 0, 1, 2, \ldots, n$.
2. There exists a morphism ρ_i of $\mathcal{C}_i(\eta(u), S_{1_A,1_B}(\phi : A \to B))$ into $\mathcal{N}_i(\eta(u), S_{1_A,1_B}(\phi : A \to B))$ associating with the identity mapping.
3. There exists a locally surjective morphism λ_i of $\mathcal{N}_i(u, S[\phi : A \to B])$ into $\mathcal{N}_i(\eta(u), S_{1_A,1_B}(\phi : A \to B))$ associated with τ, where τ is a mapping of $S \cup \{t_e \mid e \in E(A)\}$ into $S \cup \{t\}$ defined by $\tau(s) = s$ for $s \in S$ and $\tau(t_e) = t$ for $t_e \in E(A)$.
4. The morphisms λ_i are bijective on the sets of vertices.
5. In $\mathcal{N}_i(u, S[\phi : A \to B])$, for any edge y labeled by t_e, there exist loops y_1 and y_2 labeled by e and $\phi(e)$ rooted at $\alpha(y)$ and $\omega(y)$, respectively.
6. The morphisms λ_i and ρ_i make the following diagram commutative.

$$\begin{tikzcd}
\mathcal{N}_0(u, S[\phi : A \to B]) \arrow[r, "\xi_0"] \arrow[d] & \mathcal{N}_1(u, S[\phi : A \to B]) \arrow[r, "\xi_1"] \arrow[d, "\lambda_n"] & \cdots \\
\mathcal{N}_0(\eta(u), S_{1_A,1_B}(\phi : A \to B)) \arrow[r, "\nu_0"] & \mathcal{N}_1(\eta(u), S_{1_A,1_B}(\phi : A \to B)) \arrow[r, "\nu_1"] & \cdots \\
\mathcal{C}_0(\eta(u), S_{1_A,1_B}(\phi : A \to B)) \arrow[r, "\mu_0"] \arrow[u] & \mathcal{C}_1(\eta(u), S_{1_A,1_B}(\phi : A \to B)) \arrow[r, "\mu_1"] \arrow[u, "\rho_n"'] & \cdots
\end{tikzcd}$$

The construction of the automata are intricate and so we do not give the detail of the construction here. The detail will be given in the full version of the paper.

5 Proof of Main Theorem

Recall that we are assuming $\eta(u) = \eta(w)$ in $S_{1_A,1_B}(\phi : A \to B)$ and $\eta(w)$ is accepted by $\mathcal{C}_n(\eta(u), S_{1_A,1_B}(\phi : A \to B))$. It follows that $\eta(w)$ is accpeted by $\mathcal{N}_n(\eta(u), S_{1_A,1_B}(\phi : A \to B))$ as well.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ be automata and λ a morphism of $\mathcal{A}$ into $\mathcal{B}$. A path $y_1, y_2, \ldots, y_n$ in $\mathcal{A}$ is called a *lifting* of a path $z_1, z_2, \ldots, z_n$ in $\mathcal{B}$ by λ if $\lambda(y_i) = z_i$ for every $i = 1, 2, \ldots, n$. In such a case, we also say that the path $z_1, z_2, \ldots, z_n$ is lifted up to $y_1, y_2, \ldots, y_n$ by λ. The proof of the main theorem is obtained in the succession of the lemmas. The proof of these lemmas are omitted.

Lemma 5.1 *If a graph morphism λ of $\mathcal{A}$ into $\mathcal{B}$ is locally surjective and sujective on the set of vertices, every path in $\mathcal{B}$ can be lifted up to $\mathcal{A}$ by λ.*

Lemma 5.2 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are automata with the input alphabet A and B. Let λ be a morphism of $\mathcal{A}$ into $\mathcal{B}$ associated with $\tau : A \to B$. Suppose that λ is locally surjective and bijective on the set of vertices. If a word w is accepted by $\mathcal{B}$, then there exists a word w_1 in $\tau^{-1}(w)$ that is accepted by $\mathcal{A}$.*

Lemma 5.3 *If $\eta(w)$ is accepted by $\mathcal{N}_n(\eta(u), S_{1_A,1_B}(\phi : A \to B))$, then there exists w_2 in $\tau^{-1}(\eta(w))$ so that w_2 is accepted by $\mathcal{N}_n(u, S[\phi : A \to B])$.*

Lemma 5.4 *If w_2 belongs to $\tau^{-1}(\eta(w))$, then w_2 is expressed as $w[t_e \Rightarrow et_d\phi(e)]$.*

Lemma 5.5 *For every $i = 1, 2, 3, \ldots, n$, we have $L(\mathcal{N}_i(u, S_{*A,\phi})) \subset L(S\Gamma(u, S_{*A,\phi}))$.*

Lemma 5.6 *If $w[t_e \Rightarrow et_d\phi(e)]$ is accepted by $S\Gamma(u, S_{*A,\phi})$, then so is $w(t_e \Rightarrow et_e\phi(e))$.*

Proof of Theorem 2.2. Recall that we are assuming $\eta(u) = \eta(w)$ in $S_{1_A,1_B}(\phi : A \to B)$. Then $\eta(w)$ is accepted by $\mathrm{S}\Gamma(\eta(u), S_{1_A,1_B}(\phi : A \to B))$ by Lemma 3.1. We note that the morphism λ_n of $\mathcal{N}_n(u, S_{*A,\phi})$ into $\mathcal{N}_n(\eta(u), S_{1_A,1_B}(\phi : A \to B))$. By Lemmas 5.3, 5.4, 5.5, a certain word $w[t_e \Rightarrow et_d\phi(e)]$ is accepted by $\mathrm{S}\Gamma(u, S_{*A,\phi})$. By Lemma 5.6, the word $w(t_e \Rightarrow et_e\phi(e))$ is also accepted by $\mathrm{S}\Gamma(u, S_{*A,\phi})$. On the other hand, $w(t_e \Rightarrow et_e\phi(e))$ represents the element represented by w in $S[\phi : A \to B]$. This implies w is accepted by $\mathrm{S}\Gamma(u, S_{*A,\phi})$ as well.

Similarly, it is shown that u is accepted by $\mathrm{S}\Gamma(w, S_{*A,\phi})$. Therefore, we have $L(\mathrm{S}\Gamma(u, S_{*A,\phi})) = L(\mathrm{S}\Gamma(w, S_{*A,\phi}))$ and so u and w represent the same element in $S_{*A,\phi}$. Consequently, η is an embedding. □

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