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Kyoto University
Relative Difference Sets in Dihedral Groups

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1. Introduction

A \((m, n, k, \lambda)\) relative difference set (RDS) in a finite group \(G\) of order \(mn\) relative to a subgroup \(N\) of order \(n\), is a \(k\)-element subset \(R\) of \(G\) wherein every element of \(G - N\) has exactly \(\lambda\) representations as \(r_1 r_2^{-1}\) with \(r_1, r_2 \in R\). Moreover, no nonidentity element of \(N\) has such a representation. \(N\) is called the forbidden subgroup. If for a subset \(X\) of \(G\), we identify \(X\) with the group ring element \(X = \sum_{x \in X} x \in \mathbb{C}[G]\) and set \(X^{(-1)} = \sum_{x \in X} x^{-1}\), then \(R\) is a \((m, n, k, \lambda)\) RDS in \(G\) relative to \(N\) if \(RR^{(-1)} = k + \lambda(G - N)\). It follows that \(k(k - 1) = \lambda m(m - 1)\). Note that if \(N = 1\), then \(R\) is an \((m, k, \lambda)\) difference set in the usual sense.

The notion of a relative difference set was introduced by Elliot and Butson [1]. The following result which is due to them is fundamental in the study of RDS's.

Result 1.1. (1) Let \(R\) be a \((m, n, k, \lambda)\) relative difference set in a group \(G\) relative to a subgroup \(N\) and let \(U\) be a normal subgroup of \(G\) contained in \(N\). If \(\phi : G \rightarrow G/U\) is the canonical epimorphism and \(|U| = u\), then \(\phi(R)\) is a \((m, u^{-1}, k, u\lambda)\) relative difference set in \(\overline{G}(= G/U)\) with respect to \(\overline{N}(= N/U)\).

In particular, if \(N = U\), then \(\phi(R)\) is a \((m, k, n\lambda)\) ordinary difference set in \(\overline{G}(= G/N)\). We may then consider \(R\) as an "extension" of an ordinary difference set.

Although trivial ordinary difference sets with parameters of the form \((v + 2, v + 1, v)\) and \((v, v, v)\), \(v > 0\), exist in any group, it is still a question whether or not extensions of these difference sets also exist. In dihedral groups for instance, it is conjectured that only trivial ordinary difference sets exist. Hence, a problem that we would like to consider is whether extensions of these trivial difference sets exist in dihedral groups.

A relative difference set in a group \(G\) is said to be semiregular or of affine type if its parameters are of the form \((n\lambda, n, n\lambda, \lambda)\) or \((n\lambda + 2, n, n\lambda + 1, \lambda)\) respectively.
If $N$ is a normal subgroup of $G$ and $R$ is either a semiregular or affine type RDS in $G$, then $\overline{R}$ is a trivial ordinary difference set in $\overline{G}$ by Result 1.1. We say that a $(m,n,k,\lambda)$ relative difference set is trivial if $k=1$ or $(n,k) \in \{(1,m),(1,m-1)\}$.

If the conjecture mentioned above is true, then the only nontrivial RDS's that can exist in dihedral groups relative to a normal subgroup are either semiregular or of affine type.

The only nontrivial relative difference set up to equivalence in a dihedral group known to the authors is as follows:

**Example 1.2.** Let $G = \langle x, y \mid x^4 = y^2 = 1, y^{-1}xy = x^{-1} \rangle$ be the dihedral group of order 8. Then $D = \{1, xy, x^2y, x^3\}$ is a $(4, 2, 4, 2)$ relative difference set in $G$ relative to $\langle y \rangle$.

In [2], the following was shown.

**Result 1.3.** ([2]) There exists no nontrivial semiregular relative difference set in any dihedral group relative to a normal subgroup.

In section 3, we prove the following.

**Theorem 3.1.** There is no relative difference set of affine type in dihedral groups.

2. Preliminaries

We will use the following results which we mention here without proof.

**Result 2.1.** ([3]) Let $X$ be an $n \times n$ circulant matrix

$$X = \begin{bmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & \cdots & x_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_0 \end{bmatrix}$$

Then $\det(X) = \prod_{0 \leq i \leq n-1} (x_0 + \xi^i x_1 + \xi^{2i} x_2 + \cdots + \xi^{(n-1)i} x_{n-1})$, where $\xi$ is a primitive $n$th root of unity. Moreover, if $\det(x) \neq 0$, then $X^{-1}$ is also circulant.

**Result 2.2.** ([4]) (Inversion Formula). Let $G$ be an abelian group and $A = \sum_{g \in G} \alpha_g g$ be an element of the group algebra $\mathbb{C}[G]$. Then, $\alpha_g = \frac{1}{|G|} \sum_{x \in G^*} \chi(A) \chi(g^{-1})$ for each $g \in G$ where $G^*$ is the group of characters of $G$.

Throughout the rest of this paper, we will assume the following:
**Assumptions.** Let \( R \) be a \((n\lambda+2,n,n\lambda+1,\lambda)\) \((\lambda > 0)\) relative difference set in a dihedral group \( G \) relative to a subgroup \( N \). Set \( G = C(t) \) where \( C \) is a cyclic group and \( t \) is an involution which inverts \( C \). Set \( R = A + Bt \) where \( A \) and \( B \) are subsets of \( C \). By exchanging \( Rt \) for \( R \) if necessary, we may assume \(|A| \leq |B|\).

**Proposition 2.3.** Under the above assumptions, the following hold:

(i) If \( N \subset C \), then \( AA^{(-1)} + BB^{(-1)} = (n\lambda + 1) + \lambda(C - N) \) and \( AB = \frac{\lambda}{2} C \).

Furthermore, \(|A| = \frac{n\lambda}{2} \) and \(|B| = \frac{n\lambda}{2} + 1 \).

(ii) If \( N \not\subset C \), \( N_1 = N \cap C \) and \( N_2 = Nt \cap C \), then \( AA^{(-1)} + BB^{(-1)} = \frac{\lambda}{2} (C - N_2) \). Furthermore, \(|A| = \frac{(n\lambda + 1) - \sqrt{n\lambda + 1}}{2} \) and \(|B| = \frac{(n\lambda + 1) + \sqrt{n\lambda + 1}}{2} \).

**Proof.** We have \( RR^{(-1)} = (A + Bt)(A^{(-1)} + tB^{(-1)}) = AA^{(-1)} + BB^{(-1)} + 2ABt \).

Suppose \( N \subset C \). By definition, \( RR^{(-1)} = (n\lambda + 1) + \lambda(C + Ct - N) \). Thus, \( AA^{(-1)} + BB^{(-1)} = (n\lambda + 1) + \lambda(C - N) \) and \( AB = \frac{\lambda}{2} C \). If \(|A| = a \) and \(|B| = b \), it follows that \( a + b = n\lambda + 1 \) and \( ab = \frac{\lambda}{4} n(n\lambda + 2) \). Hence (i) holds.

Suppose \( N \not\subset C \). Then, \( RR^{(-1)} = (n\lambda + 1) + \lambda(C + Ct - N_1 - N_2t) \). Thus, \( AA^{(-1)} + BB^{(-1)} = (n\lambda + 1) + \lambda(C - N_1) \) and \( AB = \frac{\lambda}{2} (C - N_2) \). If \(|A| = a \) and \(|B| = b \), it follows that \( a + b = n\lambda + 1 \) and \( ab = \frac{\lambda}{4} n(n\lambda + 2) \). Hence (ii) holds.

3. **Nonexistence of Affine Type Relative Difference Sets in Dihedral Groups**

To prove our main theorem, we first show a necessary condition on the forbidden subgroup.

**Proposition 3.1.** Let \( R \) be a relative difference set of affine type in a dihedral group \( G \) relative to a subgroup \( N \) of \( G \). Then, \( N \) is normal in \( G \).

We will prove Proposition 3.1 in Lemmas 3.2 - 3.7. As mentioned in the previous section, we let \( G = C(t) \) where \( C \) is a cyclic subgroup of \( G \) and \( t \) is an element of \( G \) which inverts \( C \). Set \( R = A + Bt \) where \( A \) and \( B \) are subsets of \( C \).
Suppose the proposition is false and let $G$ be a minimal counterexample to the proposition. As every element outside $C$ is an involution and inverts $C$, we may assume that $t \in N$.

**Lemma 3.2.** $n = 2$. In particular,

(i) $G = CN$, $N = \langle t \rangle$ and $C \cong \mathbb{Z}_{2(\lambda+1)}$

(ii) $D$ is a $(2\lambda + 2, 2, 2\lambda + 1, \lambda)$ relative difference in $G$ with respect to $N$.

**Proof.** Let $L = N \cap C$. Then $[N : L] = 2$ and $G \triangleright L$ as $C$ is cyclic. Therefore, by Result 1.1, $\overline{D}$ is a difference set with parameters $(n\lambda + 2, 2, n\lambda + 1, n\lambda + 2)$ in $\overline{G} (= G/L)$ relative to $\overline{N} (= N/L \cong \mathbb{Z})$. Clearly, $\overline{G} \not\in \overline{N}$. By the minimality of $G$, $L = 1$. Thus $N = \langle t \rangle$.

By Proposition 2.3, we have

\[
AA^{-1} + BB^{-1} = (2\lambda + 1) + \lambda(C-1)
\]
\[
AB = \frac{\lambda}{2}(C-1)
\]
\[
|A| = \frac{2\lambda + 1 - \sqrt{2\lambda + 1}}{2}
\]
\[
|B| = \frac{2\lambda + 1 + \sqrt{2\lambda + 1}}{2}
\]

**Lemma 3.3.** We may assume that $C = A + B^{-1} + 1$.

**Proof.** By (2) and (3), $A \cap B^{-1} = \phi$ and $|A| + |B| = |C| - 1$. Hence $C = A \cup B^{-1}$ $\cup \{g\}$ for some $g \in C$. Exchanging $D$ for $Dg^{-1}$ if necessary, we may assume that $g = 1$.

**Lemma 3.4.** $A = A^{-1}$ and $B = B^{-1}$.

**Proof.** Let $\chi$ be a nonprincipal character of $C$ and set $\chi(A) = a$ and $\chi(B) = b$. By (1), (2) and Lemma 3.3, $a\overline{a} + b\overline{b} = \lambda + 1$, $ab = \frac{-\lambda}{2}$ and $a + \overline{b} + 1 = 0$. It follows that $2a\overline{a} + a + \overline{a} = \lambda = 2a\overline{a} + 2a$. Hence $a = \overline{a}$. By Result 2.2, $A - A^{-1} = 0$. Thus, $A = A^{-1}$ and as $B = C - A - 1$, we have $B = B^{-1}$.

By (3) above, we can set $2s + 1 = \sqrt{2\lambda + 1}$ for a positive integer $s$. Then $\lambda = 2s^2 + 2s$ and $D$ is a $(4s^2 + 4s + 2, 2, (2s + 1)^2, 2s^2 + 2)$ RDS in $G(\cong D_{4(2s^2+2s+1)})$. Moreover, by (1), (2) and Lemma 3.4, we have

\[
|A| = 2s^2 + s , \ |B| = 2s^2 + 3s + 1 , \ C = A + B + 1 , \ A^2 + B^2 = (2s + 1)^2 + (2s^2 + 2)(C - 1) , \ AB = (s^2 + s)(C - 1). \]

Hence, the following hold.
Lemma 3.5. $A^2 + A = s^2C + s^2 + s$ , $B^2 + B = (s + 1)^2 C + s^2 + s$.

Let $M$ be the unique subgroup of $C$ of index 2 and let $d$ be an involution of $C$. Then $C = M \times \langle d \rangle$. Set $\overline{C} = C/M(= \{1, d\})$.

Lemma 3.6. $|A \cap M| = s^2 + s$ , $|A \cap Md| = s^2$ , $|B \cap M| = s^2 + s$ , $|B \cap Md| = (s + 1)^2$.

Proof. Let $v = |A \cap M|$ and $w = |A \cap Md|$. Then $\overline{A} = v + w d$ and $v + w = |A| = 2 s^2 + s$. By Lemma 3.5, $(v + w d)^2 + (v + w d) = s^2(2 s^2 + 2 s + 1) (1 + d) + s^2 + s$. It follows that $v^2 + w^2 + v = 2 s^4 + 2 s^3 + 2 s^2 + s$ and $2 v w + w = 2 s^4 + 2 s^3 + s^2$ and so $(v - w)^2 + (v - w) = s^2 + s$. Thus, $v - w = s$ or $v - w = -(s + 1)$. If $v - w = -(s + 1) + 1$, then $2 w = 2 s^2 + 2 s + 1$, a contradiction. Hence $v - w = s$ and so $v = s^2 + s$ , $w = s^2$. The other equations in the Lemma can be proven similarly.

Lemma 3.7. $s$ is even.

Proof. By Lemma 3.6, $B \cap Md \neq \emptyset$. Let $g \in B \cap Md$ and set $\Omega = \{(x, y) | x, y \in B, g = x y\}$. By Lemma 3.5, $|\Omega| = (s + 1)^2 - 1$. If $s$ is odd, then $|\Omega| \equiv 1 \pmod{2}$. As $(x, y) \in \Omega$ implies $(y, x) \in \Omega$, there is an element $z \in B$ such that $(z, z) \in \Omega$. Thus, $g = z^2 \in Md$, a contradiction. Thus, $s$ is even.

Proof of Proposition 3.1:

By Lemma 3.7, $s = 2 \ell$ for some integer $\ell > 0$. By Lemma 3.6, $A \cap Md \neq \emptyset$. Let $g \in A \cap Md$ and set $\Omega = \{(x, y) | x, y \in A, x y = g\}$. By Lemma 3.5, $|\Omega| = s^2 - 1 \equiv 1 \pmod{2}$. By a similar argument as in Lemma 3.7, we have a contradiction. Thus, $G \triangleright N$.

Proposition 3.8. Let $G$ be a dihedral group and $N$ a normal subgroup of $G$. Then, there is no nontrivial relative difference set of affine type in $G$ relative to $N$.

In the rest of this section, let $G$ be a minimal counterexample to Proposition 3.8 and let $R$ be a $(p \lambda + 2, p, p \lambda + 1, \lambda)$ RDS in $G$. By the minimality condition, $p$ is a prime. As mentioned in Section 2, we let $G = C(t)$ where $t$ inverts the cyclic group $C$ and let $R = A + Bt$ where $A$ and $B$ are subsets of $C$. Exchanging $R$ for its translate, if necessary, we may assume $R \cap N = \emptyset$ and $R \cup \{1\}$ is a complete set of coset representatives of $G/N$. Since $G \triangleright N$ , $N$ is contained in $C$. By Proposition 2.3, we have

\begin{align*}
AA^{(-1)} + BB^{(-1)} &= (p \lambda + 1) + \lambda (C - N) \quad (4) \\
AB &= \frac{\lambda}{2} C \quad (5) \\
|A| &= \frac{\lambda}{2} p \quad , \quad |B| = \frac{\lambda}{2} p + 1 \quad (6)
\end{align*}
Let $h = \frac{\lambda}{2}p + 1$. Moreover, let $C = HN$ where $N = \langle s \rangle \cong \mathbb{Z}_p$ and $H \cong \mathbb{Z}_h$. Thus, we can set

$$A = A_0 + A_1s + \cdots + A_{p-1}s^{p-1}, \quad B = B_0 + B_1s + \cdots + B_{p-1}s^{p-1}$$

(7)

for some subsets $A_0, \ldots, A_{p-1}$, $B_0, \ldots, B_{p-1}$ of $H$.

**Lemma 3.9.** The following hold:

(i) $A_i \cap A_j = B_i \cap B_j = \phi \quad \forall \, i, j$ with $0 \leq i, j \leq p - 1$, $i \neq j$.

(ii) $H = 1 + \sum_{0 \leq i \leq p-1} A_i = \sum_{0 \leq i \leq p-1} B_i$.

**Proof.** Since $N = \langle s \rangle$ and $AA^{-1} \cap N = BB^{-1} \cap N = \{1\}$ by (4), (i) holds. Hence, $|A| = \sum_{0 \leq i \leq p-1} |A_i|$ and $|B| = \sum_{0 \leq i \leq p-1} |B_i|$. By (6), $|A| = h - 1$ and $|B| = h$. Then, (ii) follows immediately.

Substituting (7) into equations (4) and (5), we have

$$A_0B_0 + A_1B_{p-1} + A_2B_{p-2} + \cdots + A_{p-1}B_1 = \frac{\lambda}{2}H$$

$$A_0B_1 + A_1B_0 + A_2B_{p-1} + \cdots + A_{p-1}B_{p-2} = \frac{\lambda}{2}H$$

$$A_0B_i + A_1B_{i-1} + A_2B_{i-2} + \cdots + A_{p-1}B_{i-p+1} = \frac{\lambda}{2}H$$

(8)

and

$$A_0A_0^{(-1)} + A_1A_0^{(-1)} + A_2A_1^{(-1)} + \cdots + A_{p-2}A_{p-3}^{(-1)} + A_{p-1}A_{p-2}^{(-1)}$$

$$+ B_0B_0^{(-1)} + B_1B_0^{(-1)} + B_2 + B_1^{(-1)} + \cdots + B_{p-2}B_{p-2}^{(-1)} = \lambda(H - 1)$$

(9)

Let $\chi$ be a character of $H$. By (8), we have

$$\begin{bmatrix}
\chi(B_0) & \chi(B_0^{(-1)}) & \cdots & \chi(B_{p-1}) \\
\chi(B_1) & \chi(B_0) & \cdots & \chi(B_{p-2}) \\
\vdots & \vdots & \ddots & \vdots \\
\chi(B_{p-2}) & \chi(B_{p-3}) & \cdots & \chi(B_{p-1}) \\
\chi(B_{p-1}) & \chi(B_{p-2}) & \cdots & \chi(B_0)
\end{bmatrix}
\begin{bmatrix}
\chi(A_0) \\
\chi(A_1) \\
\vdots \\
\chi(A_{p-2}) \\
\chi(A_{p-1})
\end{bmatrix} = \frac{\lambda}{2} \chi(H)
\begin{bmatrix} 1 \\
1 \\
\vdots \\
1 \\
1 \end{bmatrix}$$

(10)

**Lemma 3.10.** The following hold.

(i) $|A_0| = |A_1| = \cdots = |A_{p-1}| = \frac{\lambda}{2}$. 

\[ |B_0||B_{p-1}| + |B_1||B_0| + \cdots + |B_{p-1}||B_{p-2}| = \frac{p\lambda^2}{4}. \]

**Proof.** Let \(|A_i| = a_i\) and \(|B_i| = b_i\) for \(i = 0, 1, \ldots, p-1\). If \(\chi\) is the principal character of \(H\), then by (10),

\[
\begin{bmatrix}
 b_0 & b_{p-1} & \cdots & b_1 \\
 b_1 & b_0 & \cdots & b_2 \\
 \vdots \\
 b_{p-2} & b_{p-3} & \cdots & b_{p-1} \\
 b_{p-1} & b_{p-2} & \cdots & b_0
\end{bmatrix}
\begin{bmatrix}
 a_0 \\
 a_1 \\
 \vdots \\
 a_{p-2} \\
 a_{p-1}
\end{bmatrix}
= \begin{bmatrix}
 1 \\
 \frac{\lambda h}{2} \\
 \vdots \\
 1 \\
 \frac{\lambda h}{2}
\end{bmatrix}
\]  

(11)

Let \(P\) be the \(p \times p\) matrix in the above equation (11). By Result 2.1, 
\[
\det(P) = \Pi_{0 \leq i \leq p-1} (\alpha_0 + \alpha_{p-1}^i + \alpha_{p-2}^{2i} + \cdots + \alpha_1^{(p-1)i}),
\]
where \(\alpha\) is a primitive \(p\)-th root of unity. By Lemma 3.9 (ii), \(b_0 + b_{p-1} + b_{p-2} + \cdots + b_1 = |H| \neq 0\). Suppose \(b_0 + b_{p-1} + b_{p-2} + \cdots + b_1 = 0\) for some \(i \neq 0\). Set \(\theta = \zeta^i\). Then, \(\theta\) is a primitive \(p\)-th root of unity and \(x^{p-1} + x^{p-2} + \cdots + x + 1\) is a minimal polynomial of \(\theta\) over \(\mathbb{Q}\). Hence \(b_0 = b_{p-1} = b_{p-2} = \cdots = b_1\). However, \(pb_0 = \sum_{0 \leq i \leq p-1} b_i = |H| = \frac{\lambda}{2}p + 1\), a contradiction. Thus \(\det(P) \neq 0\). By Result 2.1, \(P^{-1}\) is also circulant.

Since \((a_0, a_1, \ldots, a_{p-1})^T = \frac{\lambda h}{2} P^{-1} (1, 1, \ldots 1)^T\), it follows that \(a_0 = a_1 = \cdots = a_{p-1}\). Hence, by Lemma 3.9, \(a_0 = a_1 = \cdots = a_{p-1} = \frac{\lambda}{2}\). Thus (i) holds and (ii) follows from (9) and (i).

**Lemma 3.11.** Let \(\chi\) be a non-principal character of \(H\). Then \(\chi(B_0) = \chi(B_1) = \cdots = \chi(B_{p-1}) = 0\).

**Proof.** Set \(\chi(A_i) = \alpha_i\) and \(\chi(B_i) = \beta_i\) for \(i = 0, 1, \ldots, p-1\). By (10)

\[
\begin{bmatrix}
 \alpha_0 & \alpha_{p-1} & \cdots & \alpha_1 \\
 \alpha_1 & \alpha_0 & \cdots & \alpha_2 \\
 \vdots \\
 \alpha_{p-2} & \alpha_{p-3} & \cdots & \alpha_{p-1} \\
 \alpha_{p-1} & \alpha_{p-2} & \cdots & \alpha_0
\end{bmatrix}
\begin{bmatrix}
 \beta_0 \\
 \beta_1 \\
 \vdots \\
 \beta_{p-2} \\
 \beta_{p-1}
\end{bmatrix}
= \begin{bmatrix}
 0 \\
 0 \\
 \vdots \\
 0 \\
 0
\end{bmatrix}
\]  

(12)

Let \(Q\) be the \(p \times p\) matrix in the equation (12) above. By Result 2.1, 
\[
\det(Q) = \Pi_{0 \leq i \leq p-1} (\alpha_0 + \alpha_{p-1}^i + \alpha_{p-2}^{2i} + \cdots + \alpha_1^{(p-1)i}),
\]
where \(\alpha\) is a primitive \(p\)-th root of unity. Let \(i \neq 0\), \(i \in \{0, 1, \ldots, p-1\}\) and let \(\eta = \alpha_0 + \alpha_{p-1}^i + \alpha_{p-2}^{2i} + \cdots + \alpha_1^{(p-1)i}\) where \(\theta = \zeta^i\). Then, we have \(\eta^p = \alpha_0^p + \alpha_{p-1}^{ip} + \alpha_{p-2}^{ip} + \cdots + \alpha_1^{ip} \equiv \sum_{0 \leq i \leq p-1} (\sum_{x \in A_i} \chi(x))^p \equiv \sum_{0 \leq i \leq p-1} \chi(x^p) = \chi(H^p) \equiv H^p\) by Lemma 3.9 (ii). On the other hand, \(H^p = H \mod (p, h) = 1\). Hence \(\eta^p \equiv -1\).
(mod $p$) and so $\eta^p = -1 + p\alpha$ for an algebraic integer $\alpha \in \mathbb{Z}[\theta]$. If $\eta = 0$, then $\alpha = \frac{1}{p}$, a contradiction. Hence $\det(Q) \neq 0$. Thus, the lemma holds.

**Proof of Proposition 3.8:**

By Lemma 3.11 and Result 2.2, there exist $c_0, c_1, \ldots, c_{p-1} \in \mathbb{C}$ such that $B_0 = c_0 H, B_1 = c_1 H, \ldots, B_{p-1} = c_{p-1} H$. Since each $B_i$ is a subset of $H$, $B_{i_0} = H$ and $B_i = \phi(\forall i \neq i_0)$ for some $i_0 \in \{0, 1, \ldots, p - 1\}$. By Lemma 3.10, $\frac{p\lambda^2}{4} = 0$. Thus $\lambda = 0$, a contradiction. \(\blacksquare\)

By Propositions 3.1 and 3.8, we have the following.

**Theorem 3.12.** There is no nontrivial relative difference set of affine type in dihedral groups.

**References**


