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Green functors and Bouc's construction

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1 Introduction

This is a survey of the preprint [Od]. We study the Grothendieck (character) rings of the Drinfel'd double of a finite group G over the complex field $\mathbb{C}$. Witherspoon studied the representation rings of the Drinfel'd double of the group algebra in positive characteristic [Wi96]. In particular, she gave a direct sum decomposition of the representation ring into ideals involving Green rings of subgroups by using Thévenaz' twin functor construction for Green functors [Th88]. Dress introduced how to construct a *Mackey functor* M_Γ from a Mackey functor M by simply setting $M_\Gamma(X) := M(X \times \Gamma)$ for all finite G-set X when Γ is a finite G-set [Dr73]. This construction for Mackey functors is called Dress construction. Bouc introduced the Dress construction for a *Green functor* ([Bo03a] Theorem 5.1): If A is a Green functor for G over a commutative ring $\mathcal{O}$, and Γ is a crossed G-monoid, then the Mackey functor A_Γ obtained by the Dress construction has a natural structure of Green functor, and its evaluation $A_\Gamma(G)$ is an $\mathcal{O}$-algebra. The Bouc's construction involves as special cases the construction of the crossed Burnside ring obtained from the Burnside ring Green functor, the Hochschild cohomology ring of G obtained from the group cohomology Green functor, and the Grothendieck ring of the Drinfel'd double of G obtained from the Grothendieck ring Green functor for a group algebra. We also point out that Bouc's construction is discussed in [Wi04]. In this note, we show an induction theorem for Drinfel'd double for G by using a formula of primitive idempotents of the crossed Burnside ring [OY01], Bouc's construction, and some properties of Witherspoon's Green functor $R(D_G(*))$. The theorem implies Artin induction theorem for a group algebra over $\mathbb{C}$. This is a new proof of Artin induction theorem.

We refer the reader to [Bo97], [Bo00], [TW95] or [We00] for standard definitions and results regarding Burnside rings and Green functors, and to [Bo03a], [Bo03b], [OY01], and [OY04] for basic results about crossed G-sets and crossed Burnside rings.

2 Results

(2.1) Burnside Green functors. We recall the crossed Burnside ring Green functor $X\Omega(*, G^c)$ in terms of subgroups of G (see 4.1 of [OY04]). Let $S(H)$ be the family of all subgroups of $H \leq G$ and $C_G(D)$ a centralizer of $D \leq H$. Then the assignment

$$H(\leq G) \longmapsto X\Omega(H, G^c) = \langle (H/D)_s \mid D \in [H \setminus S(H)]\, s \in [H \setminus C_G(D)] \rangle_{\mathbb{Z}}$$

gives a Green functor for G over $\mathbb{Z}$ equipped with

$$\begin{array}{llll} \mathrm{ind}_L^H & : X\Omega(L,G^c) \longrightarrow X\Omega(H,G^c) & : (L/D)_s \longmapsto (H/D)_s, \\ \mathrm{res}_L^H & : X\Omega(H,G^c) \longrightarrow X\Omega(L,G^c) & : (H/D)_s \longmapsto \sum_{g\in[L\backslash H/D]} (L/L\cap {}^gD)_{{}^gs}, \\ \mathrm{con}_{H,g} & : X\Omega(H,G^c) \longrightarrow X\Omega({}^gH,G^c) & : (H/D)_s \longmapsto ({}^gH/{}^gD)_{{}^gs}, \end{array}$$

where $D \leq L \leq H \leq G$ and $g \in G$.

(2.2) Witherspoon's Green functor. Witherspoon gave a Green functor $R_{\mathbb{C}}(D_G(*))$ for G over $\mathbb{Z}$ (see [Wi96] Section 5). For each subgroup H of G, there is a subalgebra

$$D_G(H) = \sum_{g\in G,\, h\in H} \mathbb{C}\phi_g h$$

of Drinfel'd (quantum) double $D(G)$ of $\mathbb{C}G$ [Dr86], where ϕ_g is an element of the basis $\{\phi_g\}_{g\in G}$ of the dual space $(\mathbb{C}G)^* = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}G,\mathbb{C})$. Note that $D_G(G) = D(G)$ and $R(D(G))$ is the representation ring of $D(G)$ or equivalentary the Grothendieck ring of Hopf bimodules for the Hopf algebra $\mathbb{C}G$ ([?], [Bo03a], [OY04]). Let $R_{\mathbb{C}}(D_G(H))$ be the Grothendieck (representation) ring of $D_G(H)$ for subgroup H of G. Then the assignment

$$H(\leq G) \longmapsto R_{\mathbb{C}}(D_G(H))$$

gives a Green functor for G over $\mathbb{Z}$ equipped with

$$\begin{array}{llllll} \mathrm{Dres}_L^H & : R_{\mathbb{C}}(D_G(H)) & \longrightarrow & R_{\mathbb{C}}(D_G(L)) & : U & \longmapsto U\downarrow_{D_G(L)}, \\ \mathrm{Dind}_L^H & : R_{\mathbb{C}}(D_G(L)) & \longrightarrow & R_{\mathbb{C}}(D_G(H)) & : V & \longmapsto D_G(H)\otimes_{D_G(L)} V, \\ \mathrm{Dconj}_{H,g} & : R_{\mathbb{C}}(D_G(H)) & \longrightarrow & R_{\mathbb{C}}(D_G({}^gH)) & : U & \longmapsto {}^gU = gD_G(H)\otimes_{D_G(H)} U, \end{array}$$

where $U\downarrow_{D_G(L)}$ is a $D_G(L)$-module by restriction of the action from $D_G(H)$ to $D_G(L)$, $L \leq H \leq G$ and $g \in G$. We use the equivalence of the category of H-vector bundle on G^c with the category of $D_G(H)$-modules (see [Wi96] Section 2).

The following theorem obtained by Bouc is the essential tool of the proof of our theorem.

(2.3) Theorem. ([Bo03a] 5.1) *Let A be a Green functor for G over a commutative ring $\mathcal{O}$ and Γ a crossed G-monoid. Then the functor A_Γ is a Green functor for G over $\mathcal{O}$, with unit ε_{A_Γ}. Moreover the correspondence $A \mapsto A_\Gamma$ is an endo-functor of the category of Green functor for G over $\mathcal{O}$.*

(2.4) Sub-Green functors. There is a sub-Green functor $X\Omega(*,G^c)_1$ which assigns to each subgroup H of G to a subring $X\Omega(H,G^c)_1$ of $X\Omega(H,G^c)$ generated by the elements $(H/L)_{1_G}$. There is also a sub-Green functor $R_{\mathbb{C}}(D_G(*)_1)$ which assigns to each subgroup H of G to a subring $R_{\mathbb{C}}(D_G(H)_1)$ of $R_{\mathbb{C}}(D_G(H))$ generated by $\mathrm{Incl}_{H,1_G}(V)'s$, where $\mathrm{Incl}_{H,1_G}$ is a functor embedding the category of $\mathbb{C}H$-modules as a full subcategory of the category of $D_G(H)$-module (see, [Wi96] Section 1) and V is a $\mathbb{C}H$-module. It is easy to see that $X\Omega(H,G^c)_1$ is isomorphic to the Burnside ring $\Omega(H)$ and $R_{\mathbb{C}}(D_G(H)_1)$ is isomorphic to the ordinary character ring $R_{\mathbb{C}}(H)$. The homomorphism $\theta_{G^c}\downarrow_{X\Omega(H,G^c)_1}$ is the natural ring homomorphism from $\Omega(H)$ to $R_{\mathbb{C}}(H)$.

The proof of the following theorem is an analogue of Theorem 3.5.2 of [Bo00].

(2.5) Theorem. *Let G be a finite group. Then*

$$\mathbb{C}R_{\mathbb{C}}(D(G)) = \sum_{H \in \mathcal{C}(G)} \mathrm{Dind}_H^G \mathbb{C}R_{\mathbb{C}}(D_G(H)),$$

where $\mathcal{C}(G)$ is the family of cyclic subgroups of G.

The previous theorem and (2.4) show the following corollary.

(2.6) Corollary (Artin). *Let G be a finite group. Then*

$$\mathbb{Q}R_{\mathbb{C}}(G) = \sum_{H \in \mathcal{C}(G)} \mathrm{Ind}_H^G \mathbb{Q}R_{\mathbb{C}}(H).$$

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