Cospectral graphs of the Grassmann graphs

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(joint work with Edwin van Dam)
Let \( q \) be a prime power, and \( V \) be a \( n \)-dimensional space over the \( GF(q) \) the field with \( q \) elements. Let \( 1 \leq e \leq n - 1 \) be an integer.

The **Grassmann Graph** \( G_q(n, e) \) has as vertices the \( e \)-dimensional subspaces and \( S \sim T \) iff their intersection is \( (e - 1) \)-dimensional.

To construct graphs with the same spectrum as \( G_q(n, e) \) we first will look at a partial linear space.
Let $n, e$ be positive integers such that $4 \leq 2e \leq n$.

Let $V$ be a $n$-dimensional vector space over $GF(q)$ and

let $H$ be a $2e$-dimensional subspace of $V$.
We first construct the partial linear space

$$LG_q(n, e, e + 1).$$

Its points are the $e$-dimensional subspaces of $V$.

There are two kinds of lines:

Lines of the first kind: $(e + 1)$-dimensional subspaces $L$ of $V$ which are not a subspace of $H$. A line $L$ as points the $e$-dimensional subspaces contained in $L$.

b Lines of the second kind: $(e - 1)$-dimensional spaces $M$ contained in $H$. A line $M$ has as points the $e$-dimensional spaces contained in $H$ which contain $M$ as a subspace.
Now $\mathcal{L}_q(n, e, e + 1)$ has
$\binom{n}{e}$ points,
$\binom{n}{e+1}$ lines,
each point is incident with $\binom{n-e}{1}$ lines
and each line is incident with $\binom{e+1}{1}$ points.

Through any pair of points there is at most one line.
If $P$ and $Q$ are points then they are on a line iff $P \cap Q$
is $(e - 1)$-dimensional.
Define $P_q(n, e+1)$ as the line graph of $L\mathcal{G}_q(n, e, e+1)$, that is its vertices are the lines of $L\mathcal{G}_q(n, e, e+1)$ and two lines are adjacent iff they have exactly one point in common.

**Theorem 1** (i) $P_q(n, e+1)$ is cospectral with $G_q(n, e+1)$,  
(ii) $P_q(n, e + 1)$ is distance-regular iff $n = 2e + 1$.  
(iii) $P_q(2e + 1, e + 1)$ is not isomorphic to the Grassmann graph $G_q(2e + 1, e + 1)$. 
(i) Let $N$ be the point-line incidence matrix. Then $NN^T - \begin{bmatrix} n-e \end{bmatrix} I$ is the adjacency matrix of the point graph. As the point graph is clearly $G_q(n, e)$, we know the spectrum of $NN^T$. Now except for the zero eigenvalue the spectrum of $NN^T$ is the same as the $N^T N$. This implies that $P_q(n, e + 1)$ is cospectral with $G_q(n, e + 1)$ as $NN^T - \begin{bmatrix} e+1 \end{bmatrix} I$ is the adjacency matrix for $P_q(n, e + 1)$. 
(ii) If $n < 2e + 1$, then there is $e + 1$-dimensional space $L$ which intersects $H$ in a $(e - 1)$-dimensional space $M$. Now in $P_q(n, e + 1)$ the distance between $L$ and $M$ is 2 and it is easy to see that they have $\begin{bmatrix} 2 \\ 1 \end{bmatrix}^{e+1}$ common neighbours where in the Grassmann graph $c_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}^2$.

If $n = 2e + 1$, then it is possible to check that it is distance-regular. An easy way to see this is true we use a result by Fiol and Garriga which states that if a graph has the same spectrum as a distance-regular graph $\Gamma$ with diameter $d$ is distance-regular iff for all vertices $x$ we have $k_d(x) = k_d(\Gamma)$. And this is easily checked.

(iii) Let $n = 2e + 1$. Let $K$ be an $(e + 2)$-dimensional space which intersects $H$ in $e + 1$ dimensions. Now the $(e + 1)$-dimensional subsapces of $K$ which are not contained in $H$ forms a maximal clique of size $\begin{bmatrix} e+2 \\ 1 \end{bmatrix} - 1$ in $P_q(2e + 1, e + 1)$, whereas the Grassmann graph $G_q(2e + 1, e + 1)$ has maximal cliques of sizes $\begin{bmatrix} e+2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} e+1 \\ 1 \end{bmatrix}$.

This shows the theorem.
(i) By looking at maximal cliques in $P_q(2e + 1, e + 1)$, it is easy to see that it is not vertex-transitive. The group $P\Gamma L(2e + 1)_{2e}$ is an automorphism group of the graph. It was shown by M. Tagami that this is the full automorphism group.

(ii) For large $q$ and $e$ we were able to show that the local graph of a line of type 1 is not cospectral to the local graph of a line of type 2. We suspect that this is always the case. This implies, for example, that the Terwilliger Algebra depends on the base vertex for $P_q(2e + 1, e + 1)$. 