Relative Difference Sets in Dihedral Groups

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1. Introduction

A $(m, n, k, \lambda)$ relative difference set (RDS) in a finite group $G$ of order $mn$ relative to a subgroup $N$ of order $n$, is a $k$-element subset $R$ of $G$ wherein every element of $G - N$ has exactly $\lambda$ representations as $r_{1}r_{2}^{-1}$ with $r_{1}, r_{2} \in R$. Moreover, no nonidentity element of $N$ has such a representation. $N$ is called the forbidden subgroup. If for a subset $X$ of $G$, we identify $X$ with the group ring element $X = \sum_{x \in X} x \in C[G]$ and set $X^{(-1)} = \sum_{x \in X} x^{-1}$, then $R$ is a $(m, n, k, \lambda)$ RDS in $G$ relative to $N$ if $RR^{(-1)} = k + \lambda(G - N)$. It follows that $k(k - 1) = \lambda n(m - 1)$. Note that if $N = 1$, then $R$ is an $(m, k, \lambda)$ difference set in the usual sense.

The notion of a relative difference set was introduced by Elliot and Butson [1]. The following result which is due to them is fundamental in the study of RDS's.

Result 1.1. ([1]) Let $R$ be a $(m, n, k, \lambda)$ relative difference set in a group $G$ relative to a subgroup $N$ and let $U$ be a normal subgroup of $G$ contained in $N$. If $\phi : G \rightarrow G/U$ is the canonical epimorphism and $|U| = u$, then $\phi(R)$ is a $(m, \frac{n}{u}, k, u\lambda)$ relative difference set in $\overline{G} (= G/U)$ with respect to $\overline{N} (= N/U)$.

In particular, if $N = U$, then $\phi(R)$ is a $(m, k, n\lambda)$ ordinary difference set in $\overline{G} (= G/N)$. We may then consider $R$ as an "extension" of an ordinary difference set.

Although trivial ordinary difference sets with parameters of the form $(v + 2, v + 1, v)$ and $(v, v, v)$, $v > 0$, exist in any group, it is still a question whether or not extensions of these difference sets also exist. In dihedral groups for instance, it is conjectured that only trivial ordinary difference sets exist. Hence, a problem that we would like to consider is whether extensions of these trivial difference sets exist in dihedral groups.

A relative difference set in a group $G$ is said to be semiregular or of affine type if its parameters are of the form $(n\lambda, n, n\lambda, \lambda)$ or $(n\lambda + 2, n, n\lambda + 1, \lambda)$ respectively.
If $N$ is a normal subgroup of $G$ and $R$ is either a semiregular or affine type RDS in $G$, then $\overline{R}$ is a trivial ordinary difference set in $\overline{G}$ by Result 1.1. We say that a $(m, n, k, \lambda)$ relative difference set is trivial if $k = 1$ or $(n, k) \in \{(1, m), (1, m - 1)\}$.

If the conjecture mentioned above is true, then the only nontrivial RDS’s that can exist in dihedral groups relative to a normal subgroup are either semiregular or of affine type.

The only nontrivial relative difference set up to equivalence in a dihedral group known to the authors is as follows:

**Example 1.2.** Let $G = \langle x, y \mid x^4 = y^2 = 1, y^{-1}xy = x^{-1} \rangle$ be the dihedral group of order 8. Then $D = \{1, xy, x^2y, x^3\}$ is a $(4, 2, 4, 2)$ relative difference set in $G$ relative to $\langle y \rangle$.

In [2], the following was shown.

**Result 1.3.** ([2]) There exists no nontrivial semiregular relative difference set in any dihedral group relative to a normal subgroup.

In section 3, we prove the following.

**Theorem 3.1.** There is no relative difference set of affine type in dihedral groups.

2. Preliminaries

We will use the following results which we mention here without proof.

**Result 2.1.** ([3]) Let $X$ be an $n \times n$ circulant matrix

\[
X = \begin{bmatrix}
x_0 & x_1 & \cdots & x_{n-1} \\
x_{n-1} & x_0 & \cdots & x_{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
x_1 & x_2 & \cdots & x_0
\end{bmatrix}
\]

Then $\det(X) = \prod_{0 \leq i \leq n-1}(x_0 + \xi^i x_1 + \xi^{2i} x_2 + \cdots + \xi^{(n-1)i} x_{n-1})$, where $\xi$ is a primitive $n$th root of unity. Moreover, if $\det(x) \neq 0$, then $X^{-1}$ is also circulant.

**Result 2.2.** ([4]) (Inversion Formula). Let $G$ be an abelian group and $A = \sum_{g \in G} \alpha_g g$ be an element of the group algebra $\mathbb{C}[G]$. Then, $\alpha_g = \frac{1}{|G|} \sum_{\chi \in G^*} \chi(A) \chi(g^{-1})$ for each $g \in G$ where $G^*$ is the group of characters of $G$.

Throughout the rest of this paper, we will assume the following:
Assumptions. Let $R$ be a $(n\lambda+2,n,n\lambda+1,\lambda)$ $(\lambda > 0)$ relative difference set in a dihedral group $G$ relative to a subgroup $N$. Set $G = C(t)$ where $C$ is a cyclic group and $t$ is an involution which inverts $C$. Set $R = A + Bt$ where $A$ and $B$ are subsets of $C$. By exchanging $Rt$ for $R$ if necessary, we may assume $|A| \leq |B|$. 

Proposition 2.3. Under the above assumptions, the following hold:

(i) If $N \subset C$, then $AA^{(-1)} + BB^{(-1)} = (n\lambda + 1) + \lambda(C - N)$ and $AB = \frac{\lambda}{2} C$. Furthermore, $|A| = \frac{n\lambda}{2}$ and $|B| = \frac{n\lambda}{2} + 1$.

(ii) If $N \not\subset C$, $N_1 = N \cap C$ and $N_2 = Nt \cap C$, then $AA^{(-1)} + BB^{(-1)} = (n\lambda + 1) + \lambda(C - N_1)$ and $AB = \frac{\lambda}{2}(C - N_2)$. Furthermore, $|A| = \frac{(n\lambda + 1) - \sqrt{n\lambda + 1}}{2}$ and $|B| = \frac{(n\lambda + 1) + \sqrt{n\lambda + 1}}{2}$.

Proof. We have $RR^{(-1)} = (A + Bt)(A^{(-1)} + tB^{(-1)}) = AA^{(-1)} + BB^{(-1)} + 2ABt$.

Suppose $N \subset C$. By definition, $RR^{(-1)} = (n\lambda + 1) + \lambda(C + Ct - N)$. Thus, $AA^{(-1)} + BB^{(-1)} = (n\lambda + 1) + \lambda(C - N)$ and $AB = \frac{\lambda}{2} C$. If $|A| = a$ and $|B| = b$, it follows that $a + b = n\lambda + 1$ and $ab = \frac{\lambda}{4} n(n\lambda + 2)$. Hence (i) holds.

Suppose $N \not\subset C$. Then, $RR^{(-1)} = (n\lambda + 1) + \lambda(C + Ct - N_1 - N_2 t)$. Thus, $AA^{(-1)} + BB^{(-1)} = (n\lambda + 1) + \lambda(C - N_1)$ and $AB = \frac{\lambda}{2} (C - N_2)$. If $|A| = a$ and $|B| = b$, it follows that $a + b = n\lambda + 1$ and $ab = \frac{\lambda}{4} n(n\lambda + 2)$. Hence (ii) holds.

3. Nonexistence of Affine Type Relative Difference Sets in Dihedral Groups

To prove our main theorem, we first show a necessary condition on the forbidden subgroup.

Proposition 3.1. Let $R$ be a relative difference set of affine type in a dihedral group $G$ relative to a subgroup $N$ of $G$. Then, $N$ is normal in $G$.

We will prove Proposition 3.1 in Lemmas 3.2 - 3.7. As mentioned in the previous section, we let $G = C(t)$ where $C$ is a cyclic subgroup of $G$ and $t$ is an element of $G$ which inverts $C$. Set $R = A + Bt$ where $A$ and $B$ are subsets of $C$. 
Suppose the proposition is false and let $G$ be a minimal counterexample to the proposition. As every element outside $C$ is an involution and inverts $C$, we may assume that $t \in N$.

**Lemma 3.2.** $n = 2$. In particular,

(i) $G = CN$, $N = \langle t \rangle$ and $C \cong Z_{2(\lambda+1)}$

(ii) $D$ is a $(2\lambda+2, 2, 2\lambda+1, \lambda)$ relative difference in $G$ with respect to $N$.

**Proof.** Let $L = N \cap C$. Then $[N : L] = 2$ and $G > L$ as $C$ is cyclic. Therefore, by Result 1.1, $\overline{D}$ is a difference set with parameters $\left(n\lambda + 2, 2, n\lambda + 1, \frac{n\lambda}{2}\right)$ in $\overline{G}(=G/L)$ relative to $\overline{N}(=N/L \cong \mathbb{Z})$. Clearly, $\overline{G} \neq \overline{N}$. By the minimality of $G$, $L = 1$. Thus $N = \langle t \rangle$.

By Proposition 2.3, we have

\[ AA^{-1} + BB^{-1} = (2\lambda + 1) + \lambda(C - 1) \]  
\[ AB = \frac{\lambda}{2}(C - 1) \]  
\[ |A| = \frac{2\lambda + 1 - \sqrt{2\lambda + 1}}{2} \quad \text{and} \quad |B| = \frac{2\lambda + 1 + \sqrt{2\lambda + 1}}{2} \]

**Lemma 3.3.** We may assume that $C = A + B^{-1} + 1$.

**Proof.** By (2) and (3), $A \cap B^{-1} = \phi$ and $|A| + |B| = |C| - 1$. Hence $C = A \cup B^{-1}$ or $C = A \cup B^{-1}$ if necessary, we may assume that $g = 1$.

**Lemma 3.4.** $A = A^{-1}$ and $B = B^{-1}$.

**Proof.** Let $\chi$ be a nonprincipal character of $C$ and set $\chi(A) = a$ and $\chi(B) = b$. By (1), (2) and Lemma 3.3, $a\overline{a} + b\overline{b} = \lambda + 1$, $ab = \frac{\lambda}{2}$ and $a + \overline{b} + 1 = 0$. It follows that $2a\overline{a} + a + \overline{a} = \lambda = 2a\overline{a} + 2a$. Hence $a = \overline{a}$. By Result 2.2, $A - A^{-1} = 0$. Thus, $A = A^{-1}$ and as $B = C - A - 1$, we have $B = B^{-1}$.

By (3) above, we can set $2s + 1 = \sqrt{2\lambda + 1}$ for a positive integer $s$. Then $\lambda = 2s^2 + 2s$ and $D$ is a $(4s^2 + 4s + 2, 2, (2s + 1)^2, 2s^2 + 2)$ RDS in $G(\cong D_{4(2s^2+1)+2})$. Moreover, by (1), (2) and Lemma 3.4, we have $|A| = 2s^2 + s$, $|B| = 2s^2 + 3s + 1$, $C = A + B + 1$, $A^2 + B^2 = (2s + 1)^2 + (2s^2 + 2)(C - 1)$, $AB = (s^2 + s)(C - 1)$. Hence, the following hold.
Lemma 3.5. \(A^2 + A = s^2C + s^2 + s\), \(B^2 + B = (s+1)^2C + s^2 + s\).

Let \(M\) be the unique subgroup of \(C\) of index 2 and let \(d\) be an involution of \(C\). Then \(C = M \times \langle d \rangle\). Set \(\overline{C} = C/M(= \{\overline{1}, \overline{d}\})\).

Lemma 3.6. \(|A \cap M| = s^2 + s\), \(|A \cap Md| = s^2\), \(|B \cap M| = s^2 + s\), \(|B \cap Md| = (s+1)^2\).

Proof. Let \(v = |A \cap M|\) and \(w = |A \cap Md|\). Then \(A = v + wd\) and \(v + w = |A| = 2s^2 + s\).

By Lemma 3.5, \((v + wd)^2 + (v + wd) = s^2(2s^2 + 2s + 1)(\overline{1} + \overline{d}) + s^2 + s\). It follows that \(v^2 + w^2 + v = 2s^4 + 2s^3 + 2s^2 + s\) and \(2vw + w = 2s^4 + 2s^3 + s^2\) and so \((v - w)^2 + (v - w) = s^2 + s\). Thus, \(v - w = s\) or \(v - w = -(s+1)\). If \(v - w = -(s+1)\), then \(2w = 2s^2 + 2s + 1\), a contradiction. Hence \(v - w = s\) and so \(v = s^2 + s\), \(w = s^2\). The other equations in the Lemma can be proven similarly.

Lemma 3.7. \(s\) is even.

Proof. By Lemma 3.6, \(B \cap Md \neq \phi\). Let \(g \in B \cap Md\) and set \(\Omega = \{(x, y) \mid x, y \in B, g = xy\}\). By Lemma 3.5, \(|\Omega| = (s+1)^2 - 1\). If \(s\) is odd, then \(|\Omega| \equiv 1\) (mod 2). As \((x, y) \in \Omega\) implies \((y, x) \in \Omega\), there is an element \(z \in B\) such that \((z, z) \in \Omega\). Thus, \(g = x^2 \in Md\), a contradiction. Thus, \(s\) is even.

Proof of Proposition 3.1:

By Lemma 3.7, \(s = 2\ell\) for some integer \(\ell > 0\). By Lemma 3.6, \(A \cap Md \neq \phi\). Let \(g \in A \cap Md\) and set \(\Omega = \{(x, y) \mid x, y \in A, xy = g\}\). By Lemma 3.5, \(|\Omega| = s^2 - 1 \equiv 1\) (mod 2). By a similar argument as in Lemma 3.7, we have a contradiction. Thus, \(G \triangleright N\).

Proposition 3.8. Let \(G\) be a dihedral group and \(N\) a normal subgroup of \(G\). Then, there is no nontrivial relative difference set of affine type in \(G\) relative to \(N\).

In the rest of this section, let \(G\) be a minimal counterexample to Proposition 3.8 and let \(R\) be a \((p\lambda + 2, p, p\lambda + 1, \lambda)\) RDS in \(G\). By the minimality condition, \(p\) is a prime. As mentioned in Section 2, we let \(G = C(t)\) where \(t\) inverts the cyclic group \(C\) and let \(R = A + Bt\) where \(A\) and \(B\) are subsets of \(C\). Exchanging \(R\) for its translate, if necessary, we may assume \(R \cap N = \phi\) and \(R \cup \{1\}\) is a complete set of coset representatives of \(G/N\). Since \(G \triangleright N\), \(N\) is contained in \(C\). By Proposition 2.3, we have

\[
AA^{(-1)} + BB^{(-1)} = (p\lambda + 1) + \lambda(C - N)
\]  

(4)

\[
AB = \frac{\lambda}{2} C
\]  

(5)

\[
|A| = \frac{\lambda}{2} p \quad , \quad |B| = \frac{\lambda}{2} p + 1
\]  

(6)
Let $h = \frac{\lambda}{2} p + 1$. Moreover, let $C = HN$ where $N = \langle s \rangle \cong \mathbb{Z}_p$ and $H \cong \mathbb{Z}_h$.

Thus, we can set

$$A = A_0 + A_1s + \cdots + A_{p-1}s^{p-1}, \quad B = B_0 + B_1s + \cdots + B_{p-1}s^{p-1}$$

(7)

for some subsets $A_0, \ldots, A_{p-1}, B_0, \ldots, B_{p-1}$ of $H$.

**Lemma 3.9.** The following hold:

(i) $A_i \cap A_j = B_i \cap B_j = \emptyset \quad \forall i, j$ with $0 \leq i, j \leq p - 1$, $i \neq j$.

(ii) $H = 1 + \sum_{0 \leq i \leq p-1} A_i = \sum_{0 \leq i \leq p-1} B_i$.

**Proof.** Since $N = \langle s \rangle$ and $AA^{(-1)} \cap N = BB^{(-1)} \cap N = \{1\}$ by (4), (i) holds. Hence, $|A| = \sum_{0 \leq i \leq p-1} |A_i|$ and $|B| = \sum_{0 \leq i \leq p-1} |B_i|$. By (6), $|A| = h - 1$ and $|B| = h$. Then, (ii) follows immediately.

Substituting (7) into equations (4) and (5), we have

$$A_0B_0 + A_1B_{p-1} + A_2B_{p-2} + \cdots + A_{p-1}B_1 = \frac{\lambda}{2} H$$

$$A_0B_1 + A_1B_0 + A_2B_{p-1} + \cdots + A_{p-1}B_{p-1} = \frac{\lambda}{2} H$$

$$A_0B_i + A_1B_{i-1} + A_2B_{i-2} + \cdots + A_{p-1}B_{i-p+1} = \frac{\lambda}{2} H$$

$$\cdots + \cdots + \cdots + \cdots = \frac{\lambda}{2} H$$

(8)

and

$$A_0A_0^{(-1)} + A_1A_0^{(-1)} + A_2A_1^{(-1)} + \cdots + A_{p-1}A_{p-2}^{(-1)} + A_{p-1}A_{p-1}^{(-1)}$$

$$+ B_0B_{p-1}^{(-1)} + B_1B_{0}^{(-1)} + B_2 + B_{1}^{(-1)} + \cdots + B_{p-1}B_{p-2}^{(-1)} = \lambda (H - 1)$$

(9)

Let $\chi$ be a character of $H$. By (8), we have

$$\begin{bmatrix}
\chi(B_0) & \chi(B_{p-1}) & \cdots & \chi(B_1) \\
\chi(B_1) & \chi(B_0) & \cdots & \chi(B_2) \\
\vdots & \vdots & \ddots & \vdots \\
\chi(B_{p-2}) & \chi(B_{p-3}) & \cdots & \chi(B_{p-1}) \\
\chi(B_{p-1}) & \chi(B_{p-2}) & \cdots & \chi(B_0)
\end{bmatrix} \begin{bmatrix}
\chi(A_0) \\
\chi(A_1) \\
\vdots \\
\chi(A_{p-1})
\end{bmatrix} = \frac{\lambda}{2} \begin{bmatrix}
|A_0| \\
|A_1| \\
\vdots \\
|A_{p-2}| \\
|A_{p-1}|
\end{bmatrix}$$

(10)

**Lemma 3.10.** The following hold.

(i) $|A_0| = |A_1| = \cdots = |A_{p-1}| = \frac{\lambda}{2}$. 
\[(ii) \quad |B_0||B_{p-1}| + |B_1||B_0| + \cdots + |B_{p-1}||B_{p-2}| = \frac{p\lambda^2}{4}.\]

**Proof.** Let \(|A_i| = a_i\) and \(|B_i| = b_i\) for \(i = 0, 1, \ldots, p - 1\). If \(\chi\) is the principal character of \(H\), then by (10),

\[
\begin{bmatrix}
a_0 & a_1 & \cdots & a_{p-1} \\
b_0 & b_1 & \cdots & b_{p-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{p-2} & a_{p-3} & \cdots & a_0 \\
b_{p-3} & b_{p-4} & \cdots & b_{p-2}
\end{bmatrix}
\begin{bmatrix}
1 \\
\lambda h \\
\vdots \\
\frac{\lambda}{2}
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}.
\]

Let \(P\) be the \(p \times p\) matrix in the above equation (11). By Result 2.1, \(\det(P) = \Pi_{0 \leq i \leq p-1}(\alpha_0 + \alpha_{p-1} \zeta^i + \alpha_{p-2} \zeta^{2i} + \cdots + \alpha_1 \zeta^{(p-1)i})\), where \(\zeta\) is a primitive \(p\)-th root of unity. Let \(\alpha_{0} + \alpha_{p-1} \theta + \alpha_{p-2} \theta^2 + \cdots + \alpha_1 \theta^{p-1}\) where \(\theta = \zeta^i\). Then, \(\theta\) is a primitive \(p\)-th root of unity and \(x^{p-1} + x^{p-2} + \cdots + x + 1\) is a minimal polynomial of \(\theta\) over \(Q\). Hence \(\alpha_0 = \alpha_{p-1} = \alpha_{p-2} = \cdots = \alpha_1\). However, \(pb_0 = \sum_{0 \leq i \leq p-1} b_i = |H| = \frac{\lambda}{2} p + 1\), a contradiction. Thus \(\det(P) \neq 0\). By Result 2.1, \(P^{-1}\) is also circulant.

Since \((a_0, a_1, \ldots, a_{p-1})^T = \frac{\lambda h}{2} P^{-1}(1,1,\ldots,1)^T\), it follows that \(a_0 = a_1 = \cdots = a_{p-1}\). Hence, by Lemma 3.9, \(a_0 = a_1 = \cdots = a_{p-1} = \frac{\lambda}{2}\). Thus (i) holds and (ii) follows from (9) and (i).

**Lemma 3.11.** Let \(\chi\) be a non-principal character of \(H\). Then \(\chi(B_0) = \chi(B_1) = \cdots = \chi(B_{p-1}) = 0\).

**Proof.** Set \(\chi(A_i) = \alpha_i\) and \(\chi(B_i) = \beta_i\) for \(i = 0, 1, \ldots, p - 1\). By (10)

\[
\begin{bmatrix}
\alpha_0 & \alpha_{p-1} & \cdots & \alpha_1 \\
\beta_0 & \beta_1 & \cdots & \beta_{p-1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{p-2} & \alpha_{p-3} & \cdots & \alpha_0 \\
\beta_{p-3} & \beta_{p-4} & \cdots & \beta_{p-2}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}.
\]

Let \(Q\) be the \(p \times p\) matrix in the above equation (12) above. By Result 2.1, \(\det(Q) = \Pi_{0 \leq i \leq p-1}(\alpha_0 + \alpha_{p-1} \zeta^i + \alpha_{p-2} \zeta^{2i} + \cdots + \alpha_1 \zeta^{(p-1)i})\), where \(\zeta\) is a primitive \(p\)-th root of unity. Let \(\eta = \alpha_0 + \alpha_{p-1} \theta + \alpha_{p-2} \theta^2 + \cdots + \alpha_1 \theta^{p-1}\) where \(\theta = \zeta^i\). Then, we have \(\eta^p = \alpha_0^p + \alpha_{p-1}^p + \alpha_{p-2}^p + \cdots + \alpha_1^p \mod p\) and \(\eta^p \equiv \sum_{0 \leq i \leq p-1} (\sum_{x \in A_i} \chi(x))^p \mod p\) by Lemma 3.9 (ii). On the other hand, \(H^p = H\) as \((p, h) = 1\). Hence \(\eta^p \equiv -1\).
\[ (\text{mod } p) \text{ and so } \eta^p = -1 + p\alpha \text{ for an algebraic integer } \alpha \in \mathbb{Z}[\theta]. \] If \( \eta = 0 \), then \( \alpha = \frac{1}{p} \), a contradiction. Hence \( \det(Q) \neq 0 \). Thus, the lemma holds.

**Proof of Proposition 3.8:**

By Lemma 3.11 and Result 2.2, there exist \( c_0, c_1, \ldots, c_{p-1} \in \mathbb{C} \) such that \( B_0 = c_0H, B_1 = c_1H, \ldots, B_{p-1} = c_{p-1}H \). Since each \( B_i \) is a subset of \( H \), \( B_{i_0} = H \) and \( B_i = \phi(\forall i \neq i_0) \) for some \( i_0 \in \{0,1,\ldots,p-1\} \). By Lemma 3.10, \( \frac{p\lambda^2}{4} = 0 \). Thus \( \lambda = 0 \), a contradiction.

By Propositions 3.1 and 3.8, we have the following.

**Theorem 3.12.** There is no nontrivial relative difference set of affine type in dihedral groups.

**References**


