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INTEGRABLE MODULES OVER $\hat{\mathfrak{gl}}_m$ AND THE DOUBLE AFFINE HECKE ALGEBRA

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Introduction

Motivated by conformal field theory on the Riemann sphere, we introduce a certain space of coinvariants obtained from tensor product of representations of the affine Lie algebra $\hat{\mathfrak{gl}}_m$.

In [AST], an action of the degenerate affine Hecke algebra $H_\kappa$ is defined on this space through the Knizhnik-Zamolodchikov connection. This construction gives a functor from the category of highest (or lowest) weight modules over $\hat{\mathfrak{gl}}_m$ to the category of $H_\kappa$-modules.

We will see that the integrable $\hat{\mathfrak{gl}}_m$-modules correspond by this functor to irreducible $H_\kappa$-modules whose structure is described combinatorially. We also focus on the symmetric part of these irreducible $H_\kappa$-modules; i.e., the subspace consisting of those elements which are invariant with respect to the action of the Weyl group. We present a spectral decomposition of the symmetric part, and a character formula, which is described by level restricted analogue of the Kostka polynomial.

1. AFFINE LIE ALGEBRA

Throughout this note, we use the notation $[i, j] = \{i, i+1, \ldots, j\}$ for $i, j \in \mathbb{Z}$.

Let $m \in \mathbb{Z}_{\geq 2}$. Let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{gl}_m$ consisting of all $n \times n$-matrices over $\mathbb{C}$. Let $\mathfrak{g}[t, t^{-1}]$ denote the Lie algebra consisting of all $n \times n$-matrices over $\mathbb{C}[t, t^{-1}]$. Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c_g$ be the affine Lie algebra with the commutation relation

$$[a \otimes t^i, b \otimes t^j] = [a, b] \otimes t^{i+j} + \text{trace}(ab)i\delta_{i+j,0}c_g$$

for $a, b \in \mathfrak{g}$, $i, j \in \mathbb{Z}$.

Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{g}$ consisting of all diagonal matrices, and let $\mathfrak{h}^*$ denote its dual space. A Cartan subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{g}}$ is given by $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c_g$. Its dual space is denoted by $\hat{\mathfrak{h}}^*$. We regard $\hat{\mathfrak{h}}^*$ as a subspace of $\hat{\mathfrak{h}}^*$ through the identification $\hat{\mathfrak{h}}^* \cong \mathfrak{h}^* \oplus \mathbb{C}c_g^*$. 
Fix $\ell \in \mathbb{C}$. For $\lambda \in \mathfrak{h}^*$, $\overline{M}_\ell(\lambda)$ denote the highest weight Verma module of highest weight $\lambda + \ell c_{\mathfrak{g}}^* \in \hat{\mathfrak{h}}^*$, and let $\overline{M}_\ell^+(\lambda)$ denote the lowest weight Verma module of lowest weight $-\lambda - \ell c_{\mathfrak{g}}^* \in \hat{\mathfrak{h}}^*$. Their irreducible quotients are denoted by $\hat{L}_\ell(\lambda)$ and $\hat{L}_\ell^+(\lambda)$ respectively.

A $\mathfrak{g}$-module $M$ is said to be of level $\ell$ if $c$ acts as a scalar $\ell$. For example, $\overline{M}_\ell(\lambda)$ and $\hat{L}_\ell(\lambda)$ are of level $\ell$, and $\overline{M}_\ell^+(\lambda)$ and $\hat{L}_\ell^+(\lambda)$ are of level $-\ell$.

We identify $\mathfrak{h}$ with $\mathbb{C}^m$, and introduce its subspaces $X_m = \mathbb{Z}^m$ and

$$X_m^+ = \{(\lambda_1, \ldots, \lambda_m) \in X_m \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m\},$$

$$X_m^+(\ell) = \{(\lambda_1, \ldots, \lambda_m) \in X_m^+ \mid \lambda_1 - \lambda_m \leq \ell\}.$$

Note that $\hat{L}_\ell(\lambda)$ and $\hat{L}_\ell^+(\lambda)$ are integrable for $\lambda \in X_m^+(\ell)$, and that $X_m^+(\ell)$ is empty unless $\ell \in \mathbb{Z}_{\geq 0}$.

Let $E = \mathbb{C}^m$ denote the vector representation of $\mathfrak{g}$. Put $E[z, z^{-1}] = E \otimes \mathbb{C}[z, z^{-1}]$, which we regard as a $\mathfrak{g}[t, t^{-1}]$-module through the correspondence $a \otimes t^k \mapsto a \otimes z^k$.

2. THE DEGENERATE DOUBLE AFFINE HECKE ALGEBRA

Let $n \in \mathbb{Z}_{\geq 2}$. Let $V$ denote the $n$-dimensional vector space over $\mathbb{C}$ with the basis $\{y_i\}_{i \in [1,n]}$: $V = \oplus_{i \in [1,n]} \mathbb{C}y_i$. Introduce the non-degenerate symmetric bilinear form $(\cdot | \cdot)$ on $V$ by $(y_i | y_j) = \delta_{ij}$. Let $V^* = \oplus_{i=1}^n \mathbb{C}x_i$ be the dual space of $V$, where $x_i$ is the dual vector of $y_i$. The natural pairing is denoted by $(\cdot | \cdot): V^* \times V \rightarrow \mathbb{C}$.

Put $\alpha_{ij} = x_i - x_j$, $\alpha_{ij}^\vee = y_i - y_j$ and $\alpha_i = \alpha_{ii+1}$, $\alpha_i^\vee = \alpha_{ii+1}$. Then $R = \{\alpha_{ij} \mid i, j \in [1, n], i \neq j\}$ and $R^+ = \{\alpha_{ij} \in R \mid i < j\}$ give a set of roots and a set of positive roots of type $A_{n-1}$ respectively.

Let $W$ denote the Weyl group associated with the root system $R$, which is isomorphic to the symmetric group $S_n$ of degree $n$. Denote by $s_\alpha$ the reflection in $W$ corresponding to $\alpha \in R$. We write $s_i = s_{\alpha_i}$ and $s_{ij} = s_{\alpha_{ij}}$.

Put $P = \oplus_{i \in [1,n]} \mathbb{Z}x_i$, which is preserved by $W$. Define the extended affine Weyl group $\overline{W}$ as the semidirect product $P \rtimes W$ with the relation $w \tau_\eta w^{-1} = \tau_{w(\eta)}$, where $\tau_{\eta}$ denotes the element of $\overline{W}$ corresponding to $\eta \in P$.

Let $S(V)$ denote the symmetric algebra of $V$, which can be identified with the polynomial ring $\mathbb{C}[y] = \mathbb{C}[y_1, \ldots, y_n]$.

Fix $\kappa \in \mathbb{C}$. The degenerate double affine Hecke algebra (degenerate DAHA) $H_\kappa$ of $GL_n$ is an associative $\mathbb{C}$-algebra generated by the algebra $\mathbb{C}P$, $CW$ and $S(V)$, and subjects to the following defining relations...
([C1]):

\[ s_i h = s_i(h)s_i - \langle \alpha_i | h \rangle \quad (i \in [1, n], \ h \in V), \]

\[ s_i e^\eta s_i = e^{s_i(\eta)} \quad (i \in [1, n], \ \eta \in P), \]

\[ [h, e^\eta] = \kappa \langle \eta | h \rangle e^\eta + \sum_{\alpha \in R^+} \langle \alpha | h \rangle \frac{(e^\eta - e^{s_\alpha(\eta)})}{1 - e^{-\alpha}} s_\alpha \quad (h \in V, \ \eta \in P), \]

where \( e^\eta \) denote the element of \( \mathbb{C}P \) corresponding to \( \eta \in P \).

It is known that \( H_\kappa \cong \mathbb{C}P \otimes \mathbb{C}W \otimes S(V) \) as a vector space. The subalgebra \( H^{\mathrm{aff}} = \mathbb{C}W \cdot S(V) \) is called the degenerate affine Hecke algebra. Note that the subalgebra \( \mathbb{C}P \cdot \mathbb{C}W \) is isomorphic to \( \mathbb{C}\overline{W} \).

3. INDUCED REPRESENTATIONS OF \( H_\kappa \)

For \( \lambda \in X_m = \mathbb{Z}^{m} \) we write \( \lambda \models n \) when \( \sum_{i \in [1, m]} \lambda_i = n \) and \( \lambda_i \in \mathbb{Z}_{\geq 0} \) for all \( i \in [1, m] \). Let \( \lambda, \mu \in X_m \) such that \( \lambda - \mu \models n \). Introduce the subalgebra \( H_\lambda = \mathbb{C}W_{\lambda - \mu} \cdot S(V) \) of \( H_\kappa \), where \( W_{\lambda - \mu} \) denote the parabolic subgroup \( \mathfrak{S}_{\lambda_1 - \mu_1} \times \cdots \times \mathfrak{S}_{\lambda_m - \mu_m} \) of \( W \).

Let \( \mathbb{C}1_{\lambda, \mu} \) denote the one dimensional representation of \( H_{\lambda - \mu} \) such that

\[ w1_{\lambda, \mu} = 1_{\lambda, \mu} \quad (w \in W_{\lambda - \mu}), \]

\[ y_i 1_{\lambda, \mu} = \langle \zeta_{\lambda, \mu} | y_i \rangle 1_{\lambda, \mu} \quad (i \in [1, n]), \]

where \( \zeta_{\lambda, \mu} \) denote the element of \( V^* \) given by

\[ \langle \zeta_{\lambda, \mu} | y_i \rangle = \mu_j + i - m_j - j - 1 \quad \text{for} \quad i \in [m_j + 1, m_{j+1}], \]

with \( m_0 = 0 \) and \( m_j = \sum_{k \in [1, j]} (\lambda_k - \mu_k) \quad (j \in [1, m]) \).

Define an \( H_\kappa \)-module by \( \mathcal{M}_\kappa(\lambda, \mu) = H_\kappa \otimes_{H_{\lambda - \mu}} \mathbb{C}1_{\lambda, \mu} \). Obviously we have

\[ \mathcal{M}_\kappa(\lambda, \mu) \cong \mathbb{C}\overline{W}/W_{\lambda - \mu} \cong \mathbb{C}P \otimes \mathbb{C}W/W_{\lambda - \mu} \]

as an \( \overline{W} \)-module.

In the rest, we often identify the group ring \( \mathbb{C}P \) with the Laurent polynomial ring \( \mathbb{C}[z^{\pm 1}] = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \) via the correspondence \( e^{x_i} \mapsto z_i \).

**Example 3.1.** Let \( m = 1 \) and let \( \lambda = (n) \) and \( \mu = (0) \). Then \( \mathcal{M}_\kappa(\lambda, \mu) \cong \mathbb{C}P = \mathbb{C}[z_1^{\pm 1}], \) which is called the (Laurent) polynomial representation. On the representation \( \mathbb{C}P \), the element \( y_i \) \( (i \in [1, n]) \) acts as the Cherednik-Dunkl operator

\[ T_i = \kappa z_i \frac{\partial}{\partial z_i} + \sum_{1 \leq j < i} \frac{z_j}{z_i - z_j} (1 - s_\alpha) + \sum_{i < j \leq n} \frac{z_i}{z_i - z_j} (1 - s_\alpha) + i - 1. \]
The simultaneous eigenvectors of $T_1, \ldots, T_n$ are called the nonsymmetric Jack polynomials.

4. THE SPACE OF COINVARIANTS AND THE DEGENERATE DOUBLE AFFINE HECKE ALGEBRA

Let $\ell \in \mathbb{C}$. Let $M$ be a highest weight module of level $\ell$ and let $N$ be a lowest weight module of level $-\ell$. We set

$$C(M, N) = \tilde{C}(M, N)/\mathfrak{g}[t, t^{-1}][\tilde{C}(M, N)],$$

Let $\sigma_{ij} \in \text{End}_{\mathbb{C}}[\underline{z}^\pm 1]$ denote the permutation of $z_i$ and $z_j$. Let $\Omega_{ij} \in \text{End}_{\mathbb{C}}(\mathbb{C}^{E^\otimes n})$ denote the permutation of $i$-th and $j$-th component of the tensor product. Note that $\tilde{C}(M, N) \cong M \otimes E^\otimes n \otimes \mathbb{C}[\underline{z}^\pm 1] \otimes N$, through which we regard $\sigma_{ij}$ and $\Omega_{ij}$ as elements in $\text{End}_{\mathbb{C}}(\tilde{C}(M, N))$.

For $i, j \in [0, n+1]$ with $i < j$, define $\theta_{ij} : \mathfrak{g}^\otimes 2 \rightarrow U(\mathfrak{g})^\otimes n+2$ by $\theta_{ij}(u \otimes v) = 1^\otimes i \otimes u \otimes 1^\otimes j-i-1 \otimes v \otimes 1^\otimes n-j+1$.

Let $e_{ab}$ denote the matrix unit of $\mathfrak{g}$ with only non-zero entries 1 at the $(a, b)$-th component. Put $r = \frac{1}{2} \sum_{a \in [1,m]} e_{aa} + \sum_{1 \leq a < b \leq m} e_{ab} \otimes e_{ba}$ and put $r_{ij} = \theta_{ij}(r)$.

For $i \in [1, n]$, put

$$r_{0i} = r_{0i} + \sum_{k \geq 1} \sum_{a, b \in [1, m], a \neq b} \theta_{0i}((e_{ab} \otimes t^k) \otimes (e_{ba} \otimes t^{-k})), \quad (4.1)$$

$$r_{in+1} = r_{in+1} + \sum_{k \geq 1} \sum_{a, b \in [1, m], a \neq b} \theta_{in+1}((e_{ab} \otimes t^k) \otimes (e_{ba} \otimes t^{-k})), \quad (4.2)$$

which are elements of some completion of $U(\mathfrak{g}[t, t^{-1}])^\otimes n+2$ and define well-defined operators on $\tilde{C}(M, N)$.

Define the linear operators on $\tilde{C}(M, N)$ by

$$D_i = \kappa z_i \frac{\partial}{\partial z_i} + \hat{r}_{0i} - \hat{r}_{in+1} + \sum_{1 \leq j < i} r_{ij} - \sum_{i < j \leq n} r_{ji} + \theta_i(\rho^\vee)$$

$$+ \sum_{1 \leq j < i} \frac{z_j}{z_i - z_j} (1 - \sigma_{ij}) \Omega_{ij} + \sum_{i < j \leq n} \frac{z_i}{z_i - z_j} (1 - \sigma_{ij}) \Omega_{ij} + i - 1,$$

where $\rho^\vee = \sum_{k \in [1, m]} \frac{1}{2} (n - 2k + 1) e_{aa} \in \mathfrak{h}$.

**Theorem 4.1** (Theorem 4.2.2 in [AST]). Let $M$ be a highest weight module of $\hat{\mathfrak{g}}$ of level $\kappa - m$ and let $N$ be a lowest weight module of level $-\kappa + m$. 
(i) There exists a unique algebra homomorphism $\varpi : H^\text{rat}_\kappa \to \text{End}_\mathbb{C}(\overline{C}(M, N))$ such that

\begin{align*}
\varpi(s_i) &= \Omega_{i,i+1}\sigma_{i,i+1} \quad (i \in [1, n - 1]), \\
\varpi(e^\pi) &= z_i \quad (i \in [1, n]), \\
\varpi(y_i) &= D_i \quad (i \in [1, n]).
\end{align*}

(ii) The $H_\kappa$-action on $C(M, N)$ above preserves the subspace $g[t, t^{-1}]\overline{C}(M, N)$:

$$
\varpi(a)g[t, t^{-1}]\overline{C}(M, N) \subseteq g[t, t^{-1}]\overline{C}(M, N)
$$

for all $a \in H_\kappa$. Therefore, $\varpi$ induces an $H_\kappa$-module structure on $C(M, N)$.

5. Images of the Functor

The following statement has been shown in [AST].

**Proposition 5.1** (Proposition 5.3.1 in [AST]). Let $\kappa \in \mathbb{C}$ and put $\ell = \kappa - m$.

(i) Let $\lambda, \mu \in X_m^+$. Then

$$
C(\overline{M}_\ell(\mu), \overline{M}_\ell^+(\lambda)) \cong \begin{cases} 
\mathcal{M}_\kappa(\lambda, \mu) & \text{if } \lambda - \mu \vdash n, \\
0 & \text{otherwise}.
\end{cases}
$$

(ii) Let $\lambda, \mu \in X_m^+(\ell)$ such that $\lambda - \mu \vdash n$. Then

$$
C(\hat{L}_\ell(\mu), \hat{L}_\ell^+(\lambda)) \cong C(\overline{M}_\ell(\mu), \overline{M}_\ell^+(\lambda)) \cong C(\hat{L}_\ell(\mu), \hat{L}_\ell^+(\lambda)).
$$

For each $\lambda \in X_m$, we have the additive functor $F_\lambda(-) = C(-, \overline{M}_\ell^+(\lambda))$ from the category of highest weight modules over $\hat{g}$ to the category of $H_\kappa$-modules. It is right exact and sends the Verma module $\overline{M}_\ell^+(\mu)$ to the induced module $\mathcal{M}_\kappa(\lambda, \mu)$ by Proposition 5.1. In the sequel, we will determine the image $F_\lambda(\hat{L}_\ell(\mu))$ of the irreducible module $\hat{L}_\ell(\mu)$ in the case where $\lambda, \mu \in X_m^+(\ell)$. Note that $F_\lambda(\hat{L}_\ell(\mu)) \cong C(\overline{M}_\ell(\mu), \overline{M}_\ell^+(\lambda))$, and note also that it is a quotient of $F_\lambda(\mathcal{M}_\kappa(\lambda, \mu))$.

Let $\ell \in \mathbb{Z}_{\geq 0}$ and $\lambda, \mu \in X_m^+(\ell)$ such that $\lambda - \mu \vdash n$. Then it is known that the $H_\kappa$-module $\mathcal{M}_\kappa(\lambda, \mu)$ has a unique simple quotient ([AST, S1]), which we will denote by $\mathcal{L}_\kappa(\lambda, \mu)$.

The irreducible modules $\mathcal{L}_\kappa(\lambda, \mu)$ for $\lambda, \mu \in X_m^+(\ell)$ are investigated in [SV], and in particular their structure is described combinatorially using tableaux on periodic skew diagrams. We give a short review of the theory of periodic tableaux and the tableaux representations of $H_\kappa$ in Appendix. By means of this combinatorial description, we can estimate the kernel of the projection $\mathcal{M}_\kappa(\lambda, \mu) \to \mathcal{L}_\kappa(\lambda, \mu)$. By comparing it with the kernel of $\mathcal{M}_\kappa(\lambda, \mu) \to F_\lambda(\hat{L}_\ell(\mu))$, we have...
Theorem 5.2. Let \( \kappa \in \mathbb{Z}_{\geq 1} \) and put \( \ell = \kappa - m \). Let \( \lambda, \mu \in X_{m}^{+}(\ell) \) such that \( \lambda - \mu \models n \). Then the \( H_{\kappa} \)-module \( C(\widehat{L}_{\ell}(\mu), \widehat{L}_{\ell}^{\dagger}(\lambda)) \) is irreducible:

\[
C(\widehat{L}_{\ell}(\mu), \widehat{L}_{\ell}^{\dagger}(\lambda)) \cong \mathcal{L}_{\kappa}(\lambda, \mu),
\]

and moreover it is semisimple over \( S(V) \). (See Theorem A.3 for the combinatorial description of the weight decomposition).

The classification of the irreducible \( H_{\kappa} \)-modules which are semisimple over \( S(V) \) is given in [C2, SV], from which (or from Theorem A.4) we have

**Corollary 5.3.** Let \( \kappa \in \mathbb{Z}_{\geq 1} \). Let \( L \) be an irreducible \( H_{\kappa} \)-module which is finitely generated and admits a weight decomposition of the form \( L = \bigoplus_{\zeta \in P} L_{\zeta} \), where \( L_{\zeta} = \{ v \in L \mid yv = \langle \zeta \mid y \rangle \forall y \in V \} \). Then there exists \( m \in [1, n] \) and \( \lambda, \mu \in X_{m}^{+}(\kappa - m) \) such that \( L \cong C(\widehat{L}_{\kappa-m}(\mu), \widehat{L}_{\kappa-m}^{\dagger}(\lambda)) \).

6. Localization and Conformal Coinvariants

We will see the relation between our space \( C(M, N) \) of coinvariants and the space of conformal coinvariants in Wess-Zumino-Witten model [TK, TUY].

Observe that the group ring \( \mathbb{C} \mathbb{P} \) can be seen as the coordinate ring \( A = \mathbb{C}[\mathcal{T}] \) of the affine variety \( \mathcal{T} = (\mathbb{C} \setminus \{0\})^{n} \). Put \( T_{o} = \mathcal{T} \setminus \triangle \), where \( \triangle = \bigcup_{i<j}\{(\xi_{1}, \ldots , \xi_{n}) \in \mathcal{T} \mid \xi_{i}/\xi_{j} = 1\} \), and put \( A_{o} = \mathbb{C}[T_{o}] \). Namely, \( A_{o} \) is the localization of \( A \) at \( \triangle \); \( A_{o} = \mathbb{C}\left[z_{1}^{\pm 1}, \ldots , z_{n}^{\pm 1}, \frac{1}{1-z_{i}/z_{j}} \right] \).

Let \( D(T_{o}) \) denote the ring of algebraic differential operators on \( T_{o} \). Then the Cherednik-Dunkl operators in Example 3.1 \( T_{1}, \ldots , T_{n} \) can be seen as elements of the ring \( D(T_{o}) \times \mathbb{C}W \). Put \( H_{\kappa, o} = A_{o} \otimes_{A} H_{\kappa} \). There exists a unique algebra structure on \( H_{\kappa, o} \) extending \( H_{\kappa} \).

**Proposition 6.1.** Let \( \kappa \in \mathbb{C}^{\times} \). There exists a unique algebra isomorphism \( H_{\kappa, o} \to D(T_{o}) \times \mathbb{C}W \) such that \( y_{i} \mapsto T_{i}, \ w \mapsto w, \ f \mapsto f \) for all \( i \in [1, n], \ w \in W \) and \( f \in A_{o} \).

For an \( H_{\kappa} \)-module \( M \), set \( M_{o} = A_{o} \otimes_{A} M \). Then via Proposition 6.1, we have a structure of \( D(T_{o}) \times \mathbb{C}W \)-module on \( M_{o} \); namely, \( M_{o} \) admits a \( W \)-equivariant integrable (algebraic) connection

\[
\nabla_{i} = \kappa^{-1} \left\{ y_{i} - \sum_{1 \leq j < i} \frac{z_{j}}{z_{i}-z_{j}} \left( 1-s_{a} \right) - \sum_{i < j \leq n} \frac{z_{i}}{z_{i}-z_{j}} \left( 1-s_{a} \right) - (i-1) \right\}.
\]

Now consider the case where \( M = C(\widehat{L}_{\ell}(\mu), \widehat{L}_{\ell}^{\dagger}(\lambda)) \) with \( \lambda, \mu \in X_{m}^{+}(\ell) \). Then it follows that the connection given above has regular singularities along \( \triangle \), and hence \( L_{\kappa}(\lambda, \mu)_{o} \) is a projective \( A_{o} \)-module, or geometrically, a vector bundle over \( T_{o} \) of finite rank ([GGOR, VV]).
For $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{I}_\circ$, let $C_{\xi}$ denote the one-dimensional right module of $A_\circ$ given by the evaluation at $\xi$. It follows that the space $C_{\xi} \otimes_{A_\circ} (L_\kappa(\lambda, \mu)_\circ)$ is isomorphic to with "the space of conformal coinvariants"

$$
\left( \hat{L}_\ell(\mu) \otimes \hat{L}_\ell(\nu_1) \otimes \hat{L}_\ell(\lambda^\dagger) \right) / g_{(0, \xi, \infty)} \left( \hat{L}_\ell(\mu) \otimes \hat{L}_\ell(\nu_1) \otimes \hat{L}_\ell(\lambda^\dagger) \right),
$$

where $\nu_1 = (1, 0, \ldots, 0) \in X_m^+(\ell)$ (the highest weight of the vector representation $E$), $\lambda^\dagger = -w_0(\lambda)$ with $w_0$ being the longest element of $W$, and $g_{(0, \xi, \infty)}$ denotes the Lie algebra of $g$-valued algebraic functions on $\mathbb{P}^1 \setminus \{0, \xi_1, \ldots, \xi_n, \infty\}$, which acts on $\hat{L}_\ell(\mu) \otimes \hat{L}_\ell(\nu_1) \otimes \hat{L}_\ell(\lambda^\dagger)$ through the Laurent expansion at each points. (See e.g. [BK] for a precise definition.)

Therefore it follows that the vector bundle $L_\kappa(\lambda, \mu)_\circ$ is equivalent to the vector bundle of conformal coinvariants (the dual of the vector bundle of conformal blocks in the sense of [TUY, BK]). Moreover, the connection $\{\nabla_i\}$ on $L_\kappa(\lambda, \mu)_\circ$ given via Proposition 6.1 coincides with the Knizhnik-Zamolodchikov connection on the vector bundle of conformal coinvariants.

7. Weight Decomposition of Symmetric Part

For an $H_\kappa$-module $M$, put

$$(7.1) \quad M^W = \{v \in M \mid wv = v \ \forall w \in W\},$$

on which the algebra $H_\kappa^W = \{u \in H_\kappa \mid wuw^{-1} = u\}$ acts. The algebra $H_\kappa^W$ is called the zonal spherical algebra and it contains a subalgebra $S(V)^W$, which coincides with the center of the degenerate affine Hecke algebra $H^{aff}$.

For $\zeta \in V^*$, let $\chi_{\zeta}$ denote the image of the projection to the quotient space $W \setminus V^*$. Identify $W \setminus V^*$ with the set $\text{Hom}_{\text{algebra}}(S(V)^W, \mathbb{C})$ of characters, and set

$$M_{[\zeta]}^W = \{v \in M^W \mid \xi v = \chi_{\zeta}(\xi)v \ \forall \xi \in S(V)^W\}.$$

In the sequel, we will give a decomposition of $L_\kappa(\lambda, \mu)^W$ into weight spaces with respect to $S(V)^W$.

Let $\lambda, \mu \in X_m^+$ such that $\lambda - \mu \models n$. Let $\lambda/\mu$ denote the skew Young diagram associated with $(\lambda, \mu)$:

$$(7.2) \quad \lambda/\mu = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \in [1, m], \ b \in [\mu_a + 1, \lambda_a]\}.$$

Let $T$ be a tableau on the diagram $\lambda/\mu$; namely $T$ is a bijection from $\lambda/\mu$ to $[1, n]$. Then it determines the sequence $\{\lambda^{(i)}_T\}_{i \in [0, n]}$ in $X_m$ by the condition $\lambda^{(0)}_T = \mu$ and $\lambda^{(i)}_T/\lambda^{(i-1)}_T = T^{-1}(i)$ ($i \in [1, n]$).
Let $\ell \in \mathbb{Z}_{\geq 0}$. A tableau $T$ is called an $\ell$-restricted standard tableau if $\lambda_{T}^{(i)} \in X_{m}^{+}(\ell)$ for all $i \in [1, n]$. Let $\text{St}_{(\ell)}(\lambda, \mu)$ denote the set of $\ell$-restricted tableaux on $\lambda$.

Let $T \in \text{St}_{(\ell)}(\lambda, \mu)$. For $i \in [1, n]$, define

$$h_{i}(T) = \begin{cases} 1 & \text{if } a < a' \\ 0 & \text{if } a \geq a' \end{cases}$$

where $T(a, b) = i$ and $T(a', b') = i + 1$. Define

$$\eta_{T} = \sum_{i \in [1, n]} \left( \sum_{j < i} h_{j}(T) \right) x_{i} \in P.$$ 

Define $\zeta_{T} \in V^{*}$ by

$$\zeta_{T}(y_{i}) = b - a$$

when $T(a, b) = i$.

From the weight decomposition of $L_{\kappa}(\lambda, \mu)$ (Theorem A.3) with respect to $S(V)$, we have

**Theorem 7.1.** (Conjecture 6.1.1 in [AST]) Let $\lambda, \mu \in X_{m}^{+}((\kappa-m)$ such that $\lambda - \mu \models n$. Then

$$L_{\kappa}(\lambda, \mu)^{W} = \bigoplus_{\nu \in P^{-}} \bigoplus_{T \in \text{St}_{(\ell)}(\lambda/\mu)} L_{\kappa}(\lambda, \mu)^{W}_{[\zeta_{T} + \kappa(\nu + \eta_{T})]}$$

where $P^{-} = \{ \zeta \in P \mid \langle \zeta \mid \alpha_{i}^{\vee} \rangle \leq 0 \forall i \in [1, n - 1] \}$, and

$$\dim L_{\kappa}(\lambda, \mu)^{W}_{[\zeta_{T} + \kappa(\nu + \eta_{T})]} = 1$$

for all $\nu \in P^{-}$ and $T \in \text{St}_{(\ell)}(\lambda/\mu)$.

**8. q-Dimension Formula**

Put $\partial = \kappa^{-1} \sum_{i \in [1, n]} y_{i} \in S(V)^{W}$. Then $\partial$ satisfies the relation

$$[\partial, z_{i}] = \kappa z_{i}, \quad [\partial, w] = 0$$

for all $i \in [1, n]$ and $w \in W$.

Our next purpose is to give a $q$-dimension formula for $L_{\kappa}(\lambda, \mu)^{W}$ with respect to the grading operator $\partial$. To this end, we need to introduce the "polynomial part" of $L_{\kappa}(\lambda, \mu)$ following [AST].

Define a subalgebra $H_{\kappa}^{\geq 0}$ of $H_{\kappa}$ by

$$H_{\kappa}^{\geq 0} = \mathbb{C}P^{\geq 0} \cdot \mathbb{C}W \cdot S(V),$$

where $P^{\geq 0} = \bigoplus_{i \in [1, n]} \mathbb{Z}_{\geq 0} x_{i}$.

Let $\kappa \in \mathbb{Z}_{\geq 1}$ and let $\lambda, \mu \in X_{m}^{+}(\kappa - m)$ such that $\lambda - \mu \models n$. Recall that the induced module $M_{\kappa}(\lambda, \mu)$ is generated by the cyclic vector $1_{\lambda, \mu}$. We denote by $1_{\lambda, \mu}$ its image under the projection $M_{\kappa}(\lambda, \mu) \rightarrow L_{\kappa}(\lambda, \mu)$. Note that $1_{\lambda, \mu} \neq 0$. Define the polynomial part of $L_{\kappa}(\lambda, \mu)$ by $L_{\kappa}^{\geq 0}(\lambda, \mu) = H_{\kappa}^{\geq 0} 1_{\lambda, \mu}$, which is an $H_{\kappa}^{\geq 0}$-submodule of $L_{\kappa}(\lambda, \mu)$.
Put $\mathcal{L}^\geq_0(\lambda, \mu)^{(k)}_W = \{ v \in \mathcal{L}^\geq_0(\lambda, \mu)^W | \partial v = kv \}$. Then we have $\dim \mathcal{L}^\geq_0(\lambda, \mu)^{(k)}_W < \infty$ and $\mathcal{L}^\geq_0(\lambda, \mu)^W = \oplus_{k \in \mathbb{Z}} \mathcal{L}^\geq_0(\lambda, \mu)^{(k)}_W$. Define
\[
\dim_q \mathcal{L}^\geq_0(\lambda, \mu)^W = \sum_{d \in \mathbb{Z}} q^d \dim \mathcal{L}^\geq_0(\lambda, \mu)^{(d)}_W.
\]
Set
\[(8.1) \quad h(T) = \kappa \langle \eta_T | \partial \rangle = \sum_{i \in [1, n]} (n-i) h_i(T).
\]
From Theorem 7.1, we have

**Theorem 8.1.** Let $\kappa \in \mathbb{Z}_{\geq 0}$ and let $\lambda, \mu \in X_m^+(\kappa-m)$ such that $\lambda - \mu \models n$. Then
\[(8.2) \quad \dim_q \mathcal{L}^\geq_0(\lambda, \mu)^W = q^{\Delta_\lambda - \Delta_\mu} F_{\lambda/\mu}^{(\ell)}(q).
\]
Here $\Delta_\lambda = \frac{1}{2\kappa}((\lambda, \lambda) + 2(\rho, \lambda))$, $(q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n)$ and $F_{\lambda/\mu}^{(\ell)}(q)$ is a polynomial of $q$ given by
\[(8.3) \quad F_{\lambda/\mu}^{(\ell)}(q) = \sum_{T \in \text{St}_\ell(\lambda/\mu)} q^{h(T)}.
\]

**Remark 8.2.** If $\ell$ is large enough then $F_{\lambda/\mu}^{(\ell)}(q)$ coincides with the Kostka polynomial $K_{(\lambda/\mu)'(1^n)}(q)$ associated to the conjugate $(\lambda/\mu)'$ of $\lambda/\mu$. Hence our polynomial $F_{\lambda/\mu}^{(\ell)}(q)$ is an $\ell$-restricted version of the Kostka polynomial (cf. [FJKLM]).

**Remark 8.3.** A bosonic formula for $F_{\lambda/\mu}^{(\ell)}(q)$ is known (Theorem 6.2.4 in [AST]), and Theorem 8.1 is equivalent to the formula in Conjecture 6.1.1 in [AST]. Note also that the bosonic formula suggests the existence of the BGG type resolution of $\mathcal{L}^\kappa(\lambda, \mu)$.

**9. Rational analogue**

For a $\mathfrak{g}[t]$-module $N$, set
\[
\tilde{C}(M) = E[z_1] \otimes \cdots \otimes E[z_n] \otimes N
\]
\[
C(N) = \tilde{C}(N)/\mathfrak{g}[t] \tilde{C}(N),
\]
where $E[z] = E \otimes \mathbb{C}[z]$. The analogous construction gives on $C(N)$ an action of the rational Cherednik algebra $H^\text{rat}_\kappa$ ([EG]), which can be
defined as the subalgebra of $H_{\kappa}$ generated by the subalgebra $C[z] \cdot CW$ and the following (pairwise commutative) elements

$$u_i = z_i^{-1} \left( y_i - \sum_{j<i} s_{ij} \right) \quad (i \in [1,n])$$

as pointed out in [S2].

It follows for $\lambda \in X_{m}^+(\ell)$ that $C(\tilde{M}_\ell^+(\lambda))$ is isomorphic to some induced module, and $C(\tilde{L}_\ell^+(\lambda))$ is isomorphic to the unique simple quotient of $C(\tilde{M}_\ell^+(\lambda))$, which we denote by $\mathcal{L}_\kappa(\lambda)$.

Let $\mathbf{0} = (0, \ldots, 0) \in X_{m}^+(\ell)$. Then it follows that the polynomial part $L_{\kappa}^{\geq 0}(\lambda, \mathbf{0})$ of the $H_{\kappa}$-module $L_{\kappa}(\lambda, \mathbf{0})$ is an $H_{\kappa}^\text{rat}$-submodule and it is isomorphic to $L_{\kappa}(\lambda)$. This leads the $q$-dimension formula

$$\dim_q L_{\kappa}(\lambda)^W = \frac{q^{\Delta_\lambda}}{(q)_n} F_\lambda^{(\ell)}(q).$$

Remark 9.1. It can be seen that the Knizhnik-Zamolodchikov functor investigated in [GGOR] transforms the irreducible representations $L_{\kappa}(\lambda)$ for $A \in X_{m}^+(\ell)$ to Wenzl's representations [W] of the affine Hecke algebra (cf. [TK]).

APPENDIX A. TABLEAUX ON PERIODIC DIAGRAMS AND REPRESENTATIONS OF THE DEGENERATE DAHA

We will review the theory of tableaux representations for $H_{\kappa}$, which is investigated in [SV] for the double affine Hecke algebra.

Fix $\kappa \in \mathbb{Z}_{\geq 1}$. Let $m \in \mathbb{Z}_{\geq 1}$.

For $\lambda, \mu \in X_{m}^+(\kappa - m)$ such that $\lambda - \mu \models n$, we introduce the following subsets of $\mathbb{Z} \times \mathbb{Z}$:

(A.1) $\lambda/\mu = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \in [1, m], \ b \in [\mu_a + 1, \lambda_a]\},$

(A.2) $\overline{\lambda/\mu} = \{(a, b) + k(m, -\kappa + m) \in \mathbb{Z} \times \mathbb{Z} \mid (a, b) \in \lambda/\mu, \ k \in \mathbb{Z}\}.$

The set $\overline{\lambda/\mu}$ is called the periodic skew diagram of period $(m, -\kappa + m)$ associated with $(\lambda, \mu)$. The following is called the skew property:

Lemma A.1. Let $(a, b), (a', b') \in \overline{\lambda/\mu}$. If $a' - a \in \mathbb{Z}_{\geq 0}$ and $b' - b \in \mathbb{Z}_{\geq 0}$ then $(a, b'), (a', b) \in \overline{\lambda/\mu}$.

A tableau $T$ on $\overline{\lambda/\mu}$ is by definition a bijection $\overline{\lambda/\mu} \to \mathbb{Z}$ satisfying $T(a + m, b - \kappa + m) = T(a, b) + n$ for all $(a, b) \in \overline{\lambda/\mu}$.

A tableau $T$ is called a standard tableau if

$$T(a, b + 1) > T(a, b)$$
for any \((a, b), (a, b + 1) \in \lambda / \mu\), and if
\[ T(a + 1, b) > T(a, b) \]
for any \((a, b), (a + 1, b) \in \lambda / \mu\). Let \( \text{Tab}(\lambda / \mu) \) and \( \text{St}(\lambda / \mu) \) denote the set of tableaux and the set of standard tableaux on \( \lambda / \mu \) respectively.

Define the elements \( \pi = \tau_{x_{1}}s_{1}s_{2}\cdots s_{n-1} \) and \( s_{0} = \tau_{\alpha_{1n}}s_{1n} \) of the group \( \hat{W} = P \rtimes W \). Then \( \{s_{0}, s_{1}, \ldots, s_{n-1}, \pi\} \) is a generator of the group \( \hat{W} \).

Define the action of \( \hat{W} \) on the set \( \mathbb{Z} \) of integers by
\[
(A.3) \quad s_{i}(j) = \begin{cases} 
  j + 1 & \text{for } j \equiv i \mod n, \\
  j - 1 & \text{for } j \equiv i + 1 \mod n, \\
  j & \text{for } j \neq i, i + 1 \mod n,
\end{cases}
\]
\[
(A.4) \quad \tau_{x_{1}}(j) = \begin{cases} 
  j + n & \text{for } j \equiv i \mod n, \\
  j & \text{for } j \not\equiv i \mod n.
\end{cases}
\]

Observe that \( \pi(j) = j + 1 \) for all \( j \).

For \( T \in \text{Tab}(\lambda / \mu) \) and \( w \in \hat{W} \), the map \( wT : \lambda / \mu \to \mathbb{Z} \) given by
\[
(wT)(u) = w(T(u)) \quad (u \in \lambda / \mu)
\]
is also a tableau on \( \lambda / \mu \), and the assignment \( T \mapsto wT \) gives an action of \( \hat{W} \) on \( \text{Tab}(\lambda / \mu) \), which preserves \( \text{St}(\lambda / \mu) \). It is easy to see that the assignment \( w \mapsto wT \) gives a one-to-one correspondence \( \hat{W} \sim \text{Tab}(\lambda / \mu) \).

Define the map \( C : \lambda / \mu \to \mathbb{Z} \) by \( C(a, b) = b - a \), and define \( C_T : \mathbb{Z} \to \mathbb{Z} \) by \( C_T(i) = C(T^{-1}(i)) \) for \( T \in \text{St}(\lambda / \mu) \). Define \( \zeta_T \in V^* \) by \( \langle \zeta_T | y_{i} \rangle = C_T(i) (i \in [1, n]) \).

The following lemma follows from the skew property and the definition of the standard tableaux:

**Lemma A.2.** Let \( T \in \text{St}(\lambda / \mu) \) and \( i \in [0, n - 1] \).

(i) \( C_T(i) - C_T(i + 1) \neq 0 \).

(ii) \( s_{i}T \in \text{St}(\lambda / \mu) \) if and only if \( C_T(i) - C_T(i + 1) \notin \{-1, 1\} \).

Now, we introduce the tableaux representation associated with \( \lambda / \mu \).

Let \( V_\kappa(\lambda / \mu) \) be the vector space with the basis \( \{v_T\}_{T \in \text{St}(\lambda / \mu)} \):
\[
V_\kappa(\lambda / \mu) = \bigoplus_{T \in \text{St}(\lambda / \mu)} \mathbb{C}v_T.
\]

By Lemma A.2 and induction argument, we have
Theorem A.3. (Theorem 3.16, Theorem 3.17 in [SV]) Let $\kappa \in \mathbb{Z}_{\geq 1}$. Let $\lambda, \mu \in X_{m}^{\dagger}(\kappa - m)$ such that $\lambda - \mu \models n$.

(i) There exists a unique $H_{\kappa}$-module structure on $\mathcal{V}_{\kappa}(\lambda/\mu)$ such that

$$y_{i}v_{T} = C_{T}(i)v_{T} \quad (i \in [1, n]),$$

$$\pi v_{T} = v_{\pi T},$$

$$s_{i}v_{T} = \begin{cases} \frac{1+a_{i}}{a_{i}} v_{s_{i}T} - \frac{1}{a_{i}} v_{T} & \text{if } s_{i}T \in \text{St}(\lambda/\mu) \\ -\frac{1}{a_{i}} v_{T} & \text{if } s_{i}T \notin \text{St}(\lambda/\mu) \end{cases} \quad (i \in [0, n-1]),$$

where $a_{i} = C_{T}(i) - C_{T}(i+1) \neq 0$ (by Lemma A.2).

(ii) $\mathcal{V}_{\kappa}(\lambda, \mu) = \oplus_{T \in \text{St}(\lambda/\mu)} \mathcal{V}_{\kappa}(\lambda, \mu)_{\zeta_{T}}$, and $\mathcal{V}_{\kappa}(\lambda, \mu)_{\zeta_{T}} \cong \mathbb{C}v_{T}$ for all $T \in \text{St}(\lambda/\mu)$.

(iii) The $H_{\kappa}$-module $\mathcal{V}_{\kappa}(\lambda/\mu)$ is irreducible.

(iv) $\mathcal{V}_{\kappa}(\lambda/\mu) \cong \mathcal{L}_{\kappa}(\lambda, \mu)$.

The following result is also announced in [C2]:

Theorem A.4. (Theorem 3.19 in [SV]) Let $\kappa \in \mathbb{Z}_{\geq 1}$. Let $L$ be ab irreducible $H_{\kappa}$-module such that $L = \oplus_{\zeta \in P} L_{\zeta}$. Then there exist $m \in [1, n]$ and $\lambda, \mu \in X_{m}^{\dagger}(\kappa - m)$ with $\lambda - \mu \models n$ such that $L \cong \mathcal{V}_{\kappa}(\lambda/\mu)$.

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