数理解析研究所講究録 1438 巻 2005 年 37-49

INTEGRABLE MODULES OVER $\widehat{\mathfrak{gl}}_m$ AND THE DOUBLE AFFINE HECKE ALGEBRA

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Introduction

Motivated by conformal field theory on the Riemann sphere, we introduce a certain space of coinvariants obtained from tensor product of representations of the affine Lie algebra $\widehat{\mathfrak{gl}}_m$.

In [AST], an action of the degenerate affine Hecke algebra H_κ is defined on this space through the Knizhnik-Zamolodchikov connection. This construction gives a functor from the category of highest (or lowest) weight modules over $\widehat{\mathfrak{gl}}_m$ to the category of H_κ-modules.

We will see that the integrable $\widehat{\mathfrak{gl}}_m$-modules correspond by this functor to irreducible H_κ-modules whose structure is described combinatorially. We also focus on the symmetric part of these irreducible H_κ-modules; i.e., the subspace consisting of those elements which are invariant with respect to the action of the Weyl group. We present a spectral decomposition of the symmetric part, and a character formula, which is described by level restricted analogue of the Kostka polynomial.

1. Affine Lie algebra

Throughout this note, we use the notation $[i,j]=\{i,i+1,\dots,j\}$ for $i,j\in\mathbb{Z}$.

Let $m\in\mathbb{Z}_{\geq 2}$. Let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{gl}_m$ consisting of all $n\times n$-matrices over $\mathbb{C}$. Let $\mathfrak{g}[t,t^{-1}]$ denote the Lie algebra consisting of all $n\times n$-matrices over $\mathbb{C}[t,t^{-1}]$. Let $\widehat{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c_{\mathfrak{g}}$ be the affine Lie algebra with the commutation relation

$$[a\otimes t^i, b\otimes t^j]=[a,b]\otimes t^{i+j}+\mathrm{trace}(ab)i\delta_{i+j,0}c_{\mathfrak{g}}$$

for $a,b\in\mathfrak{g},\ i,j\in\mathbb{Z}$.

Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{g}$ consisting of all diagonal matrices, and let $\mathfrak{h}^*$ denote its dual space. A Cartan subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}$ is given by $\widehat{\mathfrak{h}}=\mathfrak{h}\oplus\mathbb{C}c_{\mathfrak{g}}$. Its dual space is denoted by $\widehat{\mathfrak{h}}^*$. We regard $\widehat{\mathfrak{h}}^*$ as a subspace of $\widehat{\mathfrak{h}}^*$ through the identification $\widehat{\mathfrak{h}}^*\cong\mathfrak{h}^*\oplus\mathbb{C}c_{\mathfrak{g}}^*$.

Fix $\ell \in \mathbb{C}$. For $\lambda \in \mathfrak{h}^*$, $\widehat{M}_\ell(\lambda)$ denote the highest weight Verma module of highest weight $\lambda + \ell c_{\mathfrak{g}}{}^* \in \widehat{\mathfrak{h}}^*$, and let $\widehat{M}_\ell^\dagger(\lambda)$ denote the lowest weight Verma module of lowest weight $-\lambda - \ell c_{\mathfrak{g}}{}^* \in \widehat{\mathfrak{h}}^*$. Their irreducible quotients are denoted by $\widehat{L}_\ell(\lambda)$ and $\widehat{L}_\ell^\dagger(\lambda)$ respectively.

A $\widehat{\mathfrak{g}}$-module M is said to be of level ℓ if c acts as a scalar ℓ. For example, $\widehat{M}_\ell(\lambda)$ and $\widehat{L}_\ell(\lambda)$ are of level ℓ, and $\widehat{M}_\ell^\dagger(\lambda)$ and $\widehat{L}_\ell^\dagger(\lambda)$ are of level $-\ell$.

We identify $\mathfrak{h}$ with $\mathbb{C}^m$, and introduce its subspaces $X_m = \mathbb{Z}^m$ and

$$X_m^+ = \{(\lambda_1, \dots, \lambda_m) \in X_m \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m\},$$
$$X_m^+(\ell) = \{(\lambda_1, \dots, \lambda_m) \in X_m^+ \mid \lambda_1 - \lambda_m \leq \ell\}.$$

Note that $\widehat{L}_\ell(\lambda)$ and $\widehat{L}_\ell^\dagger(\lambda)$ are integrable for $\lambda \in X_m^+(\ell)$, and that $X_m^+(\ell)$ is empty unless $\ell \in \mathbb{Z}_{\geq 0}$.

Let $E = \mathbb{C}^m$ denote the vector representation of $\mathfrak{g}$. Put $E[z, z^{-1}] = E \otimes \mathbb{C}[z, z^{-1}]$, which we regard as a $\mathfrak{g}[t, t^{-1}]$-module through the correspondence $a \otimes t^k \mapsto a \otimes z^k$.

2. The degenerate double affine Hecke algebra

Let $n \in \mathbb{Z}_{\geq 2}$. Let V denote the n-dimensional vector space over $\mathbb{C}$ with the basis $\{y_i\}_{i \in [1,n]}$: $V = \oplus_{i \in [1,n]} \mathbb{C} y_i$. Introduce the non-degenerate symmetric bilinear form $(\ |\ )$ on V by $(y_i|y_j) = \delta_{ij}$. Let $V^* = \oplus_{i=1}^n \mathbb{C} x_i$ be the dual space of V, where x_i is the dual vector of y_i. The natural pairing is denoted by $\langle | \rangle : V^* \times V \to \mathbb{C}$.

Put $\alpha_{ij} = x_i - x_j$, $\alpha_{ij}^\vee = y_i - y_j$ and $\alpha_i = \alpha_{ii+1}$, $\alpha_i^\vee = \alpha_{ii+1}$ Then $R = \{\alpha_{ij} | i, j \in [1, n],\ i \neq j\}$ and $R^+ = \{\alpha_{ij} \in R \mid i < j\}$ give a set of roots and a set of positive roots of type A_{n-1} respectively.

Let W denote the Weyl group associated with the root system R, which is isomorphic to the symmetric group $\mathfrak{S}_n$ of degree n. Denote by s_α the reflection in W corresponding to $\alpha \in R$. We write $s_i = s_{\alpha_i}$ and $s_{ij} = s_{\alpha_{ij}}$.

Put $P = \oplus_{i \in [1,n]} \mathbb{Z} x_i$, which is preserved by W. Define *the extended affine Weyl group* $\widehat{W}$ as the semidirectproduct $P \rtimes W$ with the relation $w \tau_\eta w^{-1} = \tau_{w(\eta)}$, where τ_η denotes the element of $\widehat{W}$ corresponding to $\eta \in P$.

Let $S(V)$ denote the symmetric algebra of V, which can be identified with the polynomial ring $\mathbb{C}[\underline{y}] = \mathbb{C}[y_1, \dots, y_n]$.

Fix $\kappa \in \mathbb{C}$. The *degenerate double affine Hecke algebra* (degenerate DAHA) H_κ of GL_n is an associative $\mathbb{C}$-algebra generated by the algebra $\mathbb{C}P$, $\mathbb{C}W$ and $S(V)$, and subjects to the following defining relations

([C1]):

$$s_i h = s_i(h)s_i - \langle \alpha_i \mid h\rangle \quad (i \in [1,n],\ h \in V),$$
$$s_i e^{\eta} s_i = e^{s_i(\eta)} \quad (i \in [1,n],\ \eta \in P),$$
$$[h, e^{\eta}] = \kappa \langle \eta \mid h\rangle e^{\eta} + \sum_{\alpha\in R^+} \langle \alpha | h\rangle \frac{(e^{\eta} - e^{s_\alpha(\eta)})}{1 - e^{-\alpha}} s_\alpha \quad (h \in V,\ \eta \in P),$$

where e^{η} denote the element of $\mathbb{C}P$ corresponding to $\eta \in P$.

It is known that $H_\kappa \cong \mathbb{C}P \otimes \mathbb{C}W \otimes S(V)$ as a vector space. The subalgebra $H^{\mathrm{aff}} = \mathbb{C}W \cdot S(V)$ is called the degenerate affine Hecke algebra. Note that the subalgebra $\mathbb{C}P \cdot \mathbb{C}W$ is isomorphic to $\mathbb{C}\widehat{W}$.

3. Induced representations of H_κ

For $\lambda \in X_m = \mathbb{Z}^m$ we write $\lambda \models n$ when $\sum_{i\in[1,m]} \lambda_i = n$ and $\lambda_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [1, m]$. Let $\lambda, \mu \in X_m$ such that $\lambda - \mu \models n$. Introduce the subalgebra $H_\lambda = \mathbb{C}W_{\lambda-\mu}\cdot S(V)$ of H_κ, where $W_{\lambda-\mu}$ denote the parabolic subgroup $\mathfrak{S}_{\lambda_1-\mu_1} \times \cdots \times \mathfrak{S}_{\lambda_m-\mu_m}$ of W.

Let $\mathbb{C}\mathbf{1}_{\lambda,\mu}$ denote the one dimensional representation of $H_{\lambda-\mu}$ such that

$$w\mathbf{1}_{\lambda,\mu} = \mathbf{1}_{\lambda,\mu} \quad (w \in W_{\lambda-\mu}),$$
$$y_i\mathbf{1}_{\lambda,\mu} = \langle \zeta_{\lambda,\mu} \mid y_i\rangle \mathbf{1}_{\lambda,\mu} \quad (i \in [1,n]),$$

where $\zeta_{\lambda,\mu}$ denote the element of V^* given by

(3.1) $$\langle \zeta_{\lambda,\mu} | y_i\rangle = \mu_j + i - m_j - j - 1 \quad \text{for } i \in [m_j + 1, m_{j+1}],$$

with $m_0 = 0$ and $m_j = \sum_{k\in[1,j]}(\lambda_k - \mu_k)$ $(j \in [1,m])$. Define an H_κ-module by $\mathcal{M}_\kappa(\lambda,\mu) = H_\kappa \otimes_{H_{\lambda-\mu}} \mathbb{C}\mathbf{1}_{\lambda,\mu}$. Obviously we have

$$\mathcal{M}_\kappa(\lambda,\mu) \cong \mathbb{C}\widehat{W}/W_{\lambda-\mu} \cong \mathbb{C}P \otimes \mathbb{C}W/W_{\lambda-\mu}$$

as an $\widehat{W}$-module.

In the rest, we often identify the group ring $\mathbb{C}P$ with the Laurent polynomial ring $\mathbb{C}[\underline{z}^{\pm1}] = \mathbb{C}[z_1^{\pm1}, \ldots, z_n^{\pm1}]$ via the correspondence $e^{x_i} \mapsto z_i$.

Example 3.1. Let $m = 1$ and let $\lambda = (n)$ and $\mu = (0)$. Then $\mathcal{M}_\kappa(\lambda,\mu) \cong \mathbb{C}P = \mathbb{C}[\underline{z}^{\pm1}]$, which is called the (Laurent) polynomial representation. On the representation $\mathbb{C}P$, the element y_i $(i \in [1,n])$ acts as the *Cherednik-Dunkl* operator

(3.2) $$T_i = \kappa z_i \frac{\partial}{\partial z_i} + \sum_{1\leq j<i} \frac{z_j}{z_i - z_j}(1 - s_\alpha) + \sum_{i<j\leq n} \frac{z_i}{z_i - z_j}(1 - s_\alpha) + i - 1.$$

The simultaneous eigenvectors of $T_1, \dots, T_n$ are called the nonsymmetric Jack polynomials.

4. The space of coinvariants and the degenerate double affine Hecke algebra

Let $\ell \in \mathbb{C}$. Let M be a highest weight module of level ℓ and let N be a lowest weight module of level $-\ell$. We set

$$\tilde{C}(M,N) = M \otimes E[z_1, z_1^{-1}] \otimes \cdots \otimes E[z_n, z_n^{-1}] \otimes N,$$
$$C(M,N) = \tilde{C}(M,N)/\mathfrak{g}[t,t^{-1}]\tilde{C}(M,N).$$

Let $\sigma_{ij} \in \mathrm{End}_{\mathbb{C}}\mathbb{C}[\underline{z}^{\pm 1}]$ denote the permutation of z_i and z_j. Let $\Omega_{ij} \in \mathrm{End}_{\mathbb{C}}(E^{\otimes n})$ denote the permutation of i-th and j-th component of the tensor product. Note that $\tilde{C}(M,N) \cong M \otimes E^{\otimes n} \otimes \mathbb{C}[\underline{z}^{\pm 1}] \otimes N$, through which we regard σ_{ij} and Ω_{ij} as elements in $\mathrm{End}_{\mathbb{C}}(\tilde{C}(M,N))$.

For $i \in [0, n+1]$, define $\theta_i : \widehat{\mathfrak{g}} \to U(\widehat{\mathfrak{g}})^{\otimes n+2}$ by $\theta_i(u) = 1^{\otimes i} \otimes u \otimes 1^{\otimes n-i+1}$. For $i,j \in [0, n+1]$ with $i<j$, define $\theta_{ij} : \widehat{\mathfrak{g}}^{\otimes 2} \to U(\widehat{\mathfrak{g}})^{\otimes n+2}$ by $\theta_{ij}(u \otimes v) = 1^{\otimes i} \otimes u \otimes 1^{\otimes j-i-1} \otimes v \otimes 1^{\otimes n-j+1}$.

Let e_{ab} denote the matrix unit of $\mathfrak{g}$ with only non-zero entries 1 at the (a,b)-th component. Put $r = \frac{1}{2}\sum_{a \in [1,m]} e_{aa} \otimes e_{aa} + \sum_{1 \le a < b \le m} e_{ab} \otimes e_{ba}$ and put $r_{ij} = \theta_{ij}(r)$.

For $i \in [1,n]$, put

$$\widehat{r}_{0i} = r_{0i} + \sum_{k \in \mathbb{Z}_{\ge 1}} \sum_{a,b \in [1,m],\, a \ne b} \theta_{0i}((e_{ab} \otimes t^k) \otimes (e_{ba} \otimes t^{-k})), \tag{4.1}$$

$$\widehat{r}_{in+1} = r_{in+1} + \sum_{k \in \mathbb{Z}_{\ge 1}} \sum_{a,b \in [1,m],\, a \ne b} \theta_{in+1}((e_{ab} \otimes t^k) \otimes (e_{ba} \otimes t^{-k})), \tag{4.2}$$

which are elements of some completion of $U(\mathfrak{g}[t,t^{-1}])^{\otimes n+2}$ and define well-defined operators on $\tilde{C}(M,N)$.

Define the linear operators on $\tilde{C}(M,N)$ by

$$\begin{aligned} D_i = {} & \kappa z_i \frac{\partial}{\partial z_i} + \widehat{r}_{0i} - \widehat{r}_{in+1} + \sum_{1 \le j < i} r_{ij} - \sum_{i < j \le n} r_{ji} + \theta_i(\rho^\vee) \\ & + \sum_{1 \le j < i} \frac{z_j}{z_i - z_J}(1 - \sigma_{ij})\Omega_{ij} + \sum_{i < j \le n} \frac{z_i}{z_i - z_J}(1 - \sigma_{ij})\Omega_{ij} + i - 1, \end{aligned}$$

where $\rho^\vee = \sum_{k \in [1,m]} \frac{1}{2}(n - 2k + 1)e_{aa} \in \mathfrak{h}$.

Theorem 4.1 (Theorem 4.2.2 in [AST]). *Let M be a highest weight module of $\widehat{\mathfrak{g}}$ of level $\kappa - m$ and let N be a lowest weight module of level $-\kappa + m$.*

(i) There exists a unique algebra homomorphism $\varpi : H_\kappa^{\mathrm{rat}} \to \mathrm{End}_{\mathbb{C}}(\tilde{\mathcal{C}}(M,N))$ *such that*

$$\varpi(s_i) = \Omega_{i\,i+1}\sigma_{i\,i+1}\ (i \in [1, n-1]), \tag{4.3}$$

$$\varpi(e^{x_i}) = z_i\ (i \in [1, n]), \tag{4.4}$$

$$\varpi(y_i) = D_i\ (i \in [1, n]). \tag{4.5}$$

(ii) The H_κ*-action on* $\mathcal{C}(M,N)$ *above preserves the subspace* $\mathfrak{g}[t,t^{-1}]\tilde{\mathcal{C}}(M,N)$*:*

$$\varpi(a)\mathfrak{g}[t,t^{-1}]\tilde{\mathcal{C}}(M,N) \subseteq \mathfrak{g}[t,t^{-1}]\tilde{\mathcal{C}}(M,N)$$

for all $a \in H_\kappa$*. Therefore,* ϖ *induces an* H_κ*-module structure on* $\mathcal{C}(M,N)$.

5. Images of the functor

The following statement has been shown in [AST].

Proposition 5.1 (Proposition 5.3.1 in [AST]). *Let* $\kappa \in \mathbb{C}$ *and put* $\ell = \kappa - m$.
(i) *Let* $\lambda, \mu \in X_m^+$*. Then*

$$\mathcal{C}(\widehat{M}_\ell(\mu), \widehat{M}_\ell^\dagger(\lambda)) \cong \begin{cases} \mathcal{M}_\kappa(\lambda,\mu) & \text{if } \lambda - \mu \models n, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *Let* $\lambda, \mu \in X_m^+(\ell)$ *such that* $\lambda - \mu \models n$*. Then*

$$\mathcal{C}(\widehat{L}_\ell(\mu), \widehat{L}_\ell^\dagger(\lambda)) \cong \mathcal{C}(\widehat{M}_\ell(\mu), \widehat{L}_\ell^\dagger(\lambda)) \cong \mathcal{C}(\widehat{L}_\ell(\mu), \widehat{M}_\ell^\dagger(\lambda)).$$

For each $\lambda \in X_m$, we have the additive functor $F_\lambda(-) = \mathcal{C}(-, \widehat{M}_\ell^\dagger(\lambda))$ from the category of highest weight modules over $\widehat{\mathfrak{g}}$ to the category of H_κ-modules. It is right exact and sends the Verma module $\widehat{M}_\ell(\mu)$ to the induced module $\mathcal{M}_\kappa(\lambda,\mu)$ by Proposition 5.1. In the sequel, we will determine the image $F_\lambda(\widehat{L}_\ell(\mu))$ of the irreducible module $\widehat{L}_\ell(\mu)$ in the case where $\lambda, \mu \in X_m^+(\ell)$. Note that $F_\lambda(\widehat{L}_\ell(\mu)) \cong \mathcal{C}(\widehat{L}_\ell(\mu), \widehat{L}_\ell^\dagger(\lambda))$, and note also that it is a quotient of $F_\lambda(\mathcal{M}_\kappa(\lambda,\mu))$.

Let $\ell \in \mathbb{Z}_{\geq 0}$ and $\lambda, \mu \in X_m^+(\ell)$ such that $\lambda - \mu \models n$. Then it is known that the H_κ-module $\mathcal{M}_\kappa(\lambda,\mu)$ has a unique simple quotient ([AST, S1]), which we will denote by $\mathcal{L}_\kappa(\lambda,\mu)$.

The irreducible modules $\mathcal{L}_\kappa(\lambda,\mu)$ for $\lambda, \mu \in X_m^+(\ell)$ are investigated in [SV], and in particular their structure is described combinatorially using tableaux on periodic skew diagrams. We give a short review of the theory of periodic tableaux and the tableaux representations of H_κ in Appendix. By means of this combinatorial description, we can estimate the kernel of the projection $\mathcal{M}_\kappa(\lambda,\mu) \to \mathcal{L}_\kappa(\lambda,\mu)$. By comparing it with the kernel of $\mathcal{M}_\kappa(\lambda,\mu) \to F_\lambda(\widehat{L}_\ell(\mu))$, we have

Theorem 5.2. *Let $\kappa \in \mathbb{Z}_{\geq 1}$ and put $\ell = \kappa - m$. Let $\lambda, \mu \in X_m^+(\ell)$ such that $\lambda - \mu \models n$. Then the H_κ-module $\mathcal{C}(\widehat{L}_\ell(\mu), \widehat{L}_\ell^\dagger(\lambda))$ is irreducible:*

$$\mathcal{C}(\widehat{L}_\ell(\mu), \widehat{L}_\ell^\dagger(\lambda)) \cong \mathcal{L}_\kappa(\lambda, \mu),$$

and moreover it is semisimple over $S(V)$. (See Theorem A.3 for the combinatorial description of the weight decomposition).

The classification of the irreducible H_κ-modules which are semisimple over $S(V)$ is given in [C2, SV], from which (or from Theorem A.4) we have

Corollary 5.3. *Let $\kappa \in \mathbb{Z}_{\geq 1}$. Let L be an irreducible H_κ-module which is finitely generated and admits a weight decomposition of the form $L = \oplus_{\zeta \in P} L_\zeta$, where $L_\zeta = \{v \in L \mid yv = \langle \zeta \mid y \rangle \ \forall y \in V\}$. Then there exists $m \in [1, n]$ and $\lambda, \mu \in X_m^+(\kappa - m)$ such that $L \cong \mathcal{C}(\widehat{L}_{\kappa-m}(\mu), \widehat{L}_{\kappa-m}^\dagger(\lambda))$.*

6. Localization and conformal coinvariants

We will see the relation between our space $\mathcal{C}(M, N)$ of coinvariants and the space of conformal coinvariants in Wess-Zumino-Witten model [TK, TUY].

Observe that the group ring $\mathbb{C}P$ can be seen as the coordinate ring $A = \mathbb{C}[\mathcal{T}]$ of the affine variety $\mathcal{T} = (\mathbb{C} \setminus \{0\})^n$. Put $\mathcal{T}_\circ = \mathcal{T} \setminus \Delta$, where $\Delta = \cup_{i<j}\{(\xi_1, \ldots, \xi_n) \in \mathcal{T} \mid \xi_i/\xi_j = 1\}$, and put $A_\circ = \mathbb{C}[\mathcal{T}_\circ]$. Namely, $A_\circ$ is the localization of A at Δ; $A_\circ = \mathbb{C}\left[z_1^{\pm 1}, \ldots, z_n^{\pm 1}, \frac{1}{1 - z_i/z_j} \ (i < j)\right]$. Let $\mathcal{D}(\mathcal{T}_\circ)$ denote the ring of algebraic differential operators on $\mathcal{T}_\circ$. Then the Cherednik-Dunkl operators in Example 3.1 $T_1, \ldots, T_n$ can be seen as elements of the ring $\mathcal{D}(\mathcal{T}_\circ) \rtimes \mathbb{C}W$. Put $H_{\kappa,\circ} = A_\circ \otimes_A H_\kappa$. There exists a unique algebra structure on $H_{\kappa,\circ}$ extending H_κ.

Proposition 6.1. *Let $\kappa \in \mathbb{C}^\times$. There exists a unique algebra isomorphism $H_{\kappa,\circ} \to \mathcal{D}(\mathcal{T}_\circ) \rtimes \mathbb{C}W$ such that $y_i \mapsto T_i$, $w \mapsto w$, $f \mapsto f$ for all $i \in [1, n]$, $w \in W$ and $f \in A_\circ$.*

For an H_κ-module M, set $M_\circ = A_\circ \otimes_A M$. Then via Proposition 6.1, we have a structure of $\mathcal{D}(\mathcal{T}_\circ) \rtimes \mathbb{C}W$-module on $M_\circ$; namely, $M_\circ$ admits a W-equivariant integrable (algebraic) connection

$$\nabla_i = \kappa^{-1}\left\{ y_i - \sum_{1 \leq j < i} \frac{z_j}{z_i - z_j}(1 - s_\alpha) - \sum_{i < j \leq n} \frac{z_i}{z_i - z_j}(1 - s_\alpha) - (i - 1) \right\}.$$

Now consider the case where $M = \mathcal{C}(\widehat{L}_\ell^\dagger(\mu), \widehat{L}_\ell(\lambda)) = \mathcal{L}_\kappa(\lambda, \mu)$ with $\lambda, \mu \in X_m^+(\ell)$. Then it follows that the connection given above has regular singularities along Δ, and hence $\mathcal{L}_\kappa(\lambda, \mu)_\circ$ is a projective $A_\circ$-module, or geometrically, a vector bundle over $\mathcal{T}_\circ$ of finite rank ([GGOR, VV]).

For $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{T}_\circ$, let $\mathbb{C}_\xi$ denote the one-dimensional right module of $A_\circ$ given by the evaluation at ξ. It follows that the space $\mathbb{C}_\xi \otimes_{A_\circ} (\mathcal{L}_\kappa(\lambda,\mu)_\circ)$ is isomorphic to with "the space of conformal coinvariants"

$$\left(\widehat{L}_\ell(\mu) \otimes \widehat{L}_\ell(\nu_1)^{\otimes n} \otimes \widehat{L}_\ell(\lambda^\dagger)\right) / \mathfrak{g}_{(0,\xi,\infty)} \left(\widehat{L}_\ell(\mu) \otimes \widehat{L}_\ell(\nu_1)^{\otimes n} \otimes \widehat{L}_\ell(\lambda^\dagger)\right),$$

where $\nu_1 = (1, 0, \dots, 0) \in X_m^+(\ell)$ (the highest weight of the vector representation E), $\lambda^\dagger = -w_0(\lambda)$ with w_0 being the longest element of W, and $\mathfrak{g}_{(0,\xi,\infty)}$ denotes the Lie algebra of $\mathfrak{g}$-valued algebraic functions on $\mathbb{P}^1 \backslash \{0, \xi_1, \dots, \xi_n, \infty\}$, which acts on $\widehat{L}_\ell(\mu) \otimes \widehat{L}_\ell(\nu_1)^{\otimes n} \otimes \widehat{L}_\ell(\lambda^\dagger)$ through the Laurent expansion at each points. (See e.g. [BK] for a precise definition.)

Therefore it follows that the vector bundle $\mathcal{L}_\kappa(\lambda,\mu)_\circ$ is equivalent to the vector bundle of conformal coinvariants (the dual of the vector bundle of conformal blocks in the sense of [TUY, BK]). Moreover, the connection $\{\nabla_i\}$ on $\mathcal{L}_\kappa(\lambda,\mu)_\circ$ given via Proposition 6.1 coincides with the Knizhnik-Zamolodchikov connection on the vector bundle of conformal coinvariants.

7. Weight decomposition of symmetric part

For an H_κ-module M, put

(7.1) $$M^W = \{v \in M \mid wv = v \ \forall w \in W\},$$

on which the algebra $H_\kappa^W = \{u \in H_\kappa \mid wuw^{-1} = u\}$ acts. The algebra H_κ^W is called *the zonal spherical algebra* and it contains a subalgebra $S(V)^W$, which coincides with the center of the degenerate affine Hecke algebra H^{aff}.

For $\zeta \in V^*$, let χ_ζ denote the image of the projection to the quotient space $W\backslash V^*$. Identify $W\backslash V^*$ with the set $\mathrm{Hom}_{algebra}(S(V)^W, \mathbb{C})$ of characters, and set

$$M^W_{[\zeta]} = \{v \in M^W \mid \xi v = \chi_\zeta(\xi) v \ \ \forall \xi \in S(V)^W\}.$$

In the sequel, we will give a decomposition of $\mathcal{L}_\kappa(\lambda,\mu)^W$ into weight spaces with respect to $S(V)^W$.

Let $\lambda, \mu \in X_m^+$ such that $\lambda - \mu \models n$. Let λ/μ denote the skew Young diagram associated with (λ,μ):

(7.2) $$\lambda/\mu = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid \ a \in [1,m],\ b \in [\mu_a + 1, \lambda_a]\}.$$

Let T be a tableau on the diagram λ/μ; namely T is a bijection from λ/μ to $[1,n]$. Then it determines the sequence $\{\lambda_T^{(i)}\}_{i\in[0,n]}$ in X_m by the condition $\lambda_T^{(0)} = \mu$ and $\lambda_T^{(i)}/\lambda_T^{(i-1)} = T^{-1}(i)$ $(i \in [1,n])$.

Let $\ell \in \mathbb{Z}_{\geq 0}$. A tableau T is called an *ℓ-restricted standard tableau* if $\lambda_T^{(i)} \in X_m^+(\ell)$ for all $i \in [1,n]$. Let $\mathrm{St}_{(\ell)}(\lambda,\mu)$ denote the set of ℓ-restricted tableaux on λ.

Let $T \in \mathrm{St}_{(\ell)}(\lambda,\mu)$. For $i \in [1,n]$, define

$$h_i(T) = \begin{cases} 1 & \text{if } a < a', \\ 0 & \text{if } a \geq a', \end{cases} \tag{7.3}$$

where $T(a,b) = i$ and $T(a',b') = i+1$. Define

$$\eta_T = \sum\nolimits_{i\in[1,n]} \left(\sum\nolimits_{j<i} h_j(T)\right) x_i \in P. \tag{7.4}$$

Define $\zeta_T \in V^*$ by $\zeta_T(y_i) = b - a$ when $T(a,b) = i$.

¿From the weight decomposition of $\mathcal{L}_\kappa(\lambda,\mu)$ (Theorem A.3) with respect to $S(V)$, we have

Theorem 7.1. (*Conjecture 6.1.1 in* [AST]) *Let* $\lambda.\mu \in X_m^+(\ell)$ *such that* $\lambda - \mu \models n$. *Then*

$$\mathcal{L}_\kappa(\lambda,\mu)^W = \bigoplus_{\nu\in P^-} \bigoplus_{T\in \mathrm{St}_{(\ell)}(\lambda/\mu)} \mathcal{L}_\kappa(\lambda,\mu)^W_{[\zeta_T+\kappa(\nu+\eta_T)]},$$

where $P^- = \{\zeta \in P \mid \langle \zeta \mid \alpha_i^\vee\rangle \leq 0 \ \forall i \in [1,n-1]\}$, *and*

$$\dim \mathcal{L}_\kappa(\lambda,\mu)^W_{[\zeta_T+\kappa(\nu+\eta_T)]} = 1$$

for all $\nu \in P^-$ *and* $T \in \mathrm{St}_{(\ell)}(\lambda/\mu)$.

8. q-DIMENSION FORMULA

Put $\partial = \kappa^{-1}\sum_{i\in[1,n]} y_i \in S(V)^W$. Then ∂ satisfies the relation

$$[\partial, z_i] = \kappa z_i, \ [\partial, w] = 0$$

for all $i \in [1,n]$ and $w \in W$.

Our next purpose is to give a q-dimension formula for $\mathcal{L}_\kappa(\lambda,\mu)^W$ with respect to the grading operator ∂. To this end, we need to introduce the "polynomial part" of $\mathcal{L}_\kappa(\lambda,\mu)$ following [AST].

Define a subalgebra $H_\kappa^{\geq 0}$ of H_κ by

$$H_\kappa^{\geq 0} = \mathbb{C}P^{\geq 0} \cdot \mathbb{C}W \cdot S(V),$$

where $P^{\geq 0} = \oplus_{i\in[1,n]} \mathbb{Z}_{\geq 0} x_i$.

Let $\kappa \in \mathbb{Z}_{\geq 1}$ and let $\lambda,\mu \in X_m^+(\kappa - m)$ such that $\lambda - \mu \models n$. Recall that the induced module $\mathcal{M}_\kappa(\lambda,\mu)$ is generated by the cyclic vector $\mathbf{1}_{\lambda,\mu}$. We denote by $\bar{\mathbf{1}}_{\lambda,\mu}$ its image under the projection $\mathcal{M}_\kappa(\lambda,\mu) \to \mathcal{L}_\kappa(\lambda,\mu)$. Note that $\bar{\mathbf{1}}_{\lambda,\mu} \neq 0$. Define the *polynomial part* of $\mathcal{L}_\kappa(\lambda,\mu)$ by $\mathcal{L}_\kappa^{\geq 0}(\lambda,\mu) = H_\kappa^{\geq 0}\bar{\mathbf{1}}_{\lambda,\mu}$, which is an $H_\kappa^{\geq 0}$-submodule of $\mathcal{L}_\kappa(\lambda,\mu)$.

Put $\mathcal{L}_\kappa^{\geq 0}(\lambda,\mu)^W_{(k)} = \{v \in \mathcal{L}_\kappa^{\geq 0}(\lambda,\mu)^W \mid \partial v = kv\}$. Then we have $\dim \mathcal{L}_\kappa^{\geq 0}(\lambda,\mu)^W_{(k)} < \infty$ and $\mathcal{L}_\kappa^{\geq 0}(\lambda,\mu)^W = \oplus_{k\in\mathbb{Z}}\mathcal{L}_\kappa^{\geq 0}(\lambda,\mu)^W_{(k)}$. Define

$$\dim{}_q \mathcal{L}_\kappa^{\geq 0}(\lambda,\mu)^W = \sum_{d\in\mathbb{Z}} q^k \dim \mathcal{L}_\kappa^{\geq 0}(\lambda,\mu)^W_{(k)}.$$

Set

$$h(T) = \kappa\langle \eta_T \mid \partial\rangle = \sum_{i\in[1,n]} (n-i)h_i(T). \tag{8.1}$$

¿From Theorem 7.1, we have

Theorem 8.1. *Let $\kappa \in \mathbb{Z}_{\geq 0}$ and let $\lambda,\mu \in X_m^+(\kappa - m)$ such that $\lambda - \mu \models n$. Then*

$$\dim{}_q \mathcal{L}_\kappa^{\geq 0}(\lambda,\mu)^W = \frac{q^{\Delta_\lambda - \Delta_\mu}}{(q)_n} F^{(\ell)}_{\lambda/\mu}(q). \tag{8.2}$$

Here $\Delta_\lambda = \frac{1}{2\kappa}((\lambda,\lambda) + 2(\rho,\lambda))$, $(q)_n = (1-q)(1-q^2)\dots(1-q^n)$ and $F^{(\ell)}_{\lambda/\mu}(q)$ is a polynomial of q given by

$$F^{(\ell)}_{\lambda/\mu}(q) = \sum_{T\in \mathrm{St}_{(\ell)}(\lambda/\mu)} q^{h(T)}. \tag{8.3}$$

Remark 8.2. If ℓ is large enough then $F^{(\ell)}_{\lambda/\mu}(q)$ coincides with the Kostka polynomial $K_{(\lambda/\mu)'(1^n)}(q)$ associated to the conjugate $(\lambda/\mu)'$ of λ/μ. Hence our polynomial $F^{(\ell)}_{\lambda/\mu}(q)$ is an ℓ-restricted version of the Kostka polynomial (cf. [FJKLM]).

Remark 8.3. A bosonic formula for $F^{(\ell)}_{\lambda/\mu}(q)$ is known (Theorem 6.2.4 in [AST]), and Theorem 8.1 is equivalent to the formula in Conjecture 6.1.1 in [AST]. Note also that the bosonic formula suggests the existence of the BGG type resolution of $\mathcal{L}_\kappa(\lambda,\mu)$.

9. Rational analogue

For a $\mathfrak{g}[t]$-module N, set

$$\tilde{\mathcal{C}}(M) = E[z_1]\otimes\cdots\otimes E[z_n]\otimes N$$
$$\mathcal{C}(N) = \tilde{\mathcal{C}}(N)/\mathfrak{g}[t]\tilde{\mathcal{C}}(N),$$

where $E[z] = E\otimes\mathbb{C}[z]$. The analogous construction gives on $\mathcal{C}(N)$ an action of *the rational Cherednik algebra* H^{rat}_κ ([EG]), which can be defined as the subalgebra of H_κ generated by the subalgebra $\mathbb{C}[\underline{z}]\cdot\mathbb{C}W$ and the following (pairwise commutative) elements

$$u_i = z_i^{-1}\left(y_i - \sum\nolimits_{j<i} s_{ij}\right) \quad (i\in[1,n]) \tag{9.1}$$

as pointed out in [S2].

It follows for $\lambda\in X_m^+(\ell)$ that $\mathcal{C}(\widehat{M}_\ell^\dagger(\lambda))$ is isomorphic to some induced module, and $\mathcal{C}(\widehat{L}_\ell^\dagger(\lambda))$ is isomorphic to the unique simple quotient of $\mathcal{C}(\widehat{M}_\ell^\dagger(\lambda))$, which we denote by $\mathcal{L}_\kappa(\lambda)$.

Let $\mathbf{0}=(0,\dots,0)\in X_m^+(\ell)$. Then it follows that the polynomial part $\mathcal{L}_\kappa^{\geq 0}(\lambda,\mathbf{0})$ of the H_κ-module $\mathcal{L}_\kappa(\lambda,\mathbf{0})$ is an H_κ^{rat}-submodule and it is isomorphic to $\mathcal{L}_\kappa(\lambda)$. This leads the q-dimension formula

$$\dim_q \mathcal{L}_\kappa(\lambda)^W = \frac{q^{\Delta_\lambda}}{(q)_n} F_\lambda^{(\ell)}(q). \tag{9.2}$$

Remark 9.1. It can be seen that *the Knizhnik-Zamolodchikov functor* investigated in [GGOR] transforms the irreducible representations $\mathcal{L}_\kappa(\lambda)$ for $\lambda\in X_m^+(\ell)$ to Wenzl's representations [W] of the affine Hecke algebra (cf. [TK]).

Appendix A. Tableaux on periodic diagrams and representations of the degenerate DAHA

We will review the theory of tableaux representations for H_κ, which is investigated in [SV] for the double affine Hecke algebra.

Fix $\kappa\in\mathbb{Z}_{\geq 1}$. Let $m\in\mathbb{Z}_{\geq 1}$.

For $\lambda,\mu\in X_m^+(\kappa-m)$ such that $\lambda-\mu\models n$, we introduce the following subsets of $\mathbb{Z}\times\mathbb{Z}$:

$$\lambda/\mu = \{(a,b)\in\mathbb{Z}\times\mathbb{Z} \mid a\in[1,m],\ b\in[\mu_a+1,\lambda_a]\}, \tag{A.1}$$

$$\widehat{\lambda/\mu} = \{(a,b)+k(m,-\kappa+m)\in\mathbb{Z}\times\mathbb{Z} \mid (a,b)\in\lambda/\mu,\ k\in\mathbb{Z}\}. \tag{A.2}$$

The set $\widehat{\lambda/\mu}$ is called the *periodic skew diagram* of period $(m,-\kappa+m)$ associated with (λ,μ). The following is called the skew property:

Lemma A.1. *Let $(a,b),(a',b')\in\widehat{\lambda/\mu}$. If $a'-a\in\mathbb{Z}_{\geq 0}$ and $b'-b\in\mathbb{Z}_{\geq 0}$ then $(a,b'),(a',b)\in\widehat{\lambda/\mu}$.*

A *tableau* T on $\widehat{\lambda/\mu}$ is by definition a bijection $\widehat{\lambda/\mu}\to\mathbb{Z}$ satisfying $T(a+m,b-\kappa+m)=T(a,b)+n$ for all $(a,b)\in\widehat{\lambda/\mu}$.

A tableau T is called a *standard tableau* if

$$T(a,b+1) > T(a,b)$$

for any $(a,b),(a,b+1) \in \widehat{\lambda/\mu}$, and if

$$T(a+1,b) > T(a,b)$$

for any $(a,b),(a+1,b) \in \widehat{\lambda/\mu}$. Let $\mathrm{Tab}(\widehat{\lambda/\mu})$ and $\mathrm{St}(\widehat{\lambda/\mu})$ denote the set of tableaux and the set of standard tableaux on $\widehat{\lambda/\mu}$ respectively.

Define the elements $\pi = \tau_{x_1} s_1 s_2 \cdots s_{n-1}$ and $s_0 = \tau_{\alpha_{1n}} s_{1n}$ of the group $\widehat{W} = P \rtimes W$. Then $\{s_0, s_1, \ldots, s_{n-1}, \pi\}$ is a generator of the group $\widehat{W}$.

Define the action of $\widehat{W}$ on the set $\mathbb{Z}$ of integers by

$$\text{(A.3)} \qquad s_i(j) = \begin{cases} j+1 & \text{for } j \equiv i \bmod n, \\ j-1 & \text{for } j \equiv i+1 \bmod n, \\ j & \text{for } j \not\equiv i, i+1 \bmod n, \end{cases}$$

$$\text{(A.4)} \qquad \tau_{x_i}(j) = \begin{cases} j+n & \text{for } j \equiv i \bmod n, \\ j & \text{for } j \not\equiv i \bmod n. \end{cases}$$

Observe that $\pi(j) = j+1$ for all j.

For $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$ and $w \in \widehat{W}$, the map $wT : \widehat{\lambda/\mu} \to \mathbb{Z}$ given by

$$(wT)(u) = w(T(u)) \quad (u \in \widehat{\lambda/\mu})$$

is also a tableau on $\widehat{\lambda/\mu}$, and the assignment $T \mapsto wT$ gives an action of $\widehat{W}$ on $\mathrm{Tab}(\widehat{\lambda/\mu})$, which preserves $\mathrm{St}(\widehat{\lambda/\mu})$. It is easy to see that the assignment $w \mapsto wT$ gives a one-to-one correspondence $\widehat{W} \tilde{\to} \mathrm{Tab}(\widehat{\lambda/\mu})$.

Define the map $C : \widehat{\lambda/\mu} \to \mathbb{Z}$ by $C(a,b) = b-a$, and define $C_T : \mathbb{Z} \to \mathbb{Z}$ by $C_T(i) = C(T^{-1}(i))$ for $T \in \mathrm{St}(\widehat{\lambda/\mu})$. Define $\zeta_T \in V^*$ by $\langle \zeta_T \mid y_i \rangle = C_T(i)$ $(i \in [1,n])$.

The following lemma follows from the skew property and the definition of the standard tableaux:

Lemma A.2. *Let* $T \in \mathrm{St}(\widehat{\lambda/\mu})$ *and* $i \in [0, n-1]$.

(i) $C_T(i) - C_T(i+1) \neq 0$.

(ii) $s_i T \in \mathrm{St}(\widehat{\lambda/\mu})$ *if and only if* $C_T(i) - C_T(i+1) \notin \{-1, 1\}$.

Now, we introduce the tableaux representation associated with $\widehat{\lambda/\mu}$. Let $\mathcal{V}_\kappa(\widehat{\lambda/\mu})$ be the vector space with the basis $\{v_T\}_{T \in \mathrm{St}(\widehat{\lambda/\mu})}$:

$$\mathcal{V}_\kappa(\widehat{\lambda/\mu}) = \oplus_{T \in \mathrm{St}(\widehat{\lambda/\mu})} \mathbb{C} v_T.$$

By Lemma A.2 and induction argument, we have

Theorem A.3. (*Theorem 3.16, Theorem 3.17 in* [SV]) *Let* $\kappa \in \mathbb{Z}_{\geq 1}$. *Let* $\lambda, \mu \in X_m^+(\kappa - m)$ *such that* $\lambda - \mu \models n$.

(i) *There exists a unique* H_κ*-module structure on* $\mathcal{V}_\kappa(\widehat{\lambda/\mu})$ *such that*

$$y_i v_T = C_T(i) v_T \quad (i \in [1, n]),$$

$$\pi v_T = v_{\pi T},$$

$$s_i v_T = \begin{cases} \frac{1+a_i}{a_i} v_{s_i T} - \frac{1}{a_i} v_T & \textit{if } s_i T \in \mathrm{St}(\widehat{\lambda/\mu}) \\ -\frac{1}{a_i} v_T & \textit{if } s_i T \notin \mathrm{St}(\widehat{\lambda/\mu}) \end{cases} \quad (i \in [0, n-1]),$$

where $a_i = C_T(i) - C_T(i+1) \neq 0$ (*by Lemma A.2*).

(ii) $\mathcal{V}_\kappa(\lambda, \mu) = \oplus_{T \in \mathrm{St}(\widehat{\lambda/\mu})} \mathcal{V}_\kappa(\lambda, \mu)_{\zeta_T}$, *and* $\mathcal{V}_\kappa(\lambda, \mu)_{\zeta_T} = \mathbb{C} v_T$ *for all* $T \in \mathrm{St}(\widehat{\lambda/\mu})$.

(iii) *The* H_κ*-module* $\mathcal{V}_\kappa(\widehat{\lambda/\mu})$ *is irreducible.*

(iv) $\mathcal{V}_\kappa(\widehat{\lambda/\mu}) \cong \mathcal{L}_\kappa(\lambda, \mu)$.

The following result is also announced in [C2]:

Theorem A.4. (*Theorem 3.19 in* [SV]) *Let* $\kappa \in \mathbb{Z}_{\geq 1}$. *Let* L *be ab irreducible* H_κ*-module such that* $L = \oplus_{\zeta \in P} L_\zeta$. *Then there exist* $m \in [1, n]$ *and* $\lambda, \mu \in X_m^+(\kappa - m)$ *with* $\lambda - \mu \models n$ *such that* $L \cong \mathcal{V}_\kappa(\widehat{\lambda/\mu})$.

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