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FIXED-WIDTH CONFIDENCE INTERVAL ESTIMATION OF A FUNCTION OF TWO EXPONENTIAL SCALE PARAMETERS

1 Introduction

Many researchers are working in the area of sequential estimation in the two-sample exponential case. To cite some recent works, Mukhopadhyay and Chattopadhyay [4] considered the sequential estimation of the difference between means. Sen [5] treated a sequential comparison of two exponential distributions. Uno [6] provided second-order approximations of the expected sample size and the risk of the sequential procedure for the ratio parameter $\theta = \sigma_1/\sigma_2$. Isogai and Futschik [2] dealt with the same parameter $\theta$, using bounded risk estimation. Lim, et al. [3], investigated the construction of sequential confidence intervals for positive functions of the scale parameters. In this paper, we will use the results of Lim, et al. [3] for the function $h(\sigma_1, \sigma_2) = (\sigma_1/\sigma_2)^r$, $r \neq 0$. More specifically for the cases when $r = 1$ and $r = 2$.

Let $h(x, y)$ be a positive, real-valued and three-times continuously differentiable function defined on $\mathbb{R}^2_{+} = (0, +\infty) \times (0, +\infty)$ with $h_x = \frac{\partial}{\partial x} h$, $h_y = \frac{\partial}{\partial y} h$ and $h_x^2(x, y) + h_y^2(x, y) > 0$ on $\mathbb{R}^2_{+}$.

Let $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ be independent observations from two exponential populations $\Pi_1$ and $\Pi_2$, respectively, with their corresponding densities given as follows:

\[ f_1(x) = \sigma_1^{-1} \exp(-x/\sigma_1) I(x > 0) \quad \text{and} \quad f_2(y) = \sigma_2^{-1} \exp(-y/\sigma_2) I(y > 0), \]

where the scale parameters $\sigma_1 > 0$ and $\sigma_2 > 0$ are both unknown and $I(\cdot)$ stands for the indicator function of $(\cdot)$. Taking samples of size $n$ from $\Pi_1$ and $\Pi_2$, we estimate $\theta = h(\sigma_1, \sigma_2)$ by

\[ \hat{\theta}_n = h(\overline{X}_n, \overline{Y}_n), \]
where $\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i$ and $\bar{Y}_n = n^{-1} \sum_{i=1}^{n} Y_i$.

Given $d > 0$ and $\alpha \in (0, 1)$, we want to construct a confidence interval $I_n$ for $\theta = h(\sigma_1, \sigma_2)$ with length $2d$ and coverage probability $1 - \alpha$, based on samples of size $n$, $\{X_1, \cdots, X_n\}$ and $\{Y_1, \cdots, Y_n\}$, from $\Pi_1$ and $\Pi_2$, respectively. Throughout the paper, we shall assume that $\xrightarrow{i.d.}$, $\xrightarrow{P}$ and $\xrightarrow{a.s.}$ stand for convergence in distribution, convergence in probability and almost sure convergence, respectively.

Let us look at the succeeding result which gives the asymptotic distribution of $\hat{\theta}_n = h(\overline{X}_n, \overline{Y}_n)$. This result provides the asymptotic normality of $\sqrt{n}(\hat{\theta}_n - \theta)$.

**Proposition 1.** ([3]) Let a function $g$ on $\mathbb{R}^2_+$ be defined by
\[
g(x, y) = h_x^2(x, y)x^2 + h_y^2(x, y)y^2.
\]
Then
\[
\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, g(\sigma_1, \sigma_2)) \quad \text{as } n \to \infty.
\]

For a given $d > 0$ and $0 < \alpha < 1$, let $I_n = [\hat{\theta}_n - d, \hat{\theta}_n + d]$ be a confidence interval for $\theta$ with length $2d$. This interval $I_n$ must satisfy
\[
P(\theta \in I_n) = P(|\hat{\theta}_n - \theta| \leq d) \geq 1 - \alpha. \tag{1}
\]
Choose $a = a_\alpha > 0$ such that $\Phi(a) = 1 - \alpha/2$, where $\Phi$ is the standard normal distribution function. Set
\[
n^* = \frac{a^2}{d^2} g(\sigma_1, \sigma_2). \tag{2}
\]
Then it follows from Proposition 1 that for all $n \geq n^*$,
\[
P(\theta \in I_n) = P\left(\left|\frac{\sqrt{n}(\hat{\theta}_n - \theta)}{\sqrt{g(\sigma_1, \sigma_2)}}\right| \leq d\frac{\sqrt{n}}{\sqrt{g(\sigma_1, \sigma_2)}}\right) \geq P\left(\left|\frac{\sqrt{n}(\hat{\theta}_n - \theta)}{\sqrt{g(\sigma_1, \sigma_2)}}\right| \leq a\right) \approx 1 - \alpha
\]
if $n^*$ is sufficiently large. For simplicity, assume $n^*$ to be an integer. Then $n^*$ is the asymptotically smallest sample size which approximately satisfies equation (1).

## 2 Main Results

In this section, we will propose a sequential procedure and give its asymptotic properties. We have seen from the previous section that $n^*$ in (2) is the asymptotically smallest sample size. Now, since $\sigma_1$ and $\sigma_2$ are unknown, then $n^*$ is also unknown. It is known that fixed sample size procedures are not available for scale families. Thus, we propose the following stopping rule:
\[
N = N_d = \inf \left\{n \geq m : n \geq \frac{a^2}{d^2} g(\bar{X}_n, \bar{Y}_n)\right\}, \tag{3}
\]
where \( m \geq 2 \) is the initial sample size. Then in view of the SLLN and the definition of \( N_d \), we can show the lemma below.

**Lemma 1. ([3])**

(i) \( P \{ N_d < +\infty \} = 1 \) for each \( d > 0 \).

(ii) \( N_d \xrightarrow{a.s.} +\infty \) as \( d \to 0 \).

(iii) \( N_d/n^* \xrightarrow{a.s.} 1 \) as \( d \to 0 \).

The following proposition gives the asymptotic normality of \( \sqrt{N}(\hat{\theta}_N - \theta) \) which will play the important role in showing the asymptotic consistency of the sequential confidence intervals \( \{I_N\} \).

**Proposition 2. ([3])** As \( d \to 0 \),

\[
\sqrt{N}(\hat{\theta}_N - \theta) \xrightarrow{d} N(0, g(\sigma_1, \sigma_2)),
\]

where

\[
g(\sigma_1, \sigma_2) = h_x^2(\sigma_1, \sigma_2)\sigma_1^2 + h_y^2(\sigma_1, \sigma_2)\sigma_2^2.
\]

Once sampling is stopped after taking \( N \) observations from populations \( \Pi_1 \) and \( \Pi_2 \), respectively, we use the confidence interval \( I_N = [\hat{\theta}_N - d, \hat{\theta}_N + d] \) for \( \theta \). The next result shows the asymptotic consistency of the sequential confidence intervals \( \{I_N\} \).

**Theorem 1. ([3]) [Asymptotic Consistency]**

\[
\lim_{d \to 0} P\{\theta \in I_N\} = 1 - \alpha.
\]

Throughout the remainder of this section, we let

\[
U_i = (X_i - \sigma_1)/\sigma_1, \quad V_i = (Y_i - \sigma_2)/\sigma_2 \quad \text{and} \quad X_i = (U_i, V_i) \quad \text{for} \ i = 1, 2, \ldots .
\]

Consider also the following notations:

\[
Z_{1n} = \sqrt{n}(\bar{X}_n - \sigma_1)/\sigma_1, \quad Z_{2n} = \sqrt{n}(\bar{Y}_n - \sigma_2)/\sigma_2,
\]

\[
D_n = n\bar{U}_n = \sum_{i=1}^{n} U_i = n(\bar{X}_n - \sigma_1)/\sigma_1 = \sqrt{n}Z_{1n},
\]

\[
Q_n = n\bar{V}_n = \sum_{i=1}^{n} V_i = n(\bar{Y}_n - \sigma_2)/\sigma_2 = \sqrt{n}Z_{2n},
\]

\[
S_n = (D_n, Q_n) \quad \text{and} \quad c = \left(-\sigma_1 g_x(\sigma_1, \sigma_2)/g(\sigma_1, \sigma_2), -\sigma_2 g_y(\sigma_1, \sigma_2)/g(\sigma_1, \sigma_2)\right).
\]
Define the function \( f \) on \( \mathbb{R}^2_+ \) as \( f(x, y) = g(\sigma_1, \sigma_2)/g(x, y) \). Since \( g \) is positive and continuous on \( \mathbb{R}^2_+ \), so is \( f \). Then the stopping time \( N \) in (3) can be written as

\[
N = \inf\{n \geq m : Z_n \geq n^*\},
\]

where

\[
Z_n = n f(\overline{X}_n, \overline{Y}_n) = n - \sigma_1 \frac{g_x(\sigma_1, \sigma_2)}{g(\sigma_1, \sigma_2)} D_n - \sigma_2 \frac{g_y(\sigma_1, \sigma_2)}{g(\sigma_1, \sigma_2)} Q_n + \xi_n,
\]

\[
\xi_n = \frac{1}{2} \left\{ \sigma_1^2 f_{xx}(\eta_1, \eta_2) Z_{1n}^2 + 2\sigma_1 \sigma_2 f_{xy}(\eta_1, \eta_2) Z_{1n} Z_{2n} + \sigma_2^2 f_{yy}(\eta_1, \eta_2) Z_{2n}^2 \right\},
\]

\( \eta_1 \) and \( \eta_2 \) are random variables satisfying \( |\eta_1 - \sigma_1| < |\overline{X}_n - \sigma_1| \) and \( |\eta_2 - \sigma_2| < |\overline{Y}_n - \sigma_2| \). In the notations of Aras and Woodroofe [1], we can rewrite (5) as

\[
Z_n = n + \langle \mathbf{c}, S_n \rangle + \xi_n,
\]

where \( \langle \cdot, \cdot \rangle \) denotes inner product. Let

\[
T = \inf\{n \geq 1 : n + \langle \mathbf{c}, S_n \rangle > 0\} \quad \text{and} \quad \rho = \frac{E\{(T + \langle \mathbf{c}, S_T \rangle)^2\}}{2E\{T + \langle \mathbf{c}, S_T \rangle\}}.
\]

Consider the following assumptions:

(A1) \[ \left\{ \left[ \left( Z_n - \frac{n}{\epsilon_0} \right)^+ \right]^{\frac{3}{2}}, n \geq m \right\} \text{ is uniformly integrable for some } 0 < \epsilon_0 < 1, \]

where \( x^+ = \max(x, 0) \).

(A2) \[ \sum_{n=m}^{\infty} n P\{\xi_n < -\epsilon_1 n\} < \infty \quad \text{for some } 0 < \epsilon_1 < 1. \]

The following theorem gives the second-order approximation of the expected sample size \( E(N) \).

**Theorem 2.** ([3]) \( \text{If (A1) and (A2) hold, then} \)

\[
E(N) = n^* + \rho - \nu + o(1) \quad \text{as } d \to 0,
\]

where

\[
\nu = \frac{\sigma_1^2 f_{xx}(\sigma_1, \sigma_2) + \sigma_2^2 f_{yy}(\sigma_1, \sigma_2)}{2}
\]

and \( \rho \) in (6) satisfies

\[
0 < \rho < \{1 + \langle \mathbf{c}, \mathbf{c} \rangle\}/2.
\]
3 Example

We consider the estimation of the $r$th power of the ratio of two scale parameters, namely, $\theta = h(\sigma_1, \sigma_2) = (\sigma_1 / \sigma_2)^r$ for $r \neq 0$. Theorem 3 that follows, gives the expected sample size of the sequential procedure for the given function $\theta$.

**Theorem 3.** If $m > \max\{1, 6|r|\}$, then

$$E(N) = n^* + \rho - 4r^2 + o(1) \quad \text{as} \quad d \to 0,$$

where $\rho$ in (6) satisfies

$$0 < \rho < \frac{1 + 8r^2}{2}.$$

**Proof.** For this function, the stopping random variable $N$ in (4) can be written as

$$N = \inf\{n \geq m : Z_n \geq n^*\},$$

where

$$Z_n = n - 2r(D_n - Q_n) + \xi_n$$

and

$$\xi_n = r\theta^2 \left( \frac{\eta_2}{\eta_1} \right)^{2r} \left\{ (2r + 1) \frac{\sigma_1^2}{\eta_1^2} Z_{1n}^2 - 4r \frac{\sigma_1 \sigma_2}{\eta_1 \eta_2} Z_{1n} Z_{2n} + (2r - 1) \frac{\sigma_2^2}{\eta_2^2} Z_{2n}^2 \right\},$$

$\eta_1$ and $\eta_2$ are random variables satisfying $|\eta_1 - \sigma_1| < |X_n - \sigma_1|$ and $|\eta_2 - \sigma_2| < |Y_n - \sigma_2|$. In the notations of Aras and Woodroofe [1], we can rewrite (7) as

$$Z_n = n + \langle c, S_n \rangle + \xi_n,$$

where $c = (-2r, 2r)$. In order to use Theorem 2 to determine the expected sample size, we need to satisfy conditions (A1) and (A2) of the theorem. Let $u > 1$ and $v > 1$ be such that $u^{-1} + v^{-1} = 1$ and $M$ a generic positive constant.

To prove (A1), it suffices to show that

$$\sup_{n \geq m} E \left\{ \left( \frac{Z_n - n}{\epsilon_0} \right)^3 \right\} < \infty.$$

Now

$$(Z_n - n/\epsilon_0)^+ = n \left\{ \left[ (V_n + 1)/(U_n + 1) \right]^{2r} - \epsilon_0^{-1} \right\} I\left\{ \left[ (V_{n+1})/(U_{n+1}) \right]^{2r} > \epsilon_0^{-1} \right\}.$$

Thus,

$$E \left\{ (Z_n - n/\epsilon_0)^+^3 \right\} \leq n^3 E \left\{ \left[ (V_n + 1)/(U_n + 1) \right]^{6r} I\left\{ \left[ (V_{n+1})/(U_{n+1}) \right]^{2r} > \epsilon_0^{-1} \right\} \right\}$$

$$\leq n^3 E \left\{ \left[ (V_n + 1)/(U_n + 1) \right]^{6r} I\left\{ \left[ (V_{n+1})/(U_{n+1}) \right]^{2r} > \epsilon_0^{-1}, U_{n+1} < 1 - \epsilon_0 \right\} \right\}$$

$$+ n^3 E \left\{ \left[ (V_n + 1)/(U_n + 1) \right]^{6r} I\left\{ \left[ (V_{n+1})/(U_{n+1}) \right]^{2r} > \epsilon_0^{-1}, U_{n+1} \geq 1 - \epsilon_0 \right\} \right\}$$

$$\equiv K_1(n) + K_2(n),$$

say.
By the independence of $\overline{U}_n$ and $\overline{V}_n$ and by Hölder's Inequality, we have $K_1(n) \leq n^3 E(\overline{V}_n + 1)^{6r} \left\{ E(\overline{U}_n + 1)^{-6ru} \right\}^{1/u} \{ P(\overline{U}_n > \epsilon_0) \}^{1/v}$. By Lemma 1 of Uno [6], $E(\overline{V}_n + 1)^{6r} \leq M$ and $E(\overline{U}_n + 1)^{-6ru} \leq M$ for $n \geq m > 6|r|u$. By Markov's Inequality, $P(\overline{U}_n > \epsilon_0) \leq (n\epsilon_0)^{-q} E|D_n|^q$ for $q \geq 2$. But by Marcinkiewicz-Zygmund Inequality, $E|D_n|^q = O(n^{q/2})$ as $n \to \infty$. Thus, it follows that $K_1(n) \leq M n^{3-q/2v}$ for $n \geq m > 6|r|u$. Since $m > 6|r|$, we can choose $u > 1$ such that $m > 6|r|u$. Then choose $q \geq \max\{2, \frac{6u}{u-1}\}$. Thus, $3 - q/2v \leq 0$ which shows that $\sup_{n \geq m} K_1(n) < \infty$. Let $\delta = \epsilon_0^{-1/2r} (1 - \epsilon_0) > 1$ and $r > 0$ for small $0 < \epsilon_0 < 1$. Then

$$\left\{ (\overline{V}_n + 1)/\overline{U}_n + 1 \right\}^{2r} > \epsilon_0^{-1}, \overline{U}_n + 1 \geq 1 - \epsilon_0 \right\} \subset \{ \overline{V}_n + 1 \geq \delta \}.$$  

It follows that for $r > 0$,

$$K_2(n) \leq n^3 (1 - \epsilon_0)^{-6r} E \left\{ (\overline{V}_n + 1)^{6r} I_{\{\overline{V}_n + 1 \geq \delta \}} \right\}$$

$$\leq n^3 (1 - \epsilon_0)^{-6r} \left\{ E(\overline{V}_n + 1)^{6ru} \right\}^{1/u} \{ P(\overline{V}_n + 1 \geq \delta) \}^{1/v}$$

$$\leq n^3 (1 - \epsilon_0)^{-6r} \left\{ E(\overline{V}_n + 1)^{6ru} \right\}^{1/u} \{ P(|\overline{V}_n| \geq \delta - 1) \}^{1/v},$$

where $\frac{1}{u} + \frac{1}{v} = 1$ and $u > 1$. Thus, in the same way as $K_1(n)$, $\sup_{n \geq m} K_2(n) < \infty$ for $m > 6r$. For $r < 0$, by similar arguments as above, $\sup_{n \geq m} K_2(n) < \infty$ for $m > 6|r|$. This completes the proof of (A1).

By Taylor’s Theorem,

$$(\overline{V}_n + 1)^{2r} (\overline{U}_n + 1)^{-2r}$$

$$= \left( 1 + 2r \overline{V}_n + r(2r - 1) \phi_2^{2(r-1)} \overline{V}_n^2 \right) \left( 1 - 2r \overline{U}_n + r(2r + 1) \phi_1^{-2(r+1)} \overline{U}_n^2 \right),$$

where $\phi_1$ and $\phi_2$ are positive random variables between $(\overline{U}_n + 1)$ and 1, and $(\overline{V}_n + 1)$ and 1, respectively. Thus, it follows from (7) that

$$\xi_n = Z_n - n + 2r(D_n - Q_n) = n \left\{ (\overline{V}_n + 1)^{2r} (\overline{U}_n + 1)^{-2r} - 1 + 2r(\overline{U}_n - \overline{V}_n) \right\}$$

$$= n \left[ -4r^2 \overline{U}_n \overline{V}_n + r(2r + 1) \phi_1^{-2(r+1)} \overline{U}_n^2 + 2r^2(2r + 1) \phi_1^{-2(r+1)} \overline{U}_n \overline{V}_n \right]$$

$$+ n \left[ r(2r - 1) \phi_2^{2(r-1)} \overline{V}_n^2 - 2r^2(2r - 1) \phi_2^{2(r-1)} \overline{U}_n \overline{V}_n \right]$$

$$+ r^2(4r^2 - 1) \phi_1^{-2(r+1)} \phi_2^{2(r-1)} \overline{U}_n \overline{V}_n^2 \right\}.$$  

Thus, setting $\epsilon_2 = \epsilon_1/6$ for $0 < \epsilon_1 < 1$, we have
$P \{ \xi_n < -\epsilon_1 n \} \\
\leq P \left\{ \left| 4r^2 \overline{U}_n \overline{V}_n \right| > \epsilon_2 \right\} + P \left\{ \left| r(2r+1) \phi_1^{-2(r+1)} \overline{U}_n^2 \right| > \epsilon_2 \right\} \\
+ P \left\{ \left| 2r^2 (2r+1) \phi_1^{2(r+1)} \overline{U}_n \overline{V}_n \right| > \epsilon_2 \right\} + P \left\{ \left| r(2r-1) \phi_2^{2(r-1)} \overline{V}_n^2 \right| > \epsilon_2 \right\} \\
+ P \left\{ \left| 2r^2 (2r-1) \phi_2^{2(r-1)} \overline{U}_n \overline{V}_n^2 \right| > \epsilon_2 \right\} + P \left\{ \left| r^2 (4r^2 - 1) \phi_1^{-2(r+1)} \phi_2^{2(r-1)} \overline{U}_n^2 \overline{V}_n^2 \right| > \epsilon_2 \right\} \\
\equiv \sum_{i=1}^{6} I_i(n), \text{ say.}$

By the independence of $\overline{U}_n$ and $\overline{V}_n$, and by Marcinkiewicz-Zygmund Inequality, $E\{|D_n Q_n|^q\} = E\{|D_n|^q\} E\{|Q_n|^q\} \leq M n^q$, for $q \geq 2$. Thus, by Markov's Inequality,

$I_1(n) = P \left\{ 4r^2 |D_n Q_n| > n^2 \epsilon_2 \right\} \leq M n^{-2q} E\{|D_n Q_n|^q\} \leq M n^{-q}$.

Now, since $\phi_1$ is a random variable between 1 and $\overline{U}_n + 1$, then $\phi_1 > 1/2$ on the set $\{ |\overline{U}_n| \leq 1/4 \}$. Thus, for $r + 1 \geq 0$, we have

$I_2(n) \leq P \left\{ M \left| \phi_1^{-2(r+1)} \overline{U}_n^2 \right| > \epsilon_2, |\overline{U}_n| \leq 1/4 \right\} + P \left\{ |\overline{U}_n| > 1/4 \right\} \\
\leq P \left\{ M (1/2)^{-2(r+1)} (1/2)^2 |\overline{U}_n| > \epsilon_2 \right\} + P \left\{ |\overline{U}_n| > 1/4 \right\} \\
\leq P \left\{ |\overline{U}_n| > M \right\} + M n^{-q/2} \leq M n^{-q/2}$.

In a similar way, we get

$I_3(n) \leq P \left\{ M \left| \phi_1^{-2(r+1)} \overline{U}_n \overline{V}_n \right| > \epsilon_2, |\overline{U}_n| \leq 1/4 \right\} + P \left\{ |\overline{U}_n| > 1/4 \right\} \\
\leq P \left\{ M (1/2)^{-2(r+1)} (1/2)^2 |\overline{U}_n| > \epsilon_2 \right\} + P \left\{ |\overline{U}_n| > 1/4 \right\} \\
\leq P \left\{ |\overline{U}_n| > M \right\} + M n^{-q/2} \leq M n^{-q/2}$.

Suppose that $r + 1 < 0$. Then it follows by convexity and Lemma 1 of Uno [6] that for any $q \geq 2$

$E \left\{ \phi_1^{-4(r+1)q} \right\} \leq 1 + E \left[ (\overline{U}_n + 1)^{-4(r+1)q} \right] \leq M$.

Thus,

$I_2(n) \leq M E \left\{ \phi_1^{-4(r+1)q} \right\}^{1/2} E \left\{ |\overline{U}_n|^{4q} \right\}^{1/2} \leq M n^{-q}$.

From (8), we obtain

$I_3(n) \leq M E \left\{ \phi_1^{-2(r+1)} |\overline{U}_n \overline{V}_n|^2 \right\} E \left\{ |\overline{V}_n|^q \right\} \\
\leq M n^{-q} n^{-q/2} \leq M n^{-q/2}$. 

Thus, from the above relations, $I_i(n) \leq Mn^{-q/2}$ for $i = 1, 2, 3$. Hence, taking $q = 6$, we have $\sum_{n=1}^{\infty} n I_i(n) < \infty$ for $i = 1, 2, 3$. By similar arguments, we can show that $\sum_{n=1}^{\infty} n I_i(n) < \infty$ for $i = 4, 5, 6$. Therefore, (A2) is satisfied. Now, $\nu = 4r^2$. Hence, it follows from Theorem 2 that for $m > \max\{1, 6|r|\}$,

$$E(N) = n^* + \rho - 4r^2 + o(1) \quad \text{as } d \to 0,$$

where $0 < \rho < (1 + 8r^2)/2$. This completes the proof. □

To illustrate these results, let us consider two cases. For the case when $r = 1$, we consider two stopping rules; $N$ in (3) and $N^*$ given in Isogai and Futschik [2], and compare the coverage probabilities of the sequential confidence intervals, corresponding to $N$ and $N^*$. The stopping rule $N$ becomes

$$N = N_d = \inf \left\{ n \geq m : n \geq \frac{2a^2\bar{X}_n^2}{d^2\bar{Y}_n^2} \right\}.$$

Then, letting $L(n) \equiv 1$ and replacing $w$ by $d^2/a^2$, $N$ in (4) is the same as $N_w$ in Isogai and Futschik [2] who also showed that (A1) and (A2) hold with $m > 6$ and $c = (-2, 2)$. Thus, it follows from Theorem 2 that

$$E(N) = n^* + \rho - 4 + o(1) \quad \text{and} \quad 0 < \rho < 9/2.$$

By simulation, we can get $\rho = 2.03$. Thus, taking this $\rho$ into account, we consider another stopping rule:

$$N^* = N^*_d = \inf \left\{ n \geq m : n \geq L(n) \frac{2a^2\bar{X}_n^2}{d^2\bar{Y}_n^2} \right\} \quad \text{where } L(n) = 1 + \frac{1.97}{n}.$$

From Theorem 2.1 of Isogai and Futschik [2], if $m > 6$ then $E(N^*) = n^* + o(1)$ as $d \to 0$.

Now, from Proposition 2.1 of Isogai and Futschik [2] if $m > 12$, then

$$E(\hat{\theta}_N) - \theta = -\frac{3d}{a\sqrt{2n^*}} + o(d^2) \quad \text{as } d \to 0.$$

From this result, we propose the following bias-corrected sequential confidence intervals:

$$I_N^\uparrow = [\hat{\theta}_N^\uparrow - d, \hat{\theta}_N^\uparrow + d] \quad \text{and} \quad I_N^\downarrow = [\hat{\theta}_N^\downarrow - d, \hat{\theta}_N^\downarrow + d],$$

where $\hat{\theta}_N^\uparrow = \hat{\theta}_N + (3d)/(a\sqrt{2N})$ and $\hat{\theta}_N^\downarrow = \hat{\theta}_{N^*} + (3d)/(a\sqrt{2N^*})$.

For the case when $r = 2$, the stopping rule in (3) becomes

$$N = N_d = \inf \left\{ n \geq m : n \geq \frac{8a^2\bar{X}_n^4}{d^2\bar{Y}_n^4} \right\},$$
and by Theorem 3, for $m > 12$, the expected sample size is

$$E(N) = n^* + \rho - 16 + o(1) \quad \text{and} \quad 0 < \rho < 33/2.$$ 

Now, by simulation using 100,000 repetitions, we can get $\rho = 4.02$. Considering this value for $\rho$, we propose another stopping rule as follows:

$$N^* = N_d^* = \inf \left\{ n \geq m : n \geq L(n) \frac{8a^2\overline{X}_n^4}{d^2\overline{Y}_n^4} \right\}, \quad L(n) = 1 + \frac{11.98}{n}.$$ 

Simulation Results. We shall give simulation results for the case when $(\sigma_1, \sigma_2) = (2, 1)$. The coverage probability is set at $1 - \alpha = 0.95$ and the pilot sample size at $m = 13$. The following results are based on 10,000 repetitions.

<table>
<thead>
<tr>
<th>Table 1.1</th>
<th>Using $N \ (r = 1)$</th>
<th>$\theta = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n^*$</td>
<td>20</td>
<td>100</td>
</tr>
<tr>
<td>$d$</td>
<td>1.239588</td>
<td>0.554360</td>
</tr>
<tr>
<td>$E(N)$</td>
<td>21.4789</td>
<td>96.7799</td>
</tr>
<tr>
<td>$E(\hat{\theta}_N)$</td>
<td>1.865092</td>
<td>1.917183</td>
</tr>
<tr>
<td>$E(\hat{\theta}^*_N)$</td>
<td>2.172344</td>
<td>1.981733</td>
</tr>
<tr>
<td>$P(\theta \in I_N)$</td>
<td>0.9864</td>
<td>0.9079</td>
</tr>
<tr>
<td>$P(\theta \in I^*_N)$</td>
<td>0.9878</td>
<td>0.9241</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 1.2</th>
<th>Using $N^* \ (r = 1)$</th>
<th>$\theta = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n^*$</td>
<td>20</td>
<td>100</td>
</tr>
<tr>
<td>$d$</td>
<td>1.239588</td>
<td>0.554360</td>
</tr>
<tr>
<td>$E(N^*)$</td>
<td>22.7216</td>
<td>98.7350</td>
</tr>
<tr>
<td>$E(\hat{\theta}^*_N)$</td>
<td>1.860984</td>
<td>1.920277</td>
</tr>
<tr>
<td>$E(\hat{\theta}^*_N)$</td>
<td>2.160043</td>
<td>1.983678</td>
</tr>
<tr>
<td>$P(\theta \in I^*_N)$</td>
<td>0.9881</td>
<td>0.9122</td>
</tr>
<tr>
<td>$P(\theta \in I^*_N)$</td>
<td>0.9883</td>
<td>0.9271</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2.1</th>
<th>Using $N \ (r = 2)$</th>
<th>$\theta = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n^*$</td>
<td>20</td>
<td>100</td>
</tr>
<tr>
<td>$d$</td>
<td>4.958350</td>
<td>2.217442</td>
</tr>
<tr>
<td>$E(N)$</td>
<td>23.7237</td>
<td>83.0451</td>
</tr>
<tr>
<td>$E(\hat{\theta}_N)$</td>
<td>3.489305</td>
<td>3.288380</td>
</tr>
<tr>
<td>$P(\theta \in I_N)$</td>
<td>0.9992</td>
<td>0.8055</td>
</tr>
</tbody>
</table>
The tables show that the rate of convergence of the coverage probability $P(\theta \in I_N)$ to $1 - \alpha$ seems to be slow. For the case when $r = 1$, the bias-corrected sequential confidence intervals, $I_N$ and $I_{N^*}$, are more effective than the ordinary ones, $I_N$ and $I_{N^*}$. Furthermore, there seems to be no significant difference between the coverage probabilities of the intervals, $I_N$ and $I_{N^*}$. An improvement on the stopping rule in (4) is needed.

References


