INTEGRABLE MODULES OVER $\widehat{\mathfrak{gl}}_m$ AND THE DOUBLE AFFINE HECKE ALGEBRA (Combinatorial Methods in Representation Theory and their Applications)

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INTEGRABLE MODULES OVER $\hat{\mathfrak{gl}}_m$ AND THE DOUBLE AFFINE HECKE ALGEBRA

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Introduction

Motivated by conformal field theory on the Riemann sphere, we introduce a certain space of coinvariants obtained from tensor product of representations of the affine Lie algebra $\hat{\mathfrak{gl}}_m$.

In [AST], an action of the degenerate affine Hecke algebra $H_\kappa$ is defined on this space through the Knizhnik-Zamolodchikov connection. This construction gives a functor from the category of highest (or lowest) weight modules over $\hat{\mathfrak{gl}}_m$ to the category of $H_\kappa$-modules.

We will see that the integrable $\hat{\mathfrak{gl}}_m$-modules correspond by this functor to irreducible $H_\kappa$-modules whose structure is described combinatorially. We also focus on the symmetric part of these irreducible $H_\kappa$-modules; i.e., the subspace consisting of those elements which are invariant with respect to the action of the Weyl group. We present a spectral decomposition of the symmetric part, and a character formula, which is described by level restricted analogue of the Kostka polynomial.

1. AFFINE LIE ALGEBRA

Throughout this note, we use the notation $[i,j] = \{i, i+1, \ldots, j\}$ for $i,j \in \mathbb{Z}$.

Let $m \in \mathbb{Z}_{\geq 2}$. Let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{gl}_m$ consisting of all $n \times n$-matrices over $\mathbb{C}$. Let $\mathfrak{g}[t, t^{-1}]$ denote the Lie algebra consisting of all $n \times n$-matrices over $\mathbb{C}[t, t^{-1}]$. Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c_\mathfrak{g}$ be the affine Lie algebra with the commutation relation

$$[a \otimes t^i, b \otimes t^j] = [a, b] \otimes t^{i+j} + \text{trace}(ab)i\delta_{i+j,0}c_\mathfrak{g}$$

for $a, b \in \mathfrak{g}$, $i, j \in \mathbb{Z}$.

Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{g}$ consisting of all diagonal matrices, and let $\mathfrak{h}^*$ denote its dual space. A Cartan subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{g}}$ is given by $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c_\mathfrak{g}$. Its dual space is denoted by $\hat{\mathfrak{h}}^*$. We regard $\hat{\mathfrak{h}}^*$ as a subspace of $\hat{\mathfrak{h}}^*$ through the identification $\hat{\mathfrak{h}}^* \cong \mathfrak{h}^* \oplus \mathbb{C}c_\mathfrak{g}^*$. 

Fix $\ell \in \mathbb{C}$. For $\lambda \in \mathfrak{h}^*$, $\widetilde{M}_\ell(\lambda)$ denote the highest weight Verma module of highest weight $\lambda + \ell c_{\mathfrak{g}}^* \in \mathfrak{h}^*$, and let $\overline{M}_\ell(\lambda)$ denote the lowest weight Verma module of lowest weight $-\lambda - \ell c_{\mathfrak{g}}^* \in \mathfrak{h}^*$. Their irreducible quotients are denoted by $\widehat{L}_\ell(\lambda)$ and $\overline{L}_\ell(\lambda)$ respectively.

A $\mathfrak{g}$-module $M$ is said to be of level $\ell$ if $c$ acts as a scalar $\ell$. For example, $\widetilde{M}_\ell(\lambda)$ and $\widehat{L}_\ell(\lambda)$ are of level $\ell$, and $\overline{M}_\ell(\lambda)$ and $\overline{L}_\ell(\lambda)$ are of level $-\ell$.

We identify $\mathfrak{h}$ with $\mathbb{C}^m$, and introduce its subspaces $X_m = \mathbb{Z}^m$ and

$$X_m^+ = \{(\lambda_1, \ldots, \lambda_m) \in X_m \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m\},$$

$$X_m^+(\ell) = \{(\lambda_1, \ldots, \lambda_m) \in X_m^+ \mid \lambda_1 - \lambda_m \leq \ell\}.$$

Note that $\widehat{L}_\ell(\lambda)$ and $\overline{L}_\ell(\lambda)$ are integrable for $\lambda \in X_m^+(\ell)$, and that $X_m^+(\ell)$ is empty unless $\ell \in \mathbb{Z}_{\geq 0}$.

Let $E = \mathbb{C}^m$ denote the vector representation of $\mathfrak{g}$. Put $E[z, z^{-1}] = E \otimes \mathbb{C}[z, z^{-1}]$, which we regard as a $\mathfrak{g}[t, t^{-1}]$-module through the correspondence $a \otimes t^k \mapsto a \otimes z^k$.

2. THE DEGENERATE DOUBLE AFFINE HECKE ALGEBRA

Let $n \in \mathbb{Z}_{\geq 2}$. Let $V$ denote the $n$-dimensional vector space over $\mathbb{C}$ with the basis $\{y_i\}_{i \in [1, n]}$: $V = \oplus_{i \in [1, n]} \mathbb{C}y_i$. Introduce the non-degenerate symmetric bilinear form $(\cdot | \cdot)$ on $V$ by $(y_i | y_j) = \delta_{ij}$. Let $V^* = \oplus_{i=1}^n \mathbb{C}x_i$ be the dual space of $V$, where $x_i$ is the dual vector of $y_i$. The natural pairing is denoted by $\langle \cdot, \cdot \rangle: V^* \times V \to \mathbb{C}$.

Put $\alpha_{ij} = x_i - x_j, \alpha_{ij}^\vee = y_i - y_j$ and $\alpha_i = \alpha_{ii+1}, \alpha_i^\vee = \alpha_{ii+1}$. Then $R = \{\alpha_{ij} \mid i, j \in [1, n], i \neq j\}$ and $R^+ = \{\alpha_{ij} \in R \mid i < j\}$ give a set of roots and a set of positive roots of type $A_{n-1}$ respectively. Let $W$ denote the Weyl group associated with the root system $R$, which is isomorphic to the symmetric group $\mathfrak{S}_n$ of degree $n$. Denote by $s_\alpha$ the reflection in $W$ corresponding to $\alpha \in R$. We write $s_i = s_{\alpha_i}$ and $s_{ij} = s_{\alpha_{ij}}$.

Put $P = \oplus_{i \in [1, n]} \mathbb{Z}x_i$, which is preserved by $W$. Define the extended affine Weyl group $\overline{W}$ as the semidirectproduct $P \rtimes W$ with the relation $w\tau_\eta w^{-1} = \tau_{w(\eta)}$, where $\tau_\eta$ denotes the element of $\overline{W}$ corresponding to $\eta \in P$.

Let $S(V)$ denote the symmetric algebra of $V$, which can be identified with the polynomial ring $\mathbb{C}[y] = \mathbb{C}[y_1, \ldots, y_n]$.

Fix $\kappa \in \mathbb{C}$. The degenerate double affine Hecke algebra (degenerate DAHA) $H_\kappa$ of $GL_n$ is an associative $\mathbb{C}$-algebra generated by the algebra $CP, CW$ and $S(V)$, and subjects to the following defining relations
([C1]):

\[ s_i h = s_i(h)s_i - \langle \alpha_i | h \rangle \quad (i \in [1, n], \ h \in V), \]

\[ s_i e^\eta s_i = e^{s_i(\eta)} \quad (i \in [1, n], \ \eta \in P), \]

\[ [h, e^\eta] = \kappa \langle \eta | h \rangle e^\eta + \sum_{\alpha \in R^+} \langle \alpha | h \rangle \frac{(e^\eta - e^{s_\alpha(\eta)})}{1 - e^{-\alpha}} s_\alpha \quad (h \in V, \ \eta \in P), \]

where \( e^\eta \) denote the element of \( \mathbb{C}P \) corresponding to \( \eta \in P \).

It is known that \( H_\kappa \cong \mathbb{C}P \otimes \mathbb{C}W \otimes S(V) \) as a vector space. The subalgebra \( H_\text{aff} = \mathbb{C}W \cdot S(V) \) is called the degenerate affine Hecke algebra. Note that the subalgebra \( \mathbb{C}P \cdot \mathbb{C}W \) is isomorphic to \( \mathbb{C}\overline{W} \).

3. INDUCED REPRESENTATIONS OF \( H_\kappa \)

For \( \lambda \in X_m = \mathbb{Z}^m \) we write \( \lambda \models n \) when \( \sum_{i \in [1, m]} \lambda_i = n \) and \( \lambda_i \in \mathbb{Z}_{\geq 0} \) for all \( i \in [1, m] \). Let \( \lambda, \mu \in X_m \) such that \( \lambda - \mu \models n \). Introduce the subalgebra \( H_\lambda = \mathbb{C}W_{\lambda-\mu} \cdot S(V) \) of \( H_\kappa \), where \( W_{\lambda-\mu} \) denote the parabolic subgroup \( \mathfrak{S}_{\lambda_1-\mu_1} \times \cdots \times \mathfrak{S}_{\lambda_m-\mu_m} \) of \( W \).

Let \( \mathbb{C}1_{\lambda, \mu} \) denote the one dimensional representation of \( H_{\lambda-\mu} \) such that

\[ w1_{\lambda, \mu} = 1_{\lambda, \mu} \quad (w \in W_{\lambda-\mu}), \]

\[ y_i 1_{\lambda, \mu} = \langle \zeta_{\lambda, \mu} | y_i \rangle 1_{\lambda, \mu} \quad (i \in [1, n]), \]

where \( \zeta_{\lambda, \mu} \) denote the element of \( V^* \) given by

\[ \langle \zeta_{\lambda, \mu} | y_i \rangle = \mu_j + i - m_j - j - 1 \quad \text{for} \ i \in [m_j + 1, m_{j+1}], \]

with \( m_0 = 0 \) and \( m_j = \sum_{k \in [1, j]} (\lambda_k - \mu_k) \) (\( j \in [1, m] \)). Define an \( H_\kappa \)-module by \( \mathcal{M}_\kappa(\lambda, \mu) = H_\kappa \otimes_{H_{\lambda-\mu}} \mathbb{C}1_{\lambda, \mu} \). Obviously we have

\[ \mathcal{M}_\kappa(\lambda, \mu) \cong \mathbb{C}\overline{W}/W_{\lambda-\mu} \cong \mathbb{C}P \otimes \mathbb{C}W/W_{\lambda-\mu} \]

as an \( \overline{W} \)-module.

In the rest, we often identify the group ring \( \mathbb{C}P \) with the Laurent polynomial ring \( \mathbb{C}[z^\pm 1] = \mathbb{C}^z \) (Laurent) polynomial representation. On the representation \( \mathbb{C}P \), the element \( y_i \) \( (i \in [1, n]) \) acts as the Cherednik-Dunkl operator

\[ T_i = \kappa z_i \frac{\partial}{\partial z_i} + \sum_{1 \leq j < i} \frac{z_j}{z_i - z_j} (1 - s_\alpha) + \sum_{i < j \leq n} \frac{z_i}{z_i - z_j} (1 - s_\alpha) + i - 1. \]

Example 3.1. Let \( m = 1 \) and let \( \lambda = (n) \) and \( \mu = (0) \). Then \( \mathcal{M}_\kappa(\lambda, \mu) \cong \mathbb{C}P = \mathbb{C}[z^\pm 1] \), which is called the (Laurent) polynomial representation. On the representation \( \mathbb{C}P \), the element \( y_i \) \( (i \in [1, n]) \) acts as the Cherednik-Dunkl operator

\[ T_i = \kappa z_i \frac{\partial}{\partial z_i} + \sum_{1 \leq j < i} \frac{z_j}{z_i - z_j} (1 - s_\alpha) + \sum_{i < j \leq n} \frac{z_i}{z_i - z_j} (1 - s_\alpha) + i - 1. \]
The simultaneous eigenvectors of $T_1, \ldots, T_n$ are called the nonsymmetric Jack polynomials.

4. THE SPACE OF COINVARIANTS AND THE DEGENERATE DOUBLE AFFINE HECKE ALGEBRA

Let $\ell \in \mathbb{C}$. Let $M$ be a highest weight module of level $\ell$ and let $N$ be a lowest weight module of level $-\ell$. We set

$$\tilde{C}(M, N) = M \otimes E[z_1, z_1^{-1}] \otimes \cdots \otimes E[z_n, z_n^{-1}] \otimes N,$$

$$C(M, N) = \tilde{C}(M, N)/\mathfrak{g}[t, t^{-1}]\tilde{C}(M, N).$$

Let $\sigma_{ij} \in \text{End}_{\mathbb{C}}[\underline{z}^{\pm 1}]$ denote the permutation of $z_i$ and $z_j$. Let $\Omega_{ij} \in \text{End}_{\mathbb{C}}(E^\otimes n)$ denote the permutation of the $i$-th and $j$-th component of the tensor product. Note that $\tilde{C}(M, N) \cong M \otimes E^\otimes n \otimes \mathbb{C}[\underline{z}^{\pm 1}] \otimes N$, through which we regard $\sigma_{ij}$ and $\Omega_{ij}$ as elements in $\text{End}_{\mathbb{C}}(\tilde{C}(M, N))$.

For $i \in [0, n+1]$, define $\theta_i : \mathfrak{g} \rightarrow U(\mathfrak{g})^{\otimes n+2}$ by $\theta_i(u) = 1^\otimes i \otimes u \otimes 1^\otimes n-i+1$. For $i, j \in [0, n+1]$ with $i < j$, define $\theta_{ij} : \mathfrak{g}^{\otimes 2} \rightarrow U(\mathfrak{g})^{\otimes n+2}$ by $\theta_{ij}(u \otimes v) = 1^\otimes i \otimes u \otimes 1^\otimes j-i-1 \otimes v \otimes 1^\otimes n-j+1$.

Let $e_{ab}$ denote the matrix unit of $\mathfrak{g}$ with only non-zero entries 1 at the $(a, b)$-th component. Put $r = \frac{1}{2} \sum_{a \in [1,m]} e_{aa} \otimes e_{aa} + \sum_{1 \leq a < b \leq m} e_{ab} \otimes e_{ba}$ and put $r_{ij} = \theta_{ij}(r)$.

For $i \in [1, n]$, put

$$r_{0i} = r_{0i} + \sum_{k \in \mathbb{Z}_{\geq 1}} \sum_{1 \leq a < b \leq m, a \neq b} \theta_{0i}((e_{ab} \otimes t^k) \otimes (e_{ba} \otimes t^{-k})),$$

$$r_{i+1} = r_{in+1} + \sum_{k \in \mathbb{Z}_{\geq 1}} \sum_{1 \leq a < b \leq m, a \neq b} \theta_{i+1}((e_{ab} \otimes t^k) \otimes (e_{ba} \otimes t^{-k})),$$

which are elements of some completion of $U(\mathfrak{g}[t, t^{-1}])^{\otimes n+2}$ and define well-defined operators on $\tilde{C}(M, N)$.

Define the linear operators on $\tilde{C}(M, N)$ by

$$D_i = \kappa z_i \frac{\partial}{\partial z_i} + \hat{r}_{0i} - \hat{r}_{i+1} + \sum_{1 \leq j < i} r_{ij} - \sum_{i < j \leq n} r_{ji} + \theta_i(\rho^\vee) + \sum_{1 \leq j < i} \frac{z_j}{z_i - z_j} (1 - \sigma_{ij}) \Omega_{ij} + \sum_{i < j \leq n} \frac{z_i}{z_i - z_j} (1 - \sigma_{ij}) \Omega_{ij} + i - 1,$$

where $\rho^\vee = \sum_{k \in [1,m]} \frac{1}{2} (n - 2k + 1) e_{aa} \in \mathfrak{h}$.

**Theorem 4.1** (Theorem 4.2.2 in [AST]). Let $M$ be a highest weight module of $\mathfrak{g}$ of level $\kappa - m$ and let $N$ be a lowest weight module of level $-\kappa + m$. 
(i) There exists a unique algebra homomorphism $\varpi : H^\text{rat}_\kappa \to \text{End}_C(\mathcal{C}(M, N))$ such that

$$\varpi(s_i) = \Omega_{i,i+1} \sigma_{i,i+1} \quad (i \in [1, n-1]),$$

$$\varpi(c_i^{\pm 1}) = z_i \quad (i \in [1, n]),$$

$$\varpi(y_i) = D_i \quad (i \in [1, n]).$$

(ii) The $H_\kappa$-action on $C(M, N)$ above preserves the subspace $\mathfrak{g}[t, t^{-1}]\mathcal{C}(M, N)$:

$$\varpi(a)\mathfrak{g}[t, t^{-1}]\mathcal{C}(M, N) \subseteq \mathfrak{g}[t, t^{-1}]\mathcal{C}(M, N)$$

for all $a \in H_\kappa$. Therefore, $\varpi$ induces an $H_\kappa$-module structure on $C(M, N)$.

5. Images of the Functor

The following statement has been shown in [AST].

**Proposition 5.1** (Proposition 5.3.1 in [AST]). Let $\kappa \in \mathbb{C}$ and put $\ell = \kappa - m$.

(i) Let $\lambda, \mu \in X^+_m$. Then

$$C(\hat{M}_\ell(\mu), \hat{M}_\ell(\lambda)) \cong \begin{cases} M_\kappa(\lambda, \mu) & \text{if } \lambda - \mu \models n, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Let $\lambda, \mu \in X^+_m(\ell)$ such that $\lambda - \mu \models n$. Then

$$C(\hat{L}_\ell(\mu), \hat{L}_\ell(\lambda)) \cong C(\hat{M}_\ell(\mu), \hat{M}_\ell(\lambda)) \cong C(\hat{L}_\ell(\mu), \hat{L}_\ell(\lambda)).$$

For each $\lambda \in X_m$, we have the additive functor $F_\lambda(-) = C(-, \hat{M}_\ell(\lambda))$ from the category of highest weight modules over $\hat{\mathfrak{g}}$ to the category of $H_\kappa$-modules. It is right exact and sends the Verma module $\hat{M}_\ell(\mu)$ to the induced module $\mathcal{M}_\kappa(\lambda, \mu)$ by Proposition 5.1. In the sequel, we will determine the image $F_\lambda(\hat{L}_\ell(\mu))$ of the irreducible module $\hat{L}_\ell(\mu)$ in the case where $\lambda, \mu \in X^+_m(\ell)$. Note that $F_\lambda(\hat{L}_\ell(\mu)) \cong C(\hat{L}_\ell(\mu), \hat{L}_\ell(\lambda))$, and note also that it is a quotient of $F_\lambda(\mathcal{M}_\kappa(\lambda, \mu))$.

Let $\ell \in \mathbb{Z}_{\geq 0}$ and $\lambda, \mu \in X^+_m(\ell)$ such that $\lambda - \mu \models n$. Then it is known that the $H_\kappa$-module $\mathcal{M}_\kappa(\lambda, \mu)$ has a unique simple quotient ([AST, S1]), which we will denote by $\mathcal{L}_\kappa(\lambda, \mu)$.

The irreducible modules $\mathcal{L}_\kappa(\lambda, \mu)$ for $\lambda, \mu \in X^+_m(\ell)$ are investigated in [SV], and in particular their structure is described combinatorially using tableaux on periodic skew diagrams. We give a short review of the theory of periodic tableaux and the tableaux representations of $H_\kappa$ in Appendix. By means of this combinatorial description, we can estimate the kernel of the projection $\mathcal{M}_\kappa(\lambda, \mu) \to \mathcal{L}_\kappa(\lambda, \mu)$. By comparing it with the kernel of $\mathcal{M}_\kappa(\lambda, \mu) \to F_\lambda(\hat{L}_\ell(\mu))$, we have...
Theorem 5.2. Let $\kappa \in \mathbb{Z}_{\geq 1}$ and put $\ell = \kappa - m$. Let $\lambda, \mu \in X^{\pm}_m(\ell)$ such that $\lambda - \mu \vdash n$. Then the $H_\kappa$-module $C(\hat{L}_{\ell}(\mu), \hat{L}_{\ell}^\dagger(\lambda))$ is irreducible:

$$C(\hat{L}_{\ell}(\mu), \hat{L}_{\ell}^\dagger(\lambda)) \cong \mathcal{L}_\kappa(\lambda, \mu),$$

and moreover it is semisimple over $S(V)$. (See Theorem A.3 for the combinatorial description of the weight decomposition).

The classification of the irreducible $H_\kappa$-modules which are semisimple over $S(V)$ is given in [C2, SV], from which (or from Theorem A.4) we have

Corollary 5.3. Let $\kappa \in \mathbb{Z}_{\geq 1}$. Let $L$ be an irreducible $H_\kappa$-module which is finitely generated and admits a weight decomposition of the form $L = \bigoplus_{\zeta \in P} L_{\zeta}$, where $L_{\zeta} = \{v \in L \mid yv = (\zeta \cdot y) \forall y \in V\}$. Then there exists $m \in [1, n]$ and $\lambda, \mu \in X^{\pm}_m(\kappa - m)$ such that $L \cong C(\hat{L}_{\kappa-m}(\mu), \hat{L}_{\kappa-m}^\dagger(\lambda))$.

6. Localization and Conformal Coinvariants

We will see the relation between our space $C(M, N)$ of coinvariants and the space of conformal coinvariants in Wess-Zumino-Witten model [TK, TUY].

Observe that the group ring $\mathbb{C}P$ can be seen as the coordinate ring $A = \mathbb{C}[T]$ of the affine variety $T = (\mathbb{C} \setminus \{0\})^n$. Put $T_0 = T \setminus \Delta$, where $\Delta = \cup_{i<j}\{(\xi_1, \ldots, \xi_n) \in T \mid \xi_i/\xi_j = 1\}$, and put $A_o = \mathbb{C}[T_0]$. Namely, $A_o$ is the localization of $A$ at $\Delta$; $A_o = \mathbb{C}\left[z_1^{\pm 1}, \ldots, z_n^{\pm 1}, \frac{1}{1-z_i/z_j} \ (i < j)\right]$. Let $D(T_o)$ denote the ring of algebraic differential operators on $T_o$. Then the Cherednik-Dunkl operators in Example 3.1 $T_1, \ldots, T_n$ can be seen as elements of the ring $D(T_o) \rtimes \mathbb{C}W$. Put $H_{\kappa,o} = A_o \otimes_A H_\kappa$. There exists a unique algebra structure on $H_{\kappa,o}$ extending $H_\kappa$.

Proposition 6.1. Let $\kappa \in \mathbb{C}^\times$. There exists a unique algebra isomorphism $H_{\kappa,o} \rightarrow D(T_o) \rtimes \mathbb{C}W$ such that $y_i \mapsto T_i$, $w \mapsto w$, $f \mapsto f$ for all $i \in [1, n]$, $w \in W$ and $f \in A_o$.

For an $H_\kappa$-module $M$, set $M_o = A_o \otimes_A M$. Then via Proposition 6.1, we have a structure of $D(T_o) \rtimes \mathbb{C}W$-module on $M_o$; namely, $M_o$ admits a $W$-equivariant integrable (algebraic) connection

$$\nabla_i = \kappa^{-1} \left\{ y_i - \sum_{1 \leq j < i} \frac{z_j}{z_i - z_j} (1 - s_\alpha) - \sum_{i < j \leq n} \frac{z_i}{z_i - z_j} (1 - s_\alpha) - (i - 1) \right\}.$$  

Now consider the case where $M = C(\hat{L}_{\ell}(\mu), \hat{L}_{\ell}^\dagger(\lambda)) = \mathcal{L}_\kappa(\lambda, \mu)$ with $\lambda, \mu \in X^{\pm}_m(\ell)$. Then it follows that the connection given above has regular singularities along $\Delta$, and hence $\mathcal{L}_\kappa(\lambda, \mu)_o$ is a projective $A_o$-module, or geometrically, a vector bundle over $T_o$ of finite rank ([GGOR, VV]).
For $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{T}$, let $C_\xi$ denote the one-dimensional right module of $A_\circ$ given by the evaluation at $\xi$. It follows that the space $C_\xi \otimes_{A_\circ} (L_\kappa(\lambda, \mu)_{\circ})$ is isomorphic to with "the space of conformal coinvariants"

$$\left(\hat{L}_\ell(\mu) \otimes \hat{L}_\ell(\nu_1)^{\otimes n} \otimes \hat{L}_\ell(\lambda^t)\right)/g_{(0, \xi, \infty)}\left(\hat{L}_\ell(\mu) \otimes \hat{L}_\ell(\nu_1)^{\otimes n} \otimes \hat{L}_\ell(\lambda^t)\right),$$

where $\nu_1 = (1, 0, \ldots, 0) \in X_\pm^+(\ell)$ (the highest weight of the vector representation $E$), $\lambda^t = -w_0(\lambda)$ with $w_0$ being the longest element of $W$, and $g_{(0, \xi, \infty)}$ denotes the Lie algebra of $g$-valued algebraic functions on $P^1 \setminus \{0, \xi_1, \ldots, \xi_n, \infty\}$, which acts on $\hat{L}_\ell(\mu) \otimes \hat{L}_\ell(\nu_1)^{\otimes n} \otimes \hat{L}_\ell(\lambda^t)$ through the Laurent expansion at each points. (See e.g. [BK] for a precise definition.)

Therefore it follows that the vector bundle $L_\kappa(\lambda, \mu)_{\circ}$ is equivalent to the vector bundle of conformal coinvariants (the dual of the vector bundle of conformal blocks in the sense of [TY, BK]). Moreover, the connection $\{\nabla_i\}$ on $L_\kappa(\lambda, \mu)_{\circ}$ given via Proposition 6.1 coincides with the Knizhnik-Zamolodchikov connection on the vector bundle of conformal coinvariants.

7. WEIGHT DECOMPOSITION OF SYMMETRIC PART

For an $H_\kappa$-module $M$, put

$$W = \{v \in M | uw = v \forall w \in W\},$$

on which the algebra $H_\kappa^W = \{u \in H_\kappa | wuw^{-1} = u\}$ acts. The algebra $H_\kappa^W$ is called the zonal spherical algebra and it contains a subalgebra $S(V)^W$, which coincides with the center of the degenerate affine Hecke algebra $H_{\text{aff}}$.

For $\zeta \in V^*$, let $\chi_\zeta$ denote the image of the projection to the quotient space $W \setminus V^*$. Identify $W \setminus V^*$ with the set $\text{Hom}_{\text{algebra}}(S(V)^W, \mathbb{C})$ of characters, and set

$$M^W_{[\zeta]} = \{v \in M^W | \xi v = \chi_\zeta(\xi) v \forall \xi \in S(V)^W\}.$$

In the sequel, we will give a decomposition of $L_\kappa(\lambda, \mu)^W$ into weight spaces with respect to $S(V)^W$.

Let $\lambda, \mu \in X_+^m$ such that $\lambda - \mu \models n$. Let $\lambda/\mu$ denote the skew Young diagram associated with $(\lambda, \mu)$:

$$\lambda/\mu = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} | a \in [1, m], b \in [\mu_a + 1, \lambda_a]\}.$$

Let $T$ be a tableau on the diagram $\lambda/\mu$; namely $T$ is a bijection from $\lambda/\mu$ to $[1, n]$. Then it determines the sequence $\{\lambda_T^{(i)}\}_{i \in [0, n]}$ in $X_m$ by the condition $\lambda_T^{(0)} = \mu$ and $\lambda_T^{(i)}/\lambda_T^{(i-1)} = T^{-1}(i)$ ($i \in [1, n]$).
Let \( \ell \in \mathbb{Z}_{\geq 0} \). A tableau \( T \) is called an \( \ell \)-restricted standard tableau if \( \lambda_T^{(i)} \in X_m^+ (\ell) \) for all \( i \in [1, n] \). Let \( \text{St}_\ell (\lambda, \mu) \) denote the set of \( \ell \)-restricted tableaux on \( \lambda \).

Let \( T \in \text{St}_\ell (\lambda, \mu) \). For \( i \in [1, n] \), define

\begin{equation}
(7.3) \quad h_i (T) = \begin{cases} 
1 & \text{if } a < a', \\
0 & \text{if } a \geq a', 
\end{cases}
\end{equation}

where \( T(a, b) = i \) and \( T(a', b') = i + 1 \). Define

\begin{equation}
(7.4) \quad \eta_T = \sum_{i \in [1, n]} \left( \sum_{j < i} h_j (T) \right) x_i \in P.
\end{equation}

Define \( \zeta_T \in V^* \) by \( (\zeta_T (y_i) = b - a \) when \( T(a, b) = i \).

From the weight decomposition of \( \mathcal{L}_\kappa (\lambda, \mu) \) (Theorem A.3) with respect to \( S(V) \), we have

**Theorem 7.1.** (Conjecture 6.1.1 in \cite{AST}) Let \( \lambda, \mu \in X_m^+ (\ell) \) such that \( \lambda - \mu \models n \). Then

\[
\mathcal{L}_\kappa (\lambda, \mu)^W = \bigoplus_{\nu \in P^-} \bigoplus_{T \in \text{St}_\ell (\lambda/\mu)} \mathcal{L}_\kappa (\lambda, \mu)_{[\zeta_T + \kappa (\nu + \eta_T)]}^W,
\]

where \( P^- = \{ \zeta \in P \mid \langle \zeta \mid \alpha_i \rangle \leq 0 \ \forall i \in [1, n - 1] \} \), and

\[
\dim \mathcal{L}_\kappa (\lambda, \mu)_{[\zeta_T + \kappa (\nu + \eta_T)]}^W = 1
\]

for all \( \nu \in P^- \) and \( T \in \text{St}_\ell (\lambda/\mu) \).

8. **q-Dimension Formula**

Put \( \partial = \kappa^{-1} \sum_{i \in [1, n]} y_i \in S(V)^W \). Then \( \partial \) satisfies the relation

\[ [\partial, z_i] = \kappa z_i, \ [\partial, w] = 0 \]

for all \( i \in [1, n] \) and \( w \in W \).

Our next purpose is to give a \( q \)-dimension formula for \( \mathcal{L}_\kappa (\lambda, \mu)^W \) with respect to the grading operator \( \partial \). To this end, we need to introduce the "polynomial part" of \( \mathcal{L}_\kappa (\lambda, \mu) \) following \cite{AST}.

Define a subalgebra \( H^W_\kappa \) of \( H_\kappa \) by

\[
H^W_\kappa = CP^W \cdot \mathbb{C}W \cdot S(V),
\]

where \( P^W = \bigoplus_{i \in [1, n]} \mathbb{Z}_{\geq 0} x_i \).

Let \( \kappa \in \mathbb{Z}_{\geq 1} \) and let \( \lambda, \mu \in X_m^+ (\kappa - m) \) such that \( \lambda - \mu \models n \). Recall that the induced module \( \mathcal{M}_\kappa (\lambda, \mu) \) is generated by the cyclic vector \( 1_{\lambda, \mu} \). We denote by \( 1_{\lambda, \mu}^- \) its image under the projection \( \mathcal{M}_\kappa (\lambda, \mu) \rightarrow \mathcal{L}_\kappa (\lambda, \mu) \). Note that \( 1_{\lambda, \mu}^- \neq 0 \). Define the polynomial part of \( \mathcal{L}_\kappa (\lambda, \mu) \) by \( \mathcal{L}^W_\kappa (\lambda, \mu) = H^W_\kappa 1_{\lambda, \mu}^- \), which is an \( H^W_\kappa \)-submodule of \( \mathcal{L}_\kappa (\lambda, \mu) \).
Put $\mathcal{L}_{\kappa}^\geq_0(\lambda, \mu)^W_{(k)} = \{v \in \mathcal{L}_{\kappa}^\geq_0(\lambda, \mu)^W \mid \partial v = kv\}$. Then we have $\dim \mathcal{L}_{\kappa}^\geq_0(\lambda, \mu)^W_{(k)} < \infty$ and $\mathcal{L}_{\kappa}^\geq_0(\lambda, \mu)^W = \oplus_{k \in \mathbb{Z}} \mathcal{L}_{\kappa}^\geq_0(\lambda, \mu)^W_{(k)}$. Define

$$\dim_q \mathcal{L}_{\kappa}^\geq_0(\lambda, \mu)^W = \sum_{d \in \mathbb{Z}} q^k \dim \mathcal{L}_{\kappa}^\geq_0(\lambda, \mu)^W_{(k)}.$$ 

Set

$$(8.1) \quad h(T) = \kappa \langle \eta_T \mid \partial \rangle = \sum_{i \in [1, n]} (n - i) h_i(T).$$

From Theorem 7.1, we have

**Theorem 8.1.** Let $\kappa \in \mathbb{Z}_{\geq 0}$ and let $\lambda, \mu \in X_m^+(\kappa - m)$ such that $\lambda - \mu \models n$. Then

$$(8.2) \quad \dim_q \mathcal{L}_{\kappa}^\geq_0(\lambda, \mu)^W = \frac{q^{\Delta_{\lambda} - \Delta_{\mu}}}{(q)_n} F_{\lambda/\mu}^{(\ell)}(q).$$

Here $\Delta_{\lambda} = \frac{1}{2\kappa}((\lambda, \lambda) + 2(\rho, \lambda))$, $(q)_n = (1 - q)(1 - q^2) \ldots (1 - q^n)$ and $F_{\lambda/\mu}^{(\ell)}(q)$ is a polynomial of $q$ given by

$$(8.3) \quad F_{\lambda/\mu}^{(\ell)}(q) = \sum_{T \in \mathrm{St}_{(\ell)}(\lambda/\mu)} q^{h(T)}.$$

**Remark 8.2.** If $\ell$ is large enough then $F_{\lambda/\mu}^{(\ell)}(q)$ coincides with the Kostka polynomial $K_{(\lambda/\mu)'(1^n)}(q)$ associated to the conjugate $(\lambda/\mu)'$ of $\lambda/\mu$. Hence our polynomial $F_{\lambda/\mu}^{(\ell)}(q)$ is an $\ell$-restricted version of the Kostka polynomial (cf. [FJKLM]).

**Remark 8.3.** A bosonic formula for $F_{\lambda/\mu}^{(\ell)}(q)$ is known (Theorem 6.2.4 in [AST]), and Theorem 8.1 is equivalent to the formula in Conjecture 6.1.1 in [AST]. Note also that the bosonic formula suggests the existence of the BGG type resolution of $\mathcal{L}_{\kappa}(\lambda, \mu)$.

### 9. Rational Analogue

For a $\mathfrak{g}[t]$-module $N$, set

$$\tilde{\mathcal{C}}(M) = E[z_1] \otimes \ldots \otimes E[z_n] \otimes N$$

$$\mathcal{C}(N) = \tilde{\mathcal{C}}(N)/\mathfrak{g}[t]\tilde{\mathcal{C}}(N),$$

where $E[z] = E \otimes \mathbb{C}[z]$. The analogous construction gives on $\mathcal{C}(N)$ an action of the rational Cherednik algebra $H_{\kappa}^{\text{rat}} ([EG])$, which can be
defined as the subalgebra of $H_\kappa$ generated by the subalgebra $C[z] \cdot CW$ and the following (pairwise commutative) elements

$$u_i = z_i^{-1} \left( y_i - \sum_{j<i} s_{ij} \right) \quad (i \in [1, n])$$

as pointed out in [S2].

It follows for $\lambda \in X_m^+((\ell))$ that $C(M^1_\ell(\lambda))$ is isomorphic to some induced module, and $C(L^1_\ell(\lambda))$ is isomorphic to the unique simple quotient of $C(M^1_\ell(\lambda))$, which we denote by $L_\kappa(\lambda)$.

Let $0 = (0,\ldots,0) \in X_m^+((\ell))$. Then it follows that the polynomial part $L^\geq 0_\kappa(\lambda,0)$ of the $H_\kappa$-module $L_\kappa(\lambda,0)$ is an $H_\kappa^{r\text{-}}$-submodule and it is isomorphic to $L_\kappa(\lambda)$. This leads the $q$-dimension formula

$$\dim_q L_\kappa(\lambda)^{W} = \frac{q^{\Delta_\lambda}}{(q)_n} F^{(\ell)}_\lambda(q).$$

**Remark 9.1.** It can be seen that the Knizhnik-Zamolodchikov functor investigated in [GGOR] transforms the irreducible representations $L_\kappa(\lambda)$ for $\lambda \in X_m^+((\ell))$ to Wenzl’s representations [W] of the affine Hecke algebra (cf. [TK]).

**APPENDIX A. TABLEAUX ON PERIODIC DIAGRAMS AND REPRESENTATIONS OF THE DEGENERATE DAHA**

We will review the theory of tableaux representations for $H_\kappa$, which is investigated in [SV] for the double affine Hecke algebra.

Fix $\kappa \in \mathbb{Z}_{\geq 1}$. Let $m \in \mathbb{Z}_{\geq 1}$.

For $\lambda, \mu \in X_m^+(\kappa-m)$ such that $\lambda - \mu \models n$, we introduce the following subsets of $\mathbb{Z} \times \mathbb{Z}$:

(A.1) $\lambda/\mu = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \in [1, m], \ b \in [\mu_a + 1, \lambda_a]\}$,

(A.2) $\overline{\lambda/\mu} = \{(a, b) + k(m, -\kappa + m) \in \mathbb{Z} \times \mathbb{Z} \mid (a, b) \in \lambda/\mu, \ k \in \mathbb{Z}\}$.

The set $\overline{\lambda/\mu}$ is called the periodic skew diagram of period $(m, -\kappa + m)$ associated with $(\lambda, \mu)$. The following is called the skew property:

**Lemma A.1.** Let $(a, b), (a', b') \in \overline{\lambda/\mu}$. If $a' = a \in \mathbb{Z}_{\geq 0}$ and $b' = b \in \mathbb{Z}_{\geq 0}$ then $(a, b'), (a', b) \in \overline{\lambda/\mu}$.

A tableau $T$ on $\overline{\lambda/\mu}$ is by definition a bijection $\overline{\lambda/\mu} \to \mathbb{Z}$ satisfying $T(a + m, b - \kappa + m) = T(a, b) + n$ for all $(a, b) \in \overline{\lambda/\mu}$.

A tableau $T$ is called a standard tableau if

$$T(a, b + 1) > T(a, b)$$
for any \((a, b), (a, b + 1) \in \overline{\lambda/\mu}\), and if
\[
T(a + 1, b) > T(a, b)
\]
for any \((a, b), (a + 1, b) \in \overline{\lambda/\mu}\). Let \(\text{Tab}(\overline{\lambda/\mu})\) and \(\text{St}(\overline{\lambda/\mu})\) denote the set of tableaux and the set of standard tableaux on \(\lambda/\mu\) respectively.

Define the elements \(\pi = \tau_{x_1} s_1 s_2 \cdots s_{n-1}\) and \(s_0 = \tau_{\alpha_{1n}} s_{1n}\) of the group \(\overline{W} = P \rtimes W\). Then \(\{s_0, s_1, \ldots, s_{n-1}, \pi\}\) is a generator of the group \(\overline{W}\).

Define the action of \(\overline{W}\) on the set \(\mathbb{Z}\) of integers by
\[
(A.3) \quad s_i(j) = \begin{cases} 
  j + 1 & \text{for } j \equiv i \mod n, \\
  j - 1 & \text{for } j \equiv i + 1 \mod n, \\
  j & \text{for } j \not\equiv i, i + 1 \mod n,
\end{cases}
\]
\[
(A.4) \quad \tau_{x_i}(j) = \begin{cases} 
  j + n & \text{for } j \equiv i \mod n, \\
  j & \text{for } j \not\equiv i \mod n.
\end{cases}
\]
Observe that \(\pi(j) = j + 1\) for all \(j\).

For \(T \in \text{Tab}(\overline{\lambda/\mu})\) and \(w \in \overline{W}\), the map \(wT : \overline{\lambda/\mu} \to \mathbb{Z}\) given by
\[
(wT)(u) = w(T(u)) \quad (u \in \overline{\lambda/\mu})
\]
is also a tableau on \(\overline{\lambda/\mu}\), and the assignment \(T \mapsto wT\) gives an action of \(\overline{W}\) on \(\text{Tab}(\overline{\lambda/\mu})\), which preserves \(\text{St}(\overline{\lambda/\mu})\). It is easy to see that the assignment \(w \mapsto wT\) gives a one-to-one correspondence \(\overline{W} \cong \text{Tab}(\overline{\lambda/\mu})\).

Define the map \(C : \overline{\lambda/\mu} \to \mathbb{Z}\) by \(C(a, b) = b - a\), and define \(C_T : \mathbb{Z} \to \mathbb{Z}\) by \(C_T(i) = C(T^{-1}(i))\) for \(T \in \text{St}(\overline{\lambda/\mu})\). Define \(\zeta_T \in V^*\) by \(\langle \zeta_T | y_i \rangle = C_T(i) \quad (i \in [1, n])\).

The following lemma follows from the skew property and the definition of the standard tableaux:

**Lemma A.2.** Let \(T \in \text{St}(\overline{\lambda/\mu})\) and \(i \in [0, n - 1]\).

(i) \(C_T(i) - C_T(i + 1) \neq 0\).

(ii) \(s_i T \in \text{St}(\overline{\lambda/\mu})\) if and only if \(C_T(i) - C_T(i + 1) \notin \{-1, 1\}\).

Now, we introduce the tableaux representation associated with \(\overline{\lambda/\mu}\). Let \(V_\kappa(\overline{\lambda/\mu})\) be the vector space with the basis \(\{v_T\}_{T \in \text{St}(\overline{\lambda/\mu})}\):
\[
V_\kappa(\overline{\lambda/\mu}) = \bigoplus_{T \in \text{St}(\overline{\lambda/\mu})} \mathbb{C}v_T.
\]
By Lemma A.2 and induction argument, we have
Theorem A.3. (Theorem 3.16, Theorem 3.17 in [SV]) Let $\kappa \in \mathbb{Z}_{\geq 1}$. Let $\lambda, \mu \in X_{m}^{+}(\kappa - m)$ such that $\lambda - \mu \models n$.

(i) There exists a unique $H_{\kappa}$-module structure on $\mathcal{V}_{\kappa}(\overline{\lambda/\mu})$ such that
\[
y_{i}v_{T} = C_{T}(i) v_{T} \quad (i \in [1, n]),
\]
\[
\pi v_{T} = v_{\pi T},
\]
\[
s_{i}v_{T} = \begin{cases} 
\frac{1+a_{i}}{a_{i}} v_{s_{i}T} - \frac{1}{a_{i}} v_{T} & \text{if } s_{i}T \in \text{St}(\overline{\lambda/\mu}) \\
-\frac{1}{a_{i}} v_{T} & \text{if } s_{i}T \notin \text{St}(\overline{\lambda/\mu})
\end{cases} \quad (i \in [0, n - 1]),
\]
where $a_{i} = C_{T}(i) - C_{T}(i + 1) \neq 0$ (by Lemma A.2).

(ii) $\mathcal{V}_{\kappa}(\lambda, \mu) = \bigoplus_{T \in \text{St}(\overline{\lambda/\mu})} \mathcal{V}_{\kappa}(\lambda, \mu)_{\zeta_{T}}$, and $\mathcal{V}_{\kappa}(\lambda, \mu)_{\zeta_{T}} \cong \mathbb{C}v_{T}$ for all $T \in \text{St}(\overline{\lambda/\mu})$.

(iii) The $H_{\kappa}$-module $\mathcal{V}_{\kappa}(\overline{\lambda/\mu})$ is irreducible.

(iv) $\mathcal{V}_{\kappa}(\overline{\lambda/\mu}) \cong \mathcal{L}_{\kappa}(\lambda, \mu)$.

The following result is also announced in [C2]:

Theorem A.4. (Theorem 3.19 in [SV]) Let $\kappa \in \mathbb{Z}_{\geq 1}$. Let $L$ be an irreducible $H_{\kappa}$-module such that $L = \bigoplus_{\zeta \in P} L_{\zeta}$. Then there exist $m \in [1, n]$ and $\lambda, \mu \in X_{m}^{+}(\kappa - m)$ with $\lambda - \mu \models n$ such that $L \cong \mathcal{V}_{\kappa}(\overline{\lambda/\mu})$.

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