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On the structure of the party algebra of type B

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Although in the talk at Kyoto-RIMS we used the notation $\tilde{\mathcal{A}}_n$ and $\mathcal{B}_n$ to stand for the party algebra of type $\tilde{A}$ and B respectively, in the following we use the notation $P_{n,\infty}$ and $P_{n,2}(Q)$ respectively since after the talk we have obtained general definition of the party algebra $P_{n,r}(Q)$ which is defined from the centralizer algebra of the unitary reflection group $G(r,1,k)$. This relation is discussed in Section 3.

1 $P_{n,\infty}$: Party algebra of type $\tilde{A}$

Before consider the party algebra of type B $(= P_{n,2}(Q))$, we introduce $P_{n,\infty}$ the party algebra of type $\tilde{A}$.

There also exists the party algebra of type A (no tilde) called the *partition algebra* which corresponds to $P_{n,1}(Q)$ in our notation. However, we do not treat this one, since it was intensively studied in the papers [2, 5, 6, 7].

1.1 Definition of a *seat-plan* of type $\tilde{A}$

Suppose that there exist two parties each of which consists of n members. The parties hold meetings splitting into several small groups. Every group consists of the same number of members of each party. The set of such decompositions into small groups makes an algebra $P_{n,\infty}$ under a certain product and it is called *the party algebra of type* $\tilde{A}$.

More precisely we consider the following situation. Let $F = \{f_1, f_2, \ldots, f_n\}$ and $M = \{m_1, m_2, \ldots, m_n\}$ be two sets each of which consists of n distinct elements such that $F \cap M = \emptyset$. We decompose $F \sqcup M$ into subsets

$$\begin{aligned}\Sigma_n \quad = \quad & \{\{T_1, T_2, \ldots, T_n\} \mid \\ & \bigsqcup_{j=1}^{n} T_j = F \sqcup M,\ |T_1| \geq |T_2| \geq \cdots \geq |T_n|, \\ & |T_j \cap F| = |T_j \cap M| \text{ for } j = 1, 2, \ldots, n, \\ & T_i \cap T_j = \emptyset \text{ if } i \neq j\}\end{aligned}$$

We call such a partition into subsets a *seat-plan* of type $\tilde{A}$. A seat-plan of type $\tilde{A}$ is geometrically expressed. For example

$$\Sigma_5 \ni w_1 = \{\{f_1, f_3, f_4, m_1, m_2, m_4\}, \{f_2, m_5\}, \{f_5, m_3\}\}$$

will be figured as in Figure 1.

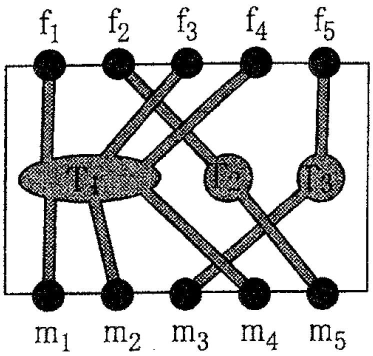

Figure 1: A seat-plan of $P_{n,\infty}$

However this geometrical expression is not unique. For example, the following two figures in Figure 2 express the same seat-plans.

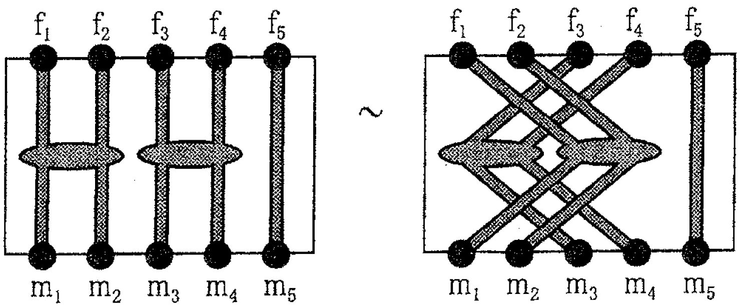

Figure 2: $\{\{f_1, f_2, m_1, m_2,\}, \{f_3, f_4, m_3, m_4\}, \{f_5, m_5\}\}$

Here we consider how many seat-plans of $P_{n,\infty}$ exist for an integer n. Let $P(n)$ be a set of partitions of n. Then there exists a partition $\lambda \in P(n)$ such that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) = (|T_1|/2, |T_2|/2, \ldots, |T_n|/2)$. Then the number of seat-plans is

$$|\sigma_n| = \sum_{\lambda \in P(n)} \left(\frac{n!}{\lambda_1! \lambda_2! \cdots \lambda_n!} \right)^2 \cdot \frac{1}{\alpha_1! \alpha_2! \cdots \alpha_n!},$$

where $\alpha_i = |\{\lambda_k; \lambda_k = i\}|$. For example, we find $|\sigma_3| = 16$ as follows:

$$\begin{aligned} |\sigma_3| &= \sum_{\lambda \in P(n)} \left(\frac{3!}{\lambda_1! \lambda_2! \lambda_3!} \right)^2 \cdot \frac{1}{\alpha_1! \alpha_2! \alpha_3!} \\ &= \left(\frac{3!}{3!0!0!} \right)^2 \cdot \frac{1}{0!0!1!} + \left(\frac{3!}{2!1!0!} \right)^2 \cdot \frac{1}{0!1!1!} + \left(\frac{3!}{1!1!1!} \right)^2 \cdot \frac{1}{3!0!0!} \\ &= 16. \end{aligned}$$

1.2 The set of seat-plans Σ_n makes an algebra $P_{n,\infty}$

For seat-plans $w_1, w_2 \in \Sigma_n$, the product is defined as in Figure 3.

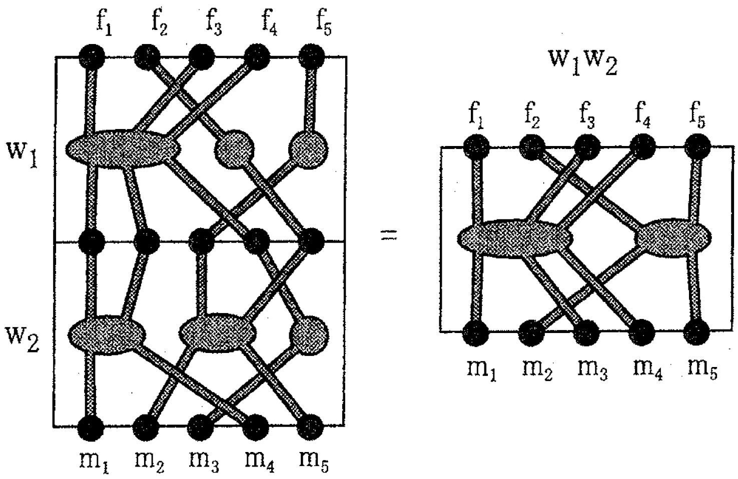

Figure 3: Product of seat-plans

It is easy to see that the identity is given by the seat-plan

$$\{\{f_1, m_1\}, \{f_2, m_2\}, \ldots, \{f_n, m_n\}\},$$

which is figured in Figure 4.

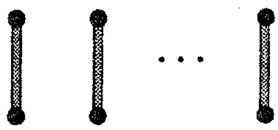

Figure 4: Identity of $\mathcal{A}_n$

We understand that $P_{0,\infty} = P_{1,\infty} = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \ldots, \sqrt{n})$.

1.3 Characterization for the party algebra $P_{n,\infty}$

In the paper [3], we gave a presentation of the party algebra $P_{n,\infty}$ by using *tokoroten* method. According to the paper, the party algebra $P_{n,\infty}$ is generated by the seat-plans in Figure 5 (Here f_i does not express a vertex on the top line), which satisfies the relation illustrated in Figure 6.

In Figure 6, the relation $s_i s_{i+1} f_i s_{i+1} s_i = f_{i+1}$ means $P_{n,\infty}$ is actually generated by $f = f_1$ and the symmetric group $\langle s_1, s_2, \ldots, s_{n-1} \rangle$. Hence we have obtain the following characterization of the party algebra $P_{n,\infty}$
by generators:

$$f, s_1, s_2, \ldots, s_{n-1}$$

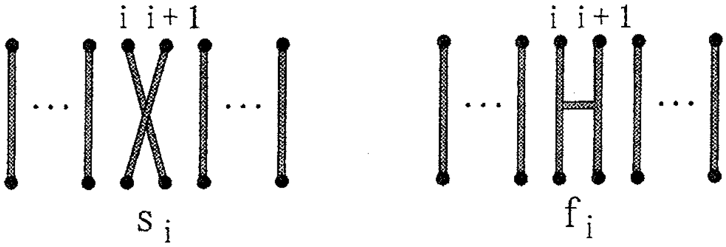

Figure 5: Special elements of $P_{n,\infty}$

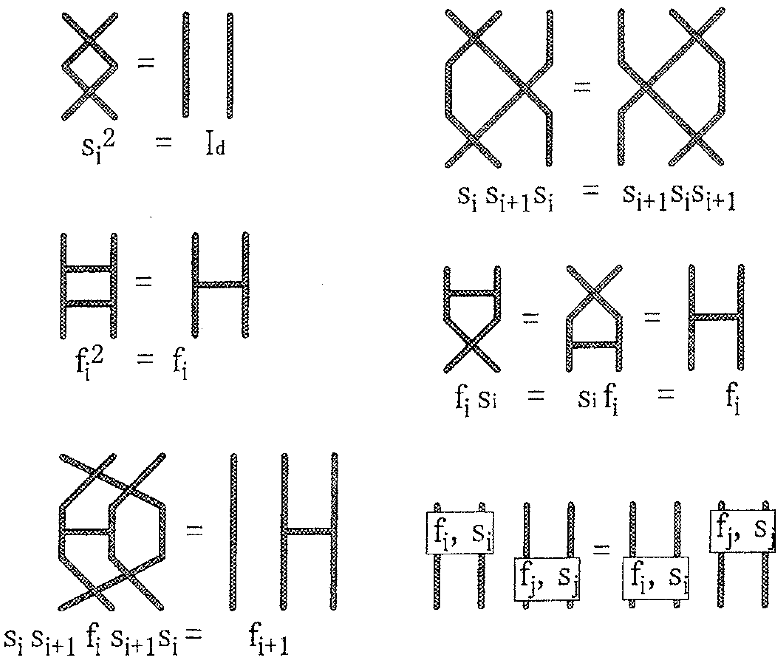

Figure 6: Relations for $P_{n,\infty}$

and relations:

$$
\begin{array}{lll}
s_i^2 = 1 & (1 \le i \le n-1), & (P1) \\
s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & (1 \le i \le n-2), & (P2) \\
s_i s_j = s_j s_i & (|i-j| \ge 2,\ 1 \le i,j \le n-1), & (P3) \\
f^2 = f, & & (P4) \\
f s_1 = s_1 f = f, & & (P5) \\
f s_i = s_i f & (3 \le i \le n-1), & (P6) \\
f s_2 f s_2 = s_2 f s_2 f, & & (P7) \\
f s_2 s_1 s_3 s_2 f s_2 s_1 s_3 s_2 = s_2 s_1 s_3 s_2 f s_2 s_1 s_3 s_2 f. & & (P8)
\end{array}
$$

1.4 Algebraic structure of $P_{n,\infty}$

The algebraic structure of $P_{n,\infty}$ is given by the following Bratteli diagram Γ_n. First we explain how to make Γ_n. Then we define $\mathbb{T}(\alpha)$ the *tableaux of shape* α

for $\boldsymbol{\alpha} \in \Lambda_n(n)$, where $\Lambda_n(n)$ is the set of vertices on the bottom of Γ_n.

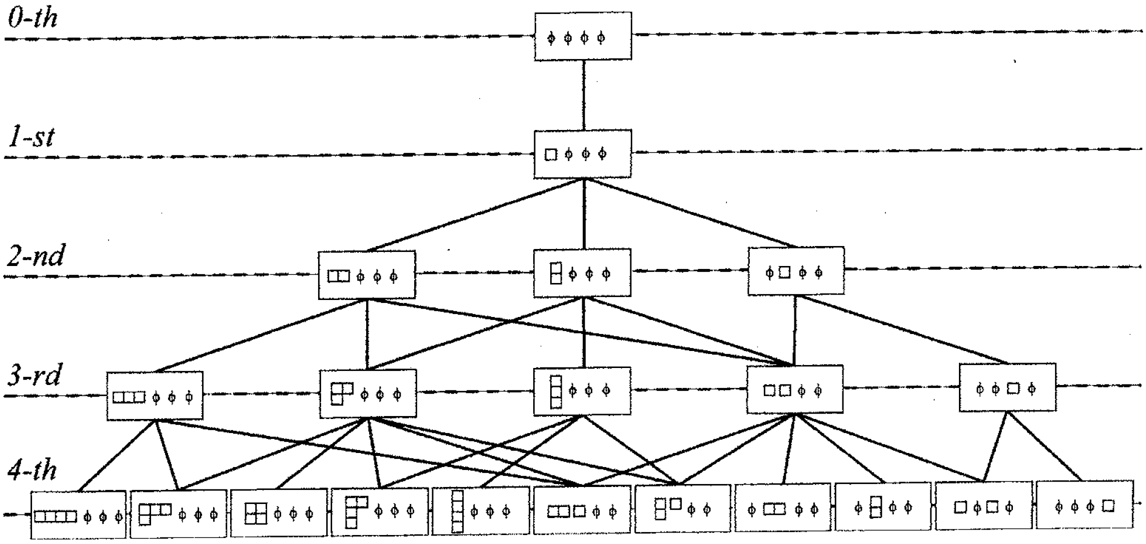

Figure 7: Γ_4 – The Bratteli diagram for the sequence $\{P_{i,\infty}\}_{i=0}^{4}$

The following is the recipe for drawing Γ_n. Fix a positive integer n. Let

$$\boldsymbol{\alpha} = [\alpha(1), \ldots, \alpha(n)]$$

be an n-tuple of Young diagrams. The j-th coordinate of the tuple is referred to *the j-th board*. The *height* $\|\boldsymbol{\alpha}\|$ of $\boldsymbol{\alpha}$ is defined as the weight sum of the sizes of all the $|\alpha(j)|$s. Namely, $\|\boldsymbol{\alpha}\|$ is defined by

$$\|\boldsymbol{\alpha}\| = \sum_{j=1}^{n} j|\alpha(j)|.$$

Let

$$\boldsymbol{\Lambda}_n(i) = \{\boldsymbol{\alpha} = [\alpha(1), \ldots, \alpha(n)] \mid \|\boldsymbol{\alpha}\| = i\}$$

be a set of n-tuples of height i. For $\boldsymbol{\alpha} \in \boldsymbol{\Lambda}_n(i)$, we set $\alpha(0) = n - i$ (the horizontal Young diagram of depth 1 and of width $n - i$) if necessary. Let $\boldsymbol{\alpha} \underset{1}{\prec} \tilde{\boldsymbol{\alpha}}$ or $\tilde{\boldsymbol{\alpha}} \underset{1}{\succ} \boldsymbol{\alpha}$ denote that $\tilde{\boldsymbol{\alpha}}$ is obtained from $\boldsymbol{\alpha}$ by removing one box from the Young diagram on the j-th board and adding the box to the Young diagram on the $(j+1)$-st board for some j $(0 \leq j \leq n-1)$. The diagram Γ_n is defined as the Hasse diagram Γ_n of $\sqcup_{i=0,\ldots,n} \boldsymbol{\Lambda}_n(i)$ with respect to the order generated by $\underset{1}{\succ}$s.

Finally we define the sets of the tableaux on Γ_n. For $\boldsymbol{\alpha} \in \boldsymbol{\Lambda}_n(n)$, The set $\mathbb{T}(\boldsymbol{\alpha})$ of *tableaux of shape* $\boldsymbol{\alpha}$ is defined by

$$\begin{aligned} \mathbb{T}(\boldsymbol{\alpha}) \quad = \quad & \{P = (\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(n)}) \mid \boldsymbol{\alpha}^{(0)} = [\emptyset, \ldots, \emptyset], \boldsymbol{\alpha}^{(n)} = \boldsymbol{\alpha}, \\ & \boldsymbol{\alpha}^{(i)} \underset{1}{\prec} \boldsymbol{\alpha}^{(i+1)} \text{ for } 0 \leq i \leq n-1\}. \end{aligned}$$

Under this preparation we obtain the following theorem.

Theorem 1. *Let $\mathbb{Q}$ be the field of rational numbers and $K_0 = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \ldots, \sqrt{n})$ its extension. If we define $\mathbb{V}(\boldsymbol{\alpha}) = \oplus_{P \in \mathbb{T}(\boldsymbol{\alpha})} K_0 v_P$ as a vector space over K_0 with the standard basis $\{v_P | P \in \mathbb{T}(\boldsymbol{\alpha})\}$, then we have*

$$P_{n,\infty} \cong \bigoplus_{\boldsymbol{\alpha} \in \Lambda_n(n)} \mathrm{End}(\mathbb{V}(\boldsymbol{\alpha})). \tag{1}$$

For the proof of the theorem above we refer the paper [4]. In the paper [4], we constructed concrete isomorphism in the equation (1).

2 $P_{n,2}(Q)$:Party algebra of type B

Next we consider $P_{n,2}(Q)$ the party algebra of type B.

2.1 Definition of a *seat-plan* of type B

Suppose again that there exist two parties each of which consists of n members. The parties hold meetings splitting into several small groups. Every group consists of *even* number of members. Some groups may consist of members of just one of the parties. The set of decompositions into small groups makes an algebra $P_{n,2}(Q)$ and it is called *the party algebra of type B*.

More precisely we consider the following situation. Let $F = \{f_1, f_2, \ldots, f_n\}$ and $M = \{m_1, m_2, \ldots, m_n\}$ be two sets each of which consists of n distinct elements such that $F \cap M = \emptyset$. We decompose $F \sqcup M$ into subsets

$$\begin{aligned} \Sigma_n^B \;=\; & \{\{T_1, T_2, \ldots, T_n\} \mid \\ & \bigsqcup_{j=1}^{n} T_j = F \sqcup M,\ |T_1| \geq |T_2| \geq \cdots \geq |T_n|, \\ & T_i \cap T_j = \emptyset \text{ if } i \neq j \\ & |T_j| \text{ :even, } j = 1, 2, \ldots, n\}. \end{aligned}$$

We call such a partition into subsets a *seat-plan* of type B. A seat-plan of type B is also geometrically expressed. For example

$$\Sigma_5^B \ni w_1 = \{\{f_1, m_1, m_2, m_4\}, \{f_2, m_5\}, \{f_3, f_4\}, \{f_5, m_3\}\}$$

will be figured as in Figure 8.

Similarly to the case of type $\tilde{A}$ we consider how many seat-plans of type B exist for a given integer n.

Since we do not have to distinguish the elements of F and M, the number of seat-plans of type B is given by the following:

$$|\Sigma_n^B| = \sum_{\lambda \in P(n)} \left(\frac{(2n)!}{(2\lambda_1)!(2\lambda_2)! \cdots (2\lambda_n)!} \right)^2 \cdot \frac{1}{\alpha_1! \alpha_2! \cdots \alpha_n!}, \tag{2}$$

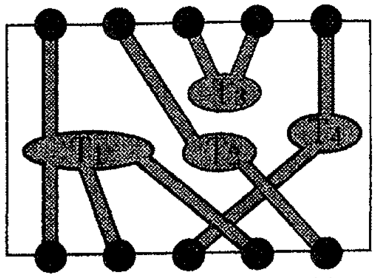

Figure 8: A seat-plan of type B

where $\alpha_i = |\{\lambda_k; \lambda_k = i\}|$.

For example,

$$\begin{aligned} |\Sigma_2^B| &= \frac{4!}{(2\cdot 2)!}\cdot\frac{1}{0!1!} + \frac{4!}{(2\cdot 1)!(2\cdot 1)!}\cdot\frac{1}{2!0!} \qquad (3)\\ &= 1+3=4, \\ |\Sigma_3^B| &= \frac{6!}{(2\cdot 3)!}\cdot\frac{6!}{(2\cdot 2)!(2\cdot 1)!} + \frac{6!}{\{(2\cdot 1)!\}^3}\cdot\frac{1}{3!} \qquad (4)\\ &= 1+15+15=31, \\ |\Sigma_4^B| &= \frac{8!}{8!}+\frac{8!}{6!2!}+\frac{8!}{4!4!}\cdot\frac{1}{2!}+\frac{8!}{4!2!2!}\cdot\frac{1}{2!}+\frac{8!}{(2!)^4}\cdot\frac{1}{4!} \qquad (5)\\ &= 1+28+35+210+105=379. \end{aligned}$$

2.2 The set of seat-plans Σ_n^B also makes an algebra $P_{n,2}(Q)$

For seat-plans $w_1, w_2 \in \Sigma_n^B$, the product is also defined as in Figure 9.

In case d shaded islands occur in the product, first remove holes in the islands (if they exist) and then multiply the resulting diagram by Q^d removing the d islands.

2.3 Characterization for the party algebra $P_{n,2}(Q)$

We can also give a presentation of the party algebra of type B by using the *tokoroten* method. The generators are given by the seat-plans as in Figure 10.

We can easily check that these generators satisfy the relations illustrated in Figure 11. More precisely, we have the following proposition.

Proposition 2. *For an integer $n > 1$, the party algebra $P_{n,2}(Q)$ is characterized*

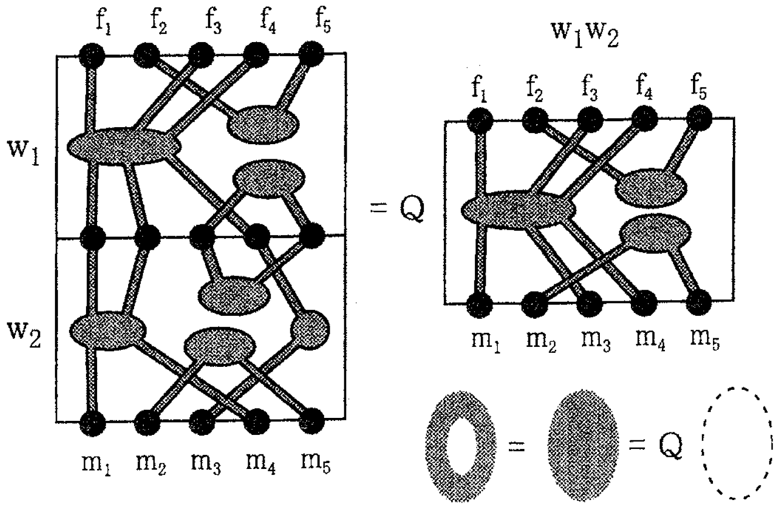

Figure 9: Product of seat-plans of type B

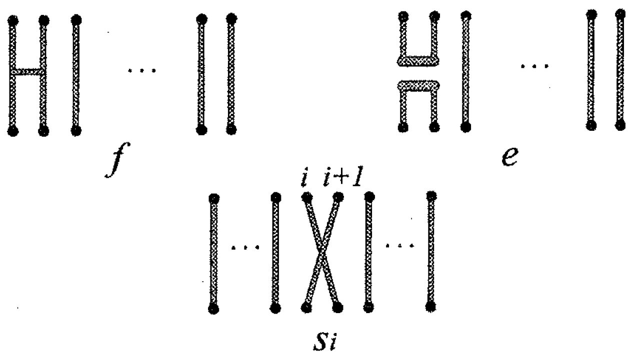

Figure 10: Generators of B

by the following generators and relations:

generators; $e, f, s_1, s_2, \ldots, s_{n-1}$,

relations; $s_i^2 = 1 \quad (1 \leq i \leq n-1)$,

$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (1 \leq i \leq n-2)$,

$s_i s_j = s_j s_i \quad (|i-j| \geq 2)$,

$e^2 = Qe,\ f^2 = f$,

$ef = fe = e,\ es_1 = s_1 e = e,\ fs_1 = s_1 f = f$,

$es_i = s_i e,\ fs_i = s_i f \quad (i \geq 3)$,

$es_2 e = e,\ fs_2 f s_2 = s_2 f s_2 f,\ fs_2 e s_2 f = fs_2 f$,

$x s_2 s_1 s_3 s_2 y s_2 s_1 s_3 s_2 = s_2 s_1 s_3 s_2 y s_2 s_1 s_3 s_2 \quad (x, y \in \{e, f\})$.

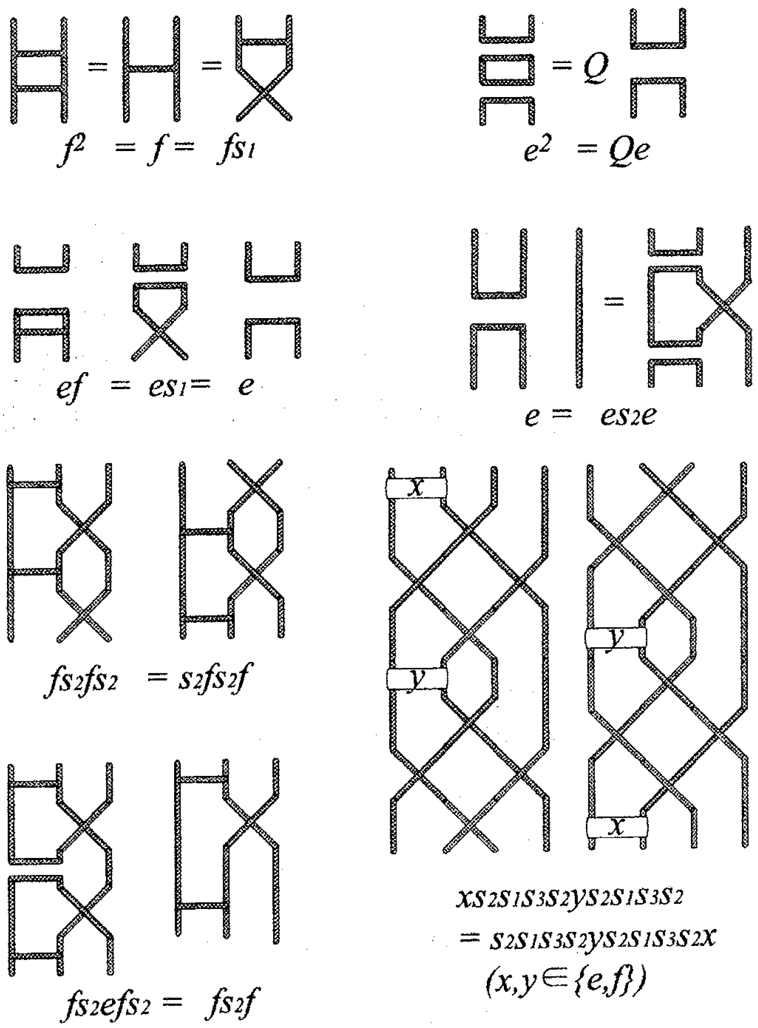

Figure 11: Relations for $P_{n,2}(Q)$

2.4 Algebraic structure of $P_{n,2}(Q)$

The algebraic structure of $P_{n,2}(Q)$ is given by the following:

$$P_{n,2}(Q) \cong \bigoplus_{\beta \in \Lambda_n^B} \mathrm{End}(W(\beta)),$$

and the Bratteli diagram of $P_{n,2}(Q)$ is given by Figure 12.

Here we explain the recipe for drawing the Bratteli diagram. In the following we fix an integer k so that $k \geq n$. First put an vertex indexed by a pair of Young diagram $[(k), \emptyset]$ on the 0-th floor. Then move the right most box in the Young diagram in the left coordinate to the right coordinate. Put a vertex indexed by the resulting pair of Young diagrams under the first vertex (1-st floor) and join these vertices by an edge. The index set of the vertices on the $(i+1)$-st floor $\Lambda_B(i+1)$ is obtained from the index set of the vertices on the i-th floor $\Lambda_B(i)$ by

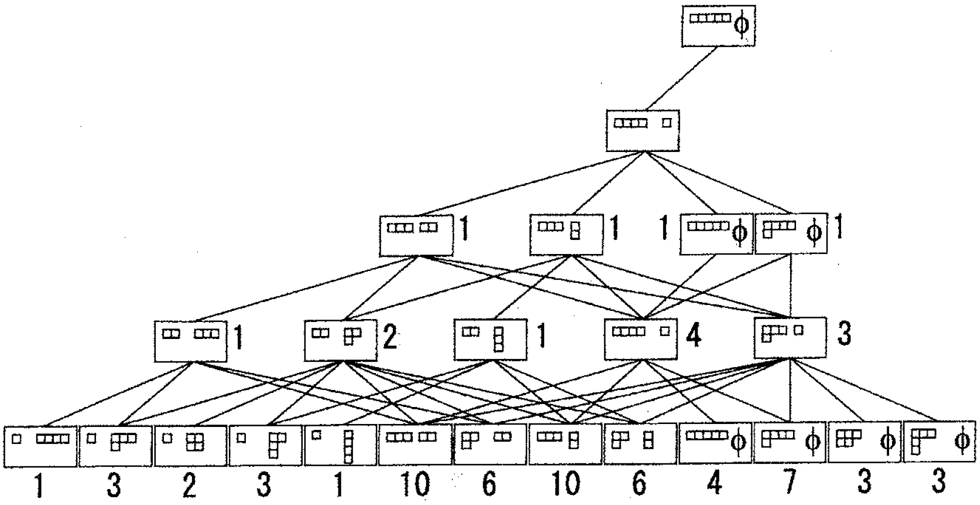

Figure 12: The Bratteli diagram of $P_{4,2}(Q)$

1. move one of the box of the Young diagram on the left coordinate to the Young diagram on the right coordinate, or,

2. move one of the box of the Young diagram on the right coordinate to the Young diagram on the left coordinate,

so that the resulting pair again become a pair of Young diagrams. A vertex on the i-th floor indexed by $\boldsymbol{\beta}_1 \in \Lambda_B(i)$ and another vertex $(i+1)$-st floor indexed by $\boldsymbol{\beta}_2 \in \Lambda_B(i+1)$ is joined by an edge if and only if $\boldsymbol{\beta}_2$ is obtained from $\boldsymbol{\beta}_1$ by the recipe above. Let $\mathbb{T}(\boldsymbol{\beta})$ be the set of paths from the top vertex $\boldsymbol{\beta} = [(k), \emptyset]$ to the vertex $\boldsymbol{\beta}$ on the n-th floor. More precisely we define

$$\begin{aligned}\mathbb{T}(\boldsymbol{\beta}) &= \{P = (\boldsymbol{\beta}^{(0)}, \boldsymbol{\beta}^{(1)}, \ldots, \boldsymbol{\beta}^{(n)}) \mid \boldsymbol{\beta}^{(i)} \in \Lambda_B(i)(0 \le i \le n), \\ &\quad \boldsymbol{\beta}^{(0)} = [(k), \emptyset],\ \boldsymbol{\beta}^{(n)} = \boldsymbol{\beta}, \\ &\quad \boldsymbol{\beta}^{(i)} \text{ and } \boldsymbol{\beta}^{(i+1)} \text{ are joined by an edge } (0 \le i \le n-1)\}.\end{aligned}$$

Let $W(\boldsymbol{\beta}) = \langle v_P | P \in \mathbb{T}(\boldsymbol{\beta})\rangle$ be a vector space over $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \ldots, \sqrt{k})$ whose standard basis is indexed by the elements of $\mathbb{T}(\boldsymbol{\beta})$.

Note that square sums of the numbers on each floor in Figure 12 is equal to the number of seat-plans of type B given in the equations (3)(4) and (5).

2.5 Construction of irreducible representations

For a generator s_i of $P_{n,2}(Q)$, we define a linear map on $V(\boldsymbol{\beta})$ giving a matrix $\mathcal{B}_i$ with respect to the basis $\{v_P | P \in \mathbb{T}(\boldsymbol{\beta})\}$. Namely, for a pair of tableaux $P = (\boldsymbol{\beta}^{(0)}, \boldsymbol{\beta}^{(1)}, \ldots, \boldsymbol{\beta}^{(n)})$ and $Q = (\boldsymbol{\beta}'^{(0)}, \boldsymbol{\beta}'^{(1)}, \ldots, \boldsymbol{\beta}'^{(n)})$ of $\mathbb{T}(\boldsymbol{\beta})$ define $\mathcal{B}_i v_P = \sum_{Q \in \mathbb{T}(\boldsymbol{\beta})} (\mathcal{B}_i)_{QP} v_Q$. If there is an $i_0 \in \{1, 2, \ldots, n-1\} \setminus \{i\}$ such that $\boldsymbol{\beta}^{(i_0)} \neq \boldsymbol{\beta}'^{(i_0)}$, then we put

$$(\mathcal{B}_i)_{QP} = 0.$$

In the following, we consider the case that $\beta^{(i_0)} = \beta'^{(i_0)}$ for $i_0 \in \{1, 2, \ldots, n-1\} \setminus \{i\}$.

First, we consider the case $\beta^{(i)}$ is obtained from $\beta^{(i-1)}$ by moving a box in the Young diagram on the left [resp. right] board to the Young diagram on the other board and $\beta^{(i+1)}$ is obtained from $\beta^{(i)}$ by moving another box in the Young diagram again on the left [resp. right] board to the Young diagram on the other board. Denote the Young diagram on the left board of $\beta^{(i-1)}$ [resp. $\beta^{(i)}$, $\beta^{(i+1)}$] by $\lambda^{(i-1)}$ [resp. $\lambda^{(i)}$, $\lambda^{(i+1)}$] and denote the Young diagram on the right board of $\beta^{(i-1)}$ [resp. $\beta^{(i)}$, $\beta^{(i+1)}$] by $\mu^{(i-1)}$ [resp. $\mu^{(i)}$, $\mu^{(i+1)}$]. Let $\lambda' \underset{1}{\subset} \lambda$ or $\lambda \underset{1}{\supset} \lambda'$ denote that λ' is obtained from λ by removing one box. Recall that if $\nu \underset{1}{\subset} \mu \underset{1}{\subset} \lambda$, then we can define the *axial distance* $d = d(\nu, \mu, \lambda)$. Namely if μ differs from ν in its r_0-th row and c_0-th column only, and if λ differs from μ in its r_1-th row and c_1-th column only, then $d = d(\nu, \mu, \lambda)$ is defined by

$$d = d(\nu, \mu, \lambda) = (c_1 - r_1) - (c_0 - r_0) = \begin{cases} h_\lambda(r_1, c_0) - 1 & \text{if } r_0 \leq r_1, \\ 1 - h_\lambda(r_0, c_1) & \text{if } r_0 > r_1. \end{cases}$$

Here $h_\lambda(i,j)$ is the *hook-length* at (i,j) in λ and for $\lambda = (\lambda_1, \lambda_2, \ldots)$ the hook-length $h_\lambda(i,j)$ is defined by

$$h_\lambda(i,j) = \lambda_i - j + |\{\lambda_l ; \lambda_l \geq j\}| - i + 1.$$

If $\lambda^{(i-1)} \underset{1}{\supset} \lambda^{(i)} \underset{1}{\supset} \lambda^{(i+1)}$, then $\mu^{(i-1)} \underset{1}{\subset} \mu^{(i)} \underset{1}{\subset} \mu^{(i+1)}$. Hence we can define the axial distance $d_1 = d(\lambda^{(i+1)}, \lambda^{(i)}, \lambda^{(i-1)})$ and $d_2 = d(\mu^{(i-1)}, \mu^{(i)}, \mu^{(i+1)})$. If $|d_1| \geq 2$ [resp. $|d_2| \geq 2$], then there is a unique Young diagram $\lambda' \neq \lambda$ [resp. $\mu' \neq \mu$] which satisfies $\lambda^{(i-1)} \underset{1}{\supset} \lambda' \underset{1}{\supset} \lambda^{(i+1)}$ [resp. $\mu^{(i-1)} \underset{1}{\subset} \mu' \underset{1}{\subset} \mu^{(i+1)}$]. Similarly, if $\lambda^{(i-1)} \underset{1}{\subset} \lambda^{(i)} \underset{1}{\subset} \lambda^{(i+1)}$, then $\mu^{(i-1)} \underset{1}{\supset} \mu^{(i)} \underset{1}{\supset} \mu^{(i+1)}$, and we can define the axial distance $d_1 = d(\lambda^{(i-1)}, \lambda^{(i)}, \lambda^{(i)})$ and $d_2 = d(\mu^{(i+1)}, \mu^{(i)}, \mu^{(i-1)})$. If $|d_1| \geq 2$ [resp. $|d_2| \geq 2$], then λ' [resp. $\mu^{(i-1)}$] is defined as before. Let Q_1, Q_2, Q_3 be tableaux of shape β which are obtained from P by replacing $\beta^{(i)} = [\lambda^{(i)}, \mu^{(i)}]$ on the j-th and the $(j+1)$-st board of $\beta^{(i)}$ with $[\lambda^{(i)}, \mu']$, $[\lambda', \mu^{(i)}]$, $[\lambda', \mu']$ respectively. For the basis elements given by the above tableaux, we define the linear map by the following matrix:

$$(v_P, v_{Q_1}, v_{Q_2}, v_{Q_3}) \longmapsto (v_P, v_{Q_1}, v_{Q_2}, v_{Q_3}) \mathcal{B}_i,$$

where

$$\mathcal{B}_i = \begin{pmatrix} \frac{1}{d_1 d_2} & \frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} & \sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} & \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} \\ \frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\frac{1}{d_1 d_2} & \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} \\ \sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} & \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\frac{1}{d_1 d_2} & -\frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} \\ \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} & -\frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} & \frac{1}{d_1 d_2} \end{pmatrix}.$$

Second, we consider the case that the only left boards of $\beta^{(i-1)}$ and $\beta^{(i+1)}$ coincide. Suppose that $\beta^{(i-1)} = [\lambda, \mu]$. Then we can write $\beta^{(i+1)} = [\lambda, \mu']$

($\mu \neq \mu'$). Let $\{\lambda^+_{(r)} | r = 1, 2, \ldots, b(\lambda)\}$ [resp. $\{\lambda^-_{(r')} | r' = 1, 2, \ldots, b(\lambda)'\}$] be the set of all the Young diagrams which satisfy $\lambda^+_{(r)} \underset{1}{\supset} \lambda$ [resp. $\lambda^-_{(r')} \underset{1}{\subset} \lambda$] and let $P_1, P_2, \ldots, P_{b(\lambda)}$ [resp. $Q_1, Q_2, \ldots, Q_{b(\lambda)'}$] be all the tableaux which are obtained from P by replacing $\beta^{(i)}$ with $[\lambda^+_{(r)}, \mu \cap \mu']$ [resp. $[\lambda^-_{(r')}, \mu \cup \mu']$]. For the basis elements given by the above tableaux, we define the linear map by the following matrix:

$$
\begin{aligned}
(\mathcal{B}_i)_{P_r, P_{r'}} &= \sqrt{\frac{h(\lambda)^2}{h(\lambda^+_{(r)}) h(\lambda^+_{(r')})}}, \\
(\mathcal{B}_i)_{P_r, Q_{r'}} &= (\mathcal{B}_i)_{Q_{r'}, P_r} = \frac{1}{d(\lambda^-_{(r')}, \lambda, \lambda^+_{(r)})} \sqrt{\frac{h(\lambda)^2}{h(\lambda^-_{(r')}) h(\lambda^+_{(r)})}}, \\
(\mathcal{B}_i)_{Q_r, Q_{r'}} &= 0.
\end{aligned}
$$

Here $h(\nu)$ is the product of all the hook-lengths in ν:

$$h(\nu) = \prod_{(i,j) \in \nu} h_\nu(i, j).$$

If $\beta^{(i-1)} = [\lambda, \mu]$ and $\beta^{(i+1)} = [\lambda', \mu]$, then the matrix $(\mathcal{B}_i)$ is similarly defined by replacing λ with μ in the argument above. For example, let

$$
\begin{aligned}
P_1 &= ([k, 0], [k-1, 1], [k-2, 2], [1(k-2), 1]) \\
P_2 &= ([k, 0], [k-1, 1], [k-2, 1^2], [1(k-2), 1]) \\
Q_1 &= ([k, 0], [k-1, 1], [1(k-1), 0], [1(k-2), 1])
\end{aligned}
$$

be the tableaux of shape $[1(k-2), 1]$. Then the matrix $\mathcal{B}_2$ with respect to this basis is

$$\begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}.$$

Next, we consider the case $\beta^{(i-1)} = \beta^{(i+1)}$. We put $\beta^{(i-1)} = \beta^{(i+1)} = [\lambda, \mu]$. Let $\{\lambda^+_{(r)}\}$, $\{\lambda^-_{(r')}\}$, $\{\mu^+_{(s)}\}$ and $\{\mu^-_{(s')}\}$ be the sets of Young diagrams previously defined and let $\{Q_{r', s}\}$ and $\{P_{r, s'}\}$ be the sets of tableaux obtained from P by replacing $\beta^{(i)}$ with $[\lambda^-_{(r')}, \mu^+_{(s)}]$ and $[\lambda^+_{(r)}, \mu^-_{(s')}]$ respectively. For the basis elements given by the above tableaux, we define the linear map by the following matrix:

$$(\mathcal{B}_i)_{P,P'} = \begin{cases} \frac{1}{d(\lambda^-_{(r')},\lambda,\lambda^+_{(r)})d(\mu^-_{(s')},\mu,\mu^+_{(s)})}\sqrt{\frac{h(\lambda)^2 h(\mu)^2}{h(\lambda^-_{(r')})h(\lambda^+_{(r)})h(\mu^-(s'))h(\mu^+_{(s)})}} & \\ \qquad\qquad\qquad \text{if } (P,P') = (P_{r,s'},Q_{r',s}) \text{ or } (Q_{r',s},P_{r,s'}), & \\ \sqrt{\frac{h(\lambda)^2}{h(\lambda^+_{(r)})h(\lambda^+_{(r')})}} \quad \text{if } (P,P') = (P_{r,s},P_{r',s}), & \\ \sqrt{\frac{h(\mu)^2}{h(\mu^+_{(s)})h(\mu^+_{(s')})}} \quad \text{if } (P,P') = (Q_{r,s},Q_{r,s'}), & \\ 0 \qquad\qquad\qquad \text{otherwise.} & \end{cases}$$

For example, let

$$\begin{aligned} Q_1 &= ([k,0],[k-1,1],[k-2,2],[k-1,1]) \\ Q_2 &= ([k,0],[k-1,1],[k-2,1^2],[k-1,1]) \\ P_1 &= ([k,0],[k-1,1],[k,0],[k-1,1]) \\ P_2 &= ([k,0],[k-1,1],[1(k-1),0],[k-1,1]) \end{aligned}$$

be the tableaux of shape $[k-1,1]$. Then the matrix $\mathcal{B}_2$ with respect to this basis is

$$\begin{pmatrix} 1/2 & 1/2 & \sqrt{k-1}/\sqrt{2k} & -1/\sqrt{2k} \\ 1/2 & 1/2 & -\sqrt{k-1}/\sqrt{2k} & 1/\sqrt{2k} \\ \sqrt{k-1}/\sqrt{2k} & -\sqrt{k-1}/\sqrt{2k} & 1/k & \sqrt{k-1}/k \\ -1/\sqrt{2k} & 1/\sqrt{2k} & \sqrt{k-1}/k & (k-1)/k \end{pmatrix}.$$

Finally, we consider the remaining cases. In these cases, we can put $\boldsymbol{\beta}^{(i-1)} = [\lambda,\mu]$ and $\boldsymbol{\beta}^{(i+1)} = [\lambda',\mu']$ ($\lambda \neq \lambda', \mu \neq \mu'$ and $|\lambda| = |\lambda'|, |\mu| = |\mu'|$). Then $\boldsymbol{\beta}^{(i)}$ must be of the form $[\lambda \cup \lambda', \mu \cap \mu']$ or $[\lambda \cap \lambda', \mu \cup \mu']$. If $\boldsymbol{\beta}^{(i)}$ if the former [resp. latter] one, then the tableau P' is obtained from P by replacing $\boldsymbol{\beta}^{(i)}$ with the latter [resp. former] one. For the basis elements given by the above tableaux, we define the linear map by the following matrix:

$$(v_P, v_{P'}) \longmapsto (v_P, v_Q)\mathcal{B}_i = (v_P, v_Q)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now we have completed the preparation, we state the following main result.

Theorem 3. *Let $\boldsymbol{\beta} = [\alpha,\beta]$ be an ordered pair of Young diagrams. If $k \geq n$, then the following statements hold:*

(1) Define $\rho_{\boldsymbol{\beta}}$ as follows:

$$
\begin{aligned}
\rho_{\boldsymbol{\beta}}(s_i)v_P &= \sum_{P' \in \mathbb{T}(\boldsymbol{\beta})} (\mathcal{B}_i)_{P'P} v_{P'}, \\
\rho_{\boldsymbol{\beta}}(f)v_P &= \begin{cases} v_P & \text{if } \boldsymbol{\beta}^{(2)} = [(k), \emptyset] \text{ or } [(k-1,1), \emptyset] \\ 0 & \text{otherwise.} \end{cases} \\
\rho_{\boldsymbol{\beta}}(e)v_P &= \begin{cases} kv_P & \text{if } \boldsymbol{\beta}^{(2)} = [(k), \emptyset] \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Then $(\rho_{\boldsymbol{\beta}}, V(\boldsymbol{\beta}))$ defines an irreducible representation of $P_{n,2}(k)$.

(2) For $\boldsymbol{\beta}, \boldsymbol{\beta}' \in \Lambda_B(n)$, the irreducible representations $\rho_{\boldsymbol{\beta}}$ and $\rho_{\boldsymbol{\beta}'}$ of $P_{n,2}(k)$ are equivalent if and only if $\boldsymbol{\beta} = \boldsymbol{\beta}'$.

(3) Conversely, for any irreducible representation ρ of $P_{n,2}(k)$, there exists an $\boldsymbol{\beta} \in \Lambda_B(n)$ such that ρ and $\rho_{\boldsymbol{\beta}}$ are equivalent.

In the process of the construction of $\rho_{\boldsymbol{\beta}}$, even if we replace the positive integer k with an indeterminate Q, the matrix elements of $(\mathcal{B}_i)_{P,P'}$ are similarly defined. This means the theorem above is valid for any generic parameter Q. More over if $Q = k$ and $k \geq n$, then by the Schur-Weyl reciprocity, we find that the dimension of $P_{n,2}(k)$ is equal to the square sum of the degree of $\rho_{\boldsymbol{\beta}}$ and it is also equal to the number of the seat-plans of type B, which is presented by the expression (2). Since the degree of $\rho_{\boldsymbol{\beta}}$ does not vary even if we replace the positive integer k with the indeterminate Q, we obtain the following.

Theorem 4. *If $Q \notin \{0, 1, \ldots, n-1\}$, then the party algebra $P_{n,2}(Q)$ is semisimple and $\{\rho_{\boldsymbol{\beta}}; \boldsymbol{\beta} \in \Lambda_B(n)\}$ gives a complete representatives of irreducible representations of $P_{n,2}(Q)$.*

3 Define $P_{n,r}(Q)$ from the centralizer of the unitary reflection group $G(r, 1, k)$

As we wrote in the beginning of this note, the party algebra $P_{n,r}(Q)$ is defined from the centralizer algebra of the unitary reflection group $G(r, 1, k)$. In this section we explain how the party algebra $P_{n,r}(Q)$ is introduced from the unitary reflection group $G(r, 1, k)$. Although in the paper [10] Tanabe studied the centralizer of the unitary reflection group even for the type $G(r, p, k)$, in the following we consider only the case $p = 1$.

The unitary reflection group $G(r, 1, k)$ is the subgroup of $GL(k, \mathbb{C})$ generated by the set of all permutation matrices of size k and $\mathrm{diag}(\zeta, 1, 1, \ldots, 1)$ where ζ is a primitive r-th root of unity. Let V be the vector space of dimension k and suppose that it has the standard basis $\{e_1, \ldots, e_k\}$. The unitary reflection group $G(r,1,k)$ acts on V naturally and it also acts on $V^{\otimes n}$ diagonally. For $X \in \mathrm{End}V^{\otimes n}$, we denote by $X^{f_1,\dots,f_n}_{m_1,\dots,m_n}$ the matrix coefficients of X with respect to the basis $\{e_{m_1}\otimes\cdots\otimes e_{m_n} \mid m_1,\dots,m_n \in [k]\}$. Since we can write $G(r,1,k) = (\mathbb{Z}/r\mathbb{Z})\wr\mathfrak{S}_k$, in order to check whether X commutes with the action of $G(r,1,k)$ or not we first examine the following action in the tensor space. For $\sigma \in \mathfrak{S}_k$, we have

$$\sigma^{-1}X\sigma(e_{m_1}\otimes\cdots\otimes e_{m_n}) = \sum_{f_1,\dots,f_n\in[k]} X^{\sigma(f_1),\dots,\sigma(f_n)}_{\sigma(m_1),\dots,\sigma(m_n)} e_{f_1}\otimes\cdots\otimes e_{f_n}$$

Hence we have the basis of $\mathrm{End}_{\mathfrak{S}_k}V^{\otimes n}$

$$\left\{T_\sim \middle| \begin{array}{l} \sim \text{ is an equivalence relation on } \{1,\dots 2n\} \\ \text{whose number of classes is less than or equal to } n \end{array}\right\},$$

where

$$(T_\sim)^{m_{n+1},\dots,m_{2n}}_{m_1,\dots,m_n} := \begin{cases} 1 & \text{if } (m_i = m_j \text{ if and only if } i\sim j), \\ 0 & \text{otherwise.} \end{cases}$$

Here we set $m_{n+i} := f_i$ $(1 \le i \le n)$. Note that $\sim$ is zero if the number of classes for $\sim$ is more than k.

In addition to the argument above, considering the action of $\xi \in \mathbb{Z}/r\mathbb{Z}$ we find that the following equivalence relation becomes a basis of the centralizer.

Lemma 5. *Let Π_{2n} be the set of all the partitions of $[2n]$ into subsets. For $B = \{B_1,\dots,B_k\} \in \Pi_{2n}$ (some of the parts may be empty), let bot$(B_i) := B_i \cap [n]$ and top$(B_i) := B_i \cap ([2n]\setminus[n])$ $(1 \le i \le k)$. Let*

$$\Pi_{2n}(r,1,k) := \{B = \{B_1,\dots,B_k\}\ ;\ |top(B_i)| \equiv |bot(B_i)| (mod\ r)(1 \le i \le k)\}.$$

Then $\{T_{\sim B}\ ;\ B \in \Pi_{2n}(r)\}$ is a basis of $\mathrm{End}_{G(r,1,k)}V^{\otimes n}$.

The set Σ_n of seat-plans of type $\tilde{A}$ is equivalent to the set $\Pi_{2n}(r,1,k)$ if $k \ge n$ and $r > n$. The set Σ^B_n of seat-plans of type B is equivalent to the case $r = 2$ and $k \ge n$. In this way we can obtain a basis of the party algebra $P_{n,r}(k)$ and its geometrical presentation. Moreover, replacing k with the parameter Q in case $k \ge n$ in the geometrical definition of the product, we obtain the party algebra $P_{n,r}(Q)$.

We further know the generator of the party algebra $P_{n,r}(Q)$ by Tanabe's paper [10].

Proposition 6. *(Tanabe [10, Theorem 3.1]) The party algebra $P_{n,r}(Q)$ is generated by the symmetric group $\langle s_1, s_2, \dots, s_{n-1}\rangle$ together with f and e_r as in Figure 13*

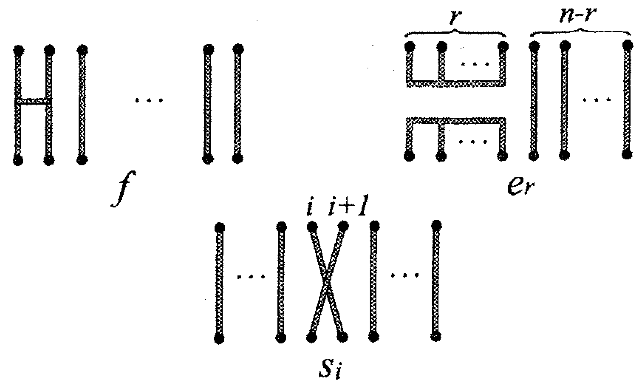

Figure 13: The generators of $P_{n,r}(Q)$

4 $\mathrm{End}_{G(2,1,3)}V^{\otimes n}$

So far, we have assumed that the left coordinate of the top vertex $\beta^{(0)} = [(k), \emptyset]$ has k boxes such that $k \geq n$. It is easy to see that the same diagram will appear even if we begin with $\beta^{(0)} = [(k_1), \emptyset]$ such that $k_1 \geq n$ and $k_1 \neq k$. On the other hand, in case $k_1 < n$, the resulting diagram vary. We mention what happens if we draw a diagram under the condition that $\beta^{(0)} = [(3), \emptyset]$ according to the same recipe. In this situation, we have Figure 14. This corresponds to the centralizer algebra $\mathrm{End}_{G(2,1,3)}V^{\otimes n}$, which is a quotient of the party algebra $P_{n,2}(3)$.

This diagram periodically grows in higher levels. This indicates that this centralizer may give an example of subfactors. Hence we can expect that using this algebra the Turaev-Viro-Ocneanu invariants of 3-dimensional manifolds will be calculated in the same way as in the papers [8, 9].

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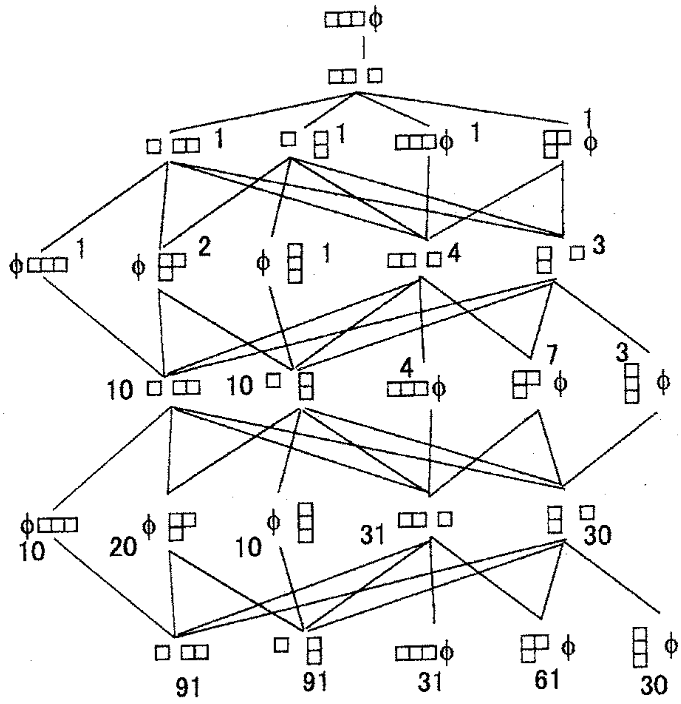

Figure 14: The Bratteli diagram of $\mathrm{End}_{G(2,1,3)}V^{\otimes n}$

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