Green functors and Bouc's construction

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1 Introduction

This is a survey of the preprint [Od]. We study the Grothendieck (character) rings of the Drinfel'd double of a finite group $G$ over the complex field $\mathbb{C}$. Witherspoon studied the representation rings of the Drinfel'd double of the group algebra in positive characteristic [Wi96]. In particular, she gave a direct sum decomposition of the representation ring into ideals involving Green rings of subgroups by using Thévenaz' twin functor construction for Green functors [Th88]. Dress introduced how to construct a Mackey functor $M_\Gamma$ from a Mackey functor $M$ by simply setting $M_\Gamma(X) := M(X \times \Gamma)$ for all finite $G$-set $X$ when $\Gamma$ is a finite $G$-set [Dr73]. This construction for Mackey functors is called Dress construction. Bouc introduced the Dress construction for a Green functor ([Bo03a] Theorem 5.1): If $A$ is a Green functor for $G$ over a commutative ring $\mathcal{O}$, and $\Gamma$ is a crossed $G$-monoid, then the Mackey functor $A_\Gamma$ obtained by the Dress construction has a natural structure of Green functor, and its evaluation $A_\Gamma(G)$ is an $\mathcal{O}$-algebra. The Bouc’s construction involves as special cases the construction of the crossed Burnside ring obtained from the Burnside ring Green functor, the Hochschild cohomology ring of $G$ obtained from the group cohomology Green functor, and the Grothendieck ring of the Drinfel’d double of $G$ obtained from the Grothendieck ring Green functor for a group algebra. We also point out that Bouc's construction is discussed in [Wi04]. In this note, we show an induction theorem for Drinfel’d double for $G$ by using a formula of primitive idempotents of the crossed Burnside ring [OY01], Bouc’s construction, and some properties of Witherspoon’s Green functor $R(D_G(*))$. The theorem implies Artin induction theorem for a group algebra over $\mathbb{C}$. This is a new proof of Artin induction theorem.

We refer the reader to [Bo97], [Bo00], [TW95] or [We00] for standard definitions and results regarding Burnside rings and Green functors, and to [Bo03a], [Bo03b], [OY01], and [OY04] for basic results about crossed $G$-sets and crossed Burnside rings.

2 Results

(2.1) Burnside Green functors. We recall the crossed Burnside ring Green functor $X\Omega(*, G^c)$ in terms of subgroups of $G$ (see 4.1 of [OY04]). Let $S(H)$ be the family of all subgroups of $H \leq G$ and $C_G(D)$ a centralizer of $D \leq H$. Then the assignment

$$H(\leq G) \mapsto X\Omega(H, G^c) = \langle(H/D)_s \mid D \in [H \setminus S(H)] \\ s \in [H \setminus C_G(D)]\rangle_2$$

is a Green functor for $G$.
gives a Green functor for $G$ over $\mathbb{Z}$ equipped with
\[
\begin{align*}
\text{ind}_L^H : & \ X\Omega(L, G^c) \rightarrow X\Omega(H, G^c) : (L/D)_s \mapsto (H/D)_s, \\
\text{res}_L^H : & \ X\Omega(H, G^c) \rightarrow X\Omega(L, G^c) : (H/D)_s \mapsto \sum_{g \in (L/D \setminus H/D)} (L/L \cap gD)_s,
\end{align*}
\]
where $D \leq L \leq H \leq G$ and $g \in G$.

(2.2) **Witherspoon’s Green functor.** Witherspoon gave a Green functor $R_C(D_G(*))$ for $G$ over $\mathbb{Z}$ (see [Wi96] Section 5). For each subgroup $H$ of $G$, there is a subalgebra
\[
D_G(H) = \sum_{g \in G, h \in H} \mathbb{C}\phi_g h
\]
of Drinfel’d (quantum) double $D(G)$ of $\mathbb{C}G$ [Dr86], where $\phi_g$ is an element of the basis $\{\phi_g\}_{g \in G}$ of the dual space $(\mathbb{C}G)^* = \text{Hom}_\mathbb{C}(\mathbb{C}G, \mathbb{C})$. Note that $D_G(G) = D(G)$ and $R(D(G))$ is the representation ring of $D(G)$ or equivalently the Grothendieck ring of Hopf bimodules for the Hopf algebra $\mathbb{C}G$ ([?], [Bo03a], [OY04]). Let $R_C(D_G(H))$ be the Grothendieck (representation) ring of $D_G(H)$ for subgroup $H$ of $G$. Then the assignment
\[
H(\leq G) \mapsto R_C(D_G(H))
\]
gives a Green functor for $G$ over $\mathbb{Z}$ equipped with
\[
\begin{align*}
\text{Dres}_L^H : & \ R_C(D_G(H)) \rightarrow R_C(D_G(L)) : U \rightarrow U \downarrow_{D_G(L)}, \\
\text{Dind}_L^H : & \ R_C(D_G(L)) \rightarrow R_C(D_G(H)) : V \rightarrow D_G(H) \otimes_{D_G(L)} V, \\
\text{Dcong}_{H,g} : & \ R_C(D_G(H)) \rightarrow R_C(D_G(gH)) : U \rightarrow gU = gD_G(H) \otimes_{D_G(H)} U,
\end{align*}
\]
where $U \downarrow_{D_G(L)}$ is a $D_G(L)$-module by restriction of the action from $D_G(H)$ to $D_G(L)$, $L \leq H \leq G$ and $g \in G$. We use the equivalence of the category of $H$-vector bundle on $G^c$ with the category of $D_G(H)$-modules (see [Wi96] Section 2).

The following theorem obtained by Bouc is the essential tool of the proof of our theorem.

(2.3) **Theorem.** ([Bo03a] 5.1) Let $A$ be a Green functor for $G$ over a commutative ring $\mathcal{O}$ and $\Gamma$ a crossed $G$-monoid. Then the functor $A$ is a Green functor for $G$ over $\mathcal{O}$, with unit $\epsilon_{A_{\Gamma}}$. Moreover the correspondence $A \mapsto A_{\Gamma}$ is an endo-functor of the category of Green functors for $G$ over $\mathcal{O}$.

(2.4) **Sub-Green functors.** There is a sub-Green functor $X\Omega(*, G^c)_1$ which assigns to each subgroup $H$ of $G$ to a subring $X\Omega(H, G^c)_1$ of $X\Omega(H, G^c)$ generated by the elements $(H/L)_{\lambda}$. There is also a sub-Green functor $R_C(D_G(*))_1$ which assigns to each subgroup $H$ of $G$ to a subring $R_C(D_G(H)_1)$ of $R_C(D_G(H))$ generated by $\text{Incl}_{H,1G}(V)'s$, where $\text{Incl}_{H,1G}$ is a functor embedding the category of $\mathcal{C}H$-modules as a full subcategory of the category of $D_G(H)$-module (see, [Wi96] Section 1) and $V$ is a $\mathcal{C}H$-module. It is easy to see that $X\Omega(H, G^c)_1$ is isomorphic to the Burnside ring $\Omega(H)$ and $R_C(D_G(H)_1)$ is isomorphic to the ordinary character ring $R_C(H)$. The homomorphism $\theta_{G^c} \downarrow_{X\Omega(H, G^c)_1}$ is the natural ring homomorphism from $\Omega(H)$ to $R_C(H)$.

The proof of the following theorem is an analogue of Theorem 3.5.2 of [Bo00].
(2.5) Theorem. Let $G$ be a finite group. Then

$$\mathbb{C}R_{\mathbb{C}}(D(G)) = \sum_{H \in C(G)} \text{Ind}_{H}^{G} \mathbb{C}R_{\mathbb{C}}(D_{G}(H)),$$

where $C(G)$ is the family of cyclic subgroups of $G$.

The previous theorem and (2.4) show the following corollary.

(2.6) Corollary (Artin). Let $G$ be a finite group. Then

$$\mathbb{Q}R(\mathbb{C}) = \sum_{H \in \text{C}(G)} \text{Ind}^{G}_{H} \mathbb{Q}R(\mathbb{C}).$$

Acknowledgements

The author would like to thank Tomoyuki Yoshida and Akihiro Munemasa for helpful advice. The author would like to thank the organizers of the meeting for the opportunity to attend the conference held in Kyoto, December 2004.

References


