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ON THE SEMISIMPLICITY OF BRAUER ALGEBRAS

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ABSTRACT. In this note, we give a necessary and sufficient condition for a Brauer algebra to be semisimple. This is the main result in [9].

The Brauer algebra $\mathscr{B}_n(\delta)$ was introduced by R. Brauer in 1937 in order to study the tensor representations of orthogonal groups and symplectic groups. In order to recall its definition, we need the notion of Brauer diagrams.

A Brauer diagram is a graph with $2n$ vertices and n arcs, arranged in two lines of n vertices each. Each arc belongs to exactly two vertices. The composite $D_1 \circ D_2$ of two Brauer diagrams D_1 and D_2 can be obtained as follows.

(1) Putting D_1 over D_2 and identifying the i-th lower vertex of D_1 to the i-th upper vertex of D_2, we get a diagram P,
(2) Removing all the closed circles which appear in P, we get the Brauer diagram $D_1 \circ D_2$.

Let $n(D_1, D_2)$ be the number of closed cycles removed above.

Definition 1. (Brauer) [1] The Brauer algebra $\mathscr{B}_n(\delta)$ is an associative algebra over the complex field $\mathbb{C}$ with a linear basis which consists of all Brauer diagrams. The multiplication is defined by setting

$$D_1 \cdot D_2 = \delta^{n(D_1,D_2)} D_1 \circ D_2.$$

Let R be a commutative ring containing δ and the identity 1. If we use R instead of $\mathbb{C}$, we will get the Brauer algebra over R. There is another definition for $\mathscr{B}_n(\delta)$ as follows. See, e.g. [8].

Definition 2. Let R be a commutative ring containing δ and the identity 1. The Brauer algebra $\tilde{\mathscr{B}}_n(\delta)$ is the unital associative R-algebra generated by $s_1, \dots, s_{n-1}$, $e_1, \dots, e_{n-1}$ which are subject to the relations:

$$\begin{array}{lll} s_i^2 = 1, & s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, & s_i s_j = s_j s_i \\ e_i^2 = \delta e_i & e_i e_{i+1} e_i = e_i, & e_i e_j = e_j e_i \\ e_i e_i = e_i = e_i e_i & e_{i+i} e_i e_{i+1} = e_{i+1} & e_i e_j = e_j e_i \\ e_i e_{i+1} e_i = e_{i+1} e_i & e_{i+1} e_i e_{i+1} = e_{i+1} e_i & \text{if } j \neq i, i+1. \end{array}$$

Using seminormal representations over $\mathbb{C}(\delta)$ in [7], one can prove

$$\mathscr{B}_n(\delta) \cong \tilde{\mathscr{B}}_n(\delta). \tag{2.1}$$

In other words, Definition 1 is equivalent to Definition 2.

Hanlon and Wales conjectured that the complex Brauer algebra $\mathscr{B}_n(\delta)$ is semisimple if $\delta \notin \mathbb{Z}$. This conjecture was proved by Wenzl [13].

Theorem 3. *(Wenzl)* [13] *The complex Brauer algebra $\mathscr{B}_n(\delta)$ is not semisimple for only finite number of integers δ.*

Another proof was later given by Doran–Wales–Hanlon in [3]. In fact, by their results together with some arguments on cellular algebras, we can give an algorithm to determine all pairs (n, δ) such that $\mathscr{B}_n(\delta)$ is semisimple. We need the notion of cellular algebras in [5].

Definition 4. (Graham-Lehrer) [5] Let R be a commutative ring with identity 1. An associative R–algebra A is called a cellular algebra over the poset $(\Lambda, \leq)$ if it has an R–basis

$$\{c_{i,j}^{\lambda} \mid i, j \in I(\lambda), \lambda \in \Lambda\}$$

which satisfies the following conditions

(1) The R-linear map $\sigma : A \to A$ with $\sigma(c_{i,j}^{\lambda}) = c_{j,i}^{\lambda}$ is an anti–involution of A.

(2) For each $a \in A$, $c_{i,j}^{\lambda} a \equiv \sum_{k \in I(\lambda)} r_a(j,k) c_{i,k}^{\lambda} \mod A^{>\lambda}$, where $A^{>\lambda}$ is the free R-module generated by $c_{l,m}^{\mu}$ with $\mu > \lambda$ and $l, m \in I(\mu)$. Furthermore, the coefficients $r_a(j,k) \in R$ do not depend on i.

For a cellular algebra, there are a class of modules $\Delta(\lambda)$, $\lambda \in \Lambda$, called cell modules. On each cell module $\Delta(\lambda)$, there is an associative, symmetric bilinear form ϕ_λ. Let G_λ be the Gram matrix for $\Delta(\lambda)$ with respect to the bilinear form ϕ_λ.

Theorem 5. *(Graham-Lehrer)*[5] *Let A be a cellular algebra over the field R.*

(1) *The quotient module $\Delta(\lambda)/Rad\ \phi_\lambda$ is either (absolutely) irreducible or zero, where $Rad\ \phi_\lambda = \{v \in \Delta(\lambda) \mid \phi_\lambda(v,w) = 0, \forall w \in \Delta(\lambda)\}$.*

(2) *The set $\{\Delta(\lambda)/Rad\ \phi_\lambda \mid \phi_\lambda \neq 0\}$ consists of all non-isomorphic irreducible A–modules.*

(3) *A is semisimple if and only if $\det G_\lambda \neq 0$ for all $\lambda \in \Lambda$.*

A partition λ of n is a weakly decreasing sequence $(\lambda_1, \lambda_2, \cdots, \lambda_r)$ of non-negative integers with $\sum_{i=1}^{r} \lambda_i = n$. In this case, write $\lambda \vdash n$. Let $\Lambda = \{(k, \lambda) \mid 0 \leq k \leq \lfloor n/2 \rfloor, \lambda \vdash n - 2k\}$. Write $(k, \lambda) \leq (l, \mu)$ if either $k < l$ or $k = l$ and $\lambda \trianglelefteq \mu$, where $\trianglelefteq$ is the dominance order defined on the set of partitions of $n - 2k$. Then $(\Lambda, \leq)$ is a poset. The following result is due to Graham–Lehrer.

Theorem 6. *(Graham-Lehrer)*[5] *Let R be a commutative ring containing δ and the identity* 1. *Then $\mathscr{B}_n(\delta)$ is a cellular algebra over the poset $(\Lambda, \leq)$*

Remark 7. *Fishel and Grojnowski constructed a basis for $\mathscr{B}_n(\delta)$ in* [4] *which is in fact a cellular basis.*

The Young diagram $Y(\lambda)$ with respect to the partition λ of n consists of n boxes placed at the matrix entries $\{(i,j) \mid 1 \leq j \leq \lambda_i\}$. If the coordinate of the box p is (i,j), define the content of p by $c(p) = j - i$. We say a partition μ is contained in the partition λ and write $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i. Define $Y(\lambda/\mu)$ to be the sub-diagram of $Y(\lambda)$, which consists of the boxes in $Y(\lambda)\backslash Y(\mu)$.

Motivated by [10] and [11], we think that the cell modules $\Delta(1,\lambda)$ of $\mathscr{B}_n(\delta)$ play the key role. This leads author to prove the following result.

Theorem 8. *(Rui)* [9] *For any positive integer n, let*

$$\mathbb{Z}(n) = \{r \in \mathbb{Z} \mid r = 1 - \sum_{p \in Y(\lambda/\mu)} c(p), \mu \vdash k-2, \lambda \vdash k, 2 \leq k \leq n\}$$

where two boxes of $Y(\lambda/\mu)$ are not in the same column.

(a) *Suppose $\delta \neq 0$. The complex Brauer algebra $\mathscr{B}_n(\delta)$ is semisimple if and only if* $\det G_{1,\lambda} \neq 0, \forall \lambda \vdash k-2, 2 \leq k \leq n$.

(b) *Suppose $\delta \neq 0$. Then $\mathscr{B}_n(\delta)$ is semisimple if and only if $\delta \notin \mathbb{Z}(n)$.*

(c) *$\mathscr{B}_n(0)$ is semisimple if and only if $n \in \{1,3,5\}$.*

By considering the pairs of partitions $\lambda = (2, 1^{k-2})$ and $\mu = (1^{k-2})$, we have $k-2 \in \mathbb{Z}(n)$ for all $2 \leq k \leq n$. Notice that $n-2$ is the maximal integer in $\mathbb{Z}(n)$, we have that $i \in \mathbb{Z}(n)$ and $i > 0$ if and only if $1 \leq i \leq n-2$. This result was proved by Brown in [2].

Theorem 9. *(Brown)* [2] *Suppose δ is a positive integer. Then complex Brauer algebra $\mathscr{B}_n(\delta)$ is semisimple if and only if $\delta \geq n-1$.*

It is not difficult to see that the minimal integer in $\mathbb{Z}(n)$ is $-2n+4$. In this case, $\lambda = (1^n)$ and $\mu = (1^{n-2})$. However, we can not say $\mathscr{B}_n(\delta)$ is not semisimple if $-2n+4 \leq \delta < 0$. For example, $\mathscr{B}_3(\delta)$ is semisimple if and only if $\delta \notin \{-2, 1\}$.

In order to prove Theorem 8, we need several results due to Doran–Wales–Hanlon in [3].

Theorem 10. *(Doran-Wales-Hanlon)*[3] *There is a non-trivial $\mathscr{B}_n(\delta)$-homomorphism $\Delta(0,\lambda') \hookrightarrow \Delta(1,\mu')$ if and only if $\delta = 1 - \sum_{p \in Y(\lambda/\mu)} c(p)$ where two boxes in $Y(\lambda/\mu)$ are not in the same column.*

Remark 11. *The cell module $\Delta(k,\lambda)$ is different from that considered in* [3]. *Doran, Wales and Hanlon use classical Specht modules for symmetric group $\mathfrak{S}_n$. We use cell* *modules for $\mathfrak{S}_n$. Therefore, we have to use λ', the dual partition of λ instead of their partition λ.*

Using certain arguments on cellular algebras together with Theorem 10, we know all the roots of the determinants $\det G_{1,\mu'}$ for $\mu \vdash n-2$. More explicitly, we have

Corollary 12. *Suppose $\mu \vdash n-2$. Then $\det G_{1,\mu'} \neq 0$ if and only if $\delta \neq 1-\sum_{p\in Y(\lambda/\mu)} c(p)$ for all $\lambda \vdash n$ with $\lambda \supset \mu$ such that two boxes in $Y(\lambda/\mu)$ are not in the same column.*

The following result sets up a relation between two Brauer algebras with different ranks.

Theorem 13. *(Doran-Wales-Hanlon)* [3] *Suppose $\delta \neq 0$. Then there is a non-trivial homomorphism $\Delta(k,\lambda) \longrightarrow \Delta(k+l,\mu)$ if and only if there is a monomorphism $\Delta(0,\lambda) \hookrightarrow \Delta(l,\mu)$.*

Doran, Wales and Hanlon also gave the result on $\delta = 0$. We do not state it here since we will not use it to prove Theorem 8.

Suppose $\lambda \vdash n$. The node $p = (i,j)$ of $Y(\lambda)$ is called a removable node if $Y(\lambda)\backslash\{p\}$ corresponds to another partition $\mu \vdash n-1$. In this case, we write either $\mu \rightarrow \lambda$ or $\lambda \leftarrow \mu$.

For any $\mathscr{B}_n(\delta)$-modules M and N, let

$$\langle M, N \rangle_{\mathscr{B}_n(\delta)} = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathscr{B}_n(\delta)}(M, N).$$

It is known that $\mathscr{B}_{n-1}(\delta)$ is a subalgebra of $\mathscr{B}_n(\delta)$. For any $\mathscr{B}_n(\delta)$-module M, let $M\downarrow$ be the restriction of M to the $\mathscr{B}_{n-1}(\delta)$-module.

Theorem 14. *(Doran-Wales-Hanlon)*[3] *Let $\lambda \vdash n$ and $\mu \vdash n-2k$. If $\langle \Delta(0,\lambda), \Delta(k,\mu) \rangle_{\mathscr{B}_n(\delta)} \neq 0$, then for every $\lambda^0 \rightarrow \lambda$, $\langle \Delta(0,\lambda^0), \Delta(k,\mu)\downarrow \rangle_{\mathscr{B}_{n-1}(\delta)} \neq 0$. Furthermore, either $\langle \Delta(0,\lambda^0), \Delta(k-1,\mu^0) \rangle_{\mathscr{B}_{n-1}(\delta)} \neq 0$ for some $\mu^0 \leftarrow \mu$ or $\langle \Delta(0,\lambda^0), \Delta(k,\mu^1) \rangle_{\mathscr{B}_{n-1}(\delta)} \neq 0$ for $\mu^1 \rightarrow \mu$.*

The following result follows from Theorems 13–14 immediately.

Corollary 15. *Suppose $\delta \neq 0$. If $\langle \Delta(k,\lambda), \Delta(l,\mu) \rangle_{\mathscr{B}_n(\delta)} \neq 0$, then there are a pair of partitions $\tilde{\lambda} \vdash m$ and $\tilde{\mu} \vdash m-2$ with $m \leq n$ such that $\Delta(0,\tilde{\lambda}) \hookrightarrow \Delta(1,\tilde{\mu})$.*

In general, we do not know the explicit relations between λ and $\tilde{\lambda}$ etc. Now, we give the sketch proof of Theorem 8.

Proof of Theorem 8 It is not difficult to see that (b) follows immediately from (a) and Corollary 12. Suppose $\delta = 0$. If n is even, then $G_{n/2,(0)} = 0$, and consequently, by Theorem 5(c), $\mathscr{B}_n(0)$ is not semisimple. Suppose $n \in \{1,3,5\}$. By computation, we see that the determinants for all Gram matrices are not equal to zero. Consequently, $\mathscr{B}_n(0)$

is semisimple. Suppose $n \geq 7$ and n is odd. It follows from Theorem 10 that there is a non-trivial homomorphism from $\Delta(0,(421)')$ to $\Delta(1,(32)')$. By Theorem 13,

$$\langle \Delta((n-7)/2,(421)'), \Delta((n-5)/2,(32)') \rangle_{\mathscr{B}_n(0)} \neq 0. \tag{15.1}$$

If $\mathscr{B}_n(0)$ is semisimple, then different cell modules must be non-isomorphic irreducible, contradicting to (15.1).

Finally, we prove (a). Suppose $\det G_{1,\lambda} \neq 0$ for all $\lambda \vdash k$, $2 \leq k \leq n$. If $\mathscr{B}_n(\delta)$ is not semisimple, then we can find a cell module, say $\Delta(k,\mu)$ such that $\det G_{k,\mu} = 0$. Take an irreducible submodule $M \subset \operatorname{Rad} \Delta(k,\mu)$. By Theorem 5, M must be isomorphic to the simple head of a cell module, say $\Delta(l,\lambda)$ with $(l,\lambda) < (k,\mu)$. Furthermore, we have a non-trivial homomorphism from $\Delta(l,\lambda)$ to $\Delta(k,\mu)$. By Corollary 15, there are $\tilde{\lambda}$ and $\tilde{\mu}$ such that $\Delta(0,\tilde{\lambda}) \hookrightarrow \Delta(1,\tilde{\mu})$ for some $\lambda \vdash m$, $2 \leq m \leq n$. Consequently, $\det G_{1,\tilde{\mu}} = 0$, a contradiction.

Suppose that $\mathscr{B}_n(\delta)$ is semisimple. If the result were false, then $\det G_{1,\mu'} = 0$ for some $\mu' \vdash k-2$ and $k < n$. By Theorem 10, We have $\Delta(0,\lambda') \hookrightarrow \Delta(1,\mu')$.

Case 1: $n-k=2f$ By Theorem 13, $\langle \Delta(f,\lambda'), \Delta(1+f,\mu') \rangle_{\mathscr{B}_n(\delta)} \neq 0$, a contradiction since different cell modules of a semisimple cellular algebra must be non-isomorphic irreducible.

Case 2: $n-k=2f+1$ Since $\Delta(0,\lambda') \hookrightarrow \Delta(1,\mu')$, $\delta = 1 - \sum_{p \in Y(\lambda/\mu)} c(p)$. Furthermore, two boxes in $Y(\lambda/\mu)$ can not be in the same column. Suppose $\lambda \neq (2,1)$ and $\mu \neq (1)$. Then we can find a pair of partitions $(\tilde{\lambda},\tilde{\mu})$ such that $\delta = 1 - \sum_{p \in Y(\tilde{\lambda}/\tilde{\mu})} c(p)$. Furthermore, $n - |\tilde{\mu}|$ is even. This is a contradiction by **Case 1:**.

If $\lambda = (21)$ and $\mu = (1)$, then n is even and $\delta = 1 - 1 + 1 = 1$. By computation, $\det G_{2,(0)} = 0$. This implies that there is a non-trivial $\mathscr{B}_4(1)$–homomorphism from $\Delta(k,\lambda')$ to $\Delta(2,(0))$. Consequently, by Theorem 13,

$$\langle \Delta(n/2+k-2,\lambda'), \Delta(n/2,0) \rangle_{\mathscr{B}_n(1)} \neq 0.$$

This implies $\mathscr{B}_n(1)$ is not semisimple, a contradiction. □

Remark 16. *When F is a filed with positive characteristic p, we can add $p \nmid n!$ in the theorem to get a criterion on the semisimplicity of $\mathscr{B}_n(\delta)$ over F. The reason is that, if $\mathscr{B}_n(\delta)$ is semisimple, then all $\Delta(0,\lambda)$ are non-isomorphic irreducible, which implies $F\mathfrak{S}_n$ is semisimple. In this case, Doran–Wales–Hanlon's results we need are still valid since their results depend on ordinary representations of symmetric groups. Our argument is about cellular algebra. This will give the proof.*

The cyclotomic Brauer algebra $\mathscr{B}_{m,n}$ was introduced by Haering–Oldenburg in [6]. In [12], we gave a result on the semisimplicity of a cyclotomic Brauer algebra, which can be considered as a generalization of Theorem 3. It would be interesting to generalize Theorem 8 to find a necessary and sufficient condition for $\mathscr{B}_{m,n}$.

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REFERENCES

[1] R. Brauer, *On algebras which are connected with the semisimple continuous groups,* Ann. of Math. **38** (1937), 854–872.
[2] W. Brown, *The semisimplicity of ω_f^n*, Ann. of Math. **63** (1956), 324–335.
[3] W. Doran, D. Wales and P. Hanlon, *On the semisimplicity of the Brauer centralizer algebras*, J. Algebra **211** (1999), 647–685.
[4] S. Fishel and I. Grojnowski, *Canonical bases for the Brauer centralizer algebra*, Math. Res. Letters, **2** (1995), 15–26.
[5] J. Graham and G. Lehrer, *Cellular algebras*, Invent. Math. **123** (1996),1–34.
[6] R. Häring-Oldenburg, *Cyclotomic Birman-Murakami-Wenzl algebras*, J. Pure Appl. Algebra **161** (2001), no. 1–2, 113–144.
[7] R. Leduc and A. Ram, *A ribbon Hopf algebra approach to the irreducible representations of centralizer algebras: The Brauer, Birman-Wenzl and type A Iwahori-Hecke algebras*, Adv. in Math. **125** (1997),1–94.
[8] M. Nazarov, *Young's orthogonal form for Brauer's centralizer algebra*, J. Algebra, **182** (1996), 664–693.
[9] H. Rui, *A criterion on the semisimplicity of Brauer algebras*, J. Comb. Theory, Series A, to appear.
[10] H. Rui and C. Xi, *The representation theory of cyclotomic Temperley-Lieb algebras*, Comment. Math. Helv. **79** (2004) 427–450.
[11] H. Rui, C. Xi and W. Yu, *On the semi-simplicity of the cyclotomic Temperley-Lieb algebras*, Michigan Math. Journal, to appear.
[12] H. Rui and W. Yu, *On the semi-simplicity of cyclotomic Brauer algebras*, J. Algebra **277**, (2004) 187–221.
[13] H. Wenzl, *On the structure of Brauer's centralizer algebras*, Ann. of Math. **128** (1988), 173–193.

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