A parameter bound on distance-biregular graphs

Sejeong Bang*
Faculty of Mathematics, Graduate School, Kyushu University
6-10-1 Hakozaki, Higashiku, Fukuoka 812-8581 Japan e-mail: sjbang@math.kyushu-u.ac.jp.

Abstract

In this note we consider distance-biregular graphs with valencies \((\sigma+1,5+1)\) with \(\sigma<\delta\). We show that bigger valency \(\delta\) is bounded by a function of smaller valency \(\sigma\).

Definition 0.1 Let \(\Gamma = (V(\Gamma), E(\Gamma))\) be a bipartite connected graph with the vertex set \(V(\Gamma)\) and the edge set \(E(\Gamma)\). Let \(D\) be a diameter of \(\Gamma\). Then \(\Gamma\) is called a distance-biregular if for any pair of vertices \(u, v \in V(\Gamma)\) with \(d(u, v) = i\), the numbers \(c_i(u, v)\) and \(b_i(u, v)\) are depend only on \(d(u, v) = i\) and the partition the vertex \(u\) belongs to, where

\[
\begin{align*}
c_i(u, v) &:= |\Gamma_{i-1}(u) \cap \Gamma_1(v)|, \\
b_i(u, v) &:= |\Gamma_{i+1}(u) \cap \Gamma_1(v)|.
\end{align*}
\]

Remark 0.2 (i) The vertices in the same partition have the same intersection array. From now on, let \(V(\Gamma) = V_\sigma \cup V_\delta\), where \(V_\sigma = \{v \in V(\Gamma) \mid \deg(v) = \sigma + 1\}\) and \(V_\delta = \{v \in V(\Gamma) \mid \deg(v) = \delta + 1\}\).
(ii) For \(u \in V_\sigma\) and \(d(u, v) = i\), define

\[
\begin{align*}
D_\sigma &:= \max\{d(u, x) \mid x \in V(\Gamma)\} \\
c_i^\sigma &:= |\Gamma_{i-1}(u) \cap \Gamma_1(v)| \\
b_i^\sigma &:= |\Gamma_{i+1}(u) \cap \Gamma_1(v)|.
\end{align*}
\]

In the same way, we could define \(D_\delta, c_i^\delta\) and \(b_i^\delta\).

Example 0.3 Bipartite distance-regular graphs are distance-biregular graphs.

Lemma 0.4 (Delorme)
For each \(i\),

\[
\begin{align*}
(i) \quad &c_{2i}^\sigma c_{2i+1}^\sigma = c_{2i+1}^\delta c_{2i+1}^\delta \\
(ii) \quad &b_{2i-1}^\sigma b_{2i+1}^\sigma = b_{2i-1}^\delta b_{2i+1}^\delta.
\end{align*}
\]

*This note is based on a joint work with Akira Hiraki(Division of Mathematical Sciences, Osaka Kyoiku University, Japan) and Jack Koolen(Department of Mathematics, Pohang University of Science and Technology, South Korea)
Proof: (i) Let $d(x, y) = 2i + 1$ and $x \in V_{\sigma}$. By counting the number of paths between $x$ and $y$ in two ways, we have

$$c_{2i+1}^\sigma c_{2i}^\sigma \cdots c_2^\sigma = c_{2i+1}^\delta c_{2i}^\delta \cdots c_2^\delta.$$ 

(ii) Consider $\{(u, w) \in V_{\sigma} \times V_{\delta} | d(u, w) = 2i + 1\}$. Then

$$v_{\sigma} b_{\sigma}^1 b_{\sigma}^2 \cdots b_{\sigma}^n = v_{\delta} b_{\delta}^1 b_{\delta}^2 \cdots b_{\delta}^n.$$ 

Proposition 0.5 Let $\Gamma$ be a distance-biregular graph with valencies $(\sigma + 1, \delta + 1)$. If $\sigma = \delta$ then $\Gamma$ is distance-regular.

Proof: By Lemma 0.4, for each $i \geq 1$, we have the following

$$(\delta + 1 - c_{2i-1}^\sigma)(\sigma + 1 - c_{2i}^\sigma) = (\sigma + 1 - c_{2i-1}^\delta)(\delta + 1 - c_{2i}^\delta);$$

$$c_{2i}^\sigma c_{2i+1}^\sigma = c_{2i}^\delta c_{2i+1}^\delta.$$ 

As $\sigma = \delta$ and $c_1^\sigma = 1 = c_1^\delta$ hold, $c_i^\sigma = c_i^\delta$ (1 $\leq i \leq D$) and $b_i^\sigma = b_i^\delta$ (0 $\leq i \leq D-1$) must be satisfied.

By the previous proposition, we may assume that $\sigma < \delta$ in this note.

Lemma 0.6 (Delorme)

Let $\sigma < \delta$. Then

(i) $D_\sigma$ is even;

(ii) $D_\delta \leq D_\sigma \leq D_\delta + 1$.

Let $g = 2n + 2$ be the girth of a distance biregular graph $\Gamma$. Note that $c_{n+1}^\sigma > c_n^\sigma = 1$ and $c_{n+1}^\delta > c_n^\delta = 1$ are satisfied.

Theorem 0.7 (Delorme)

$$\max\{D_\sigma, D_\delta\} \leq \frac{g}{2} \min\{\sigma, \delta\} + 1.$$ 

Proof:

$$\max\{D_\sigma, D_\delta\} \leq \min\{(\sigma - c_{n+1}^\sigma + 2)n + 2, (\delta - c_{n+1}^\delta + 2)n + 2\}$$

$$\leq \min\{\sigma, \delta\}n + 2$$

$$\leq \frac{g}{2} \min\{\sigma, \delta\} + 1.$$
We denote by $\Sigma$ (resp. $\Delta$) the graph with vertices the vertices of degree $\sigma+1$ (resp. $\delta+1$) and two vertices are adjacent if and only if they are at distance 2 in $\Gamma$. Note that $\Sigma$ (resp. $\Delta$) is a distance-regular graph of order $(\delta, \sigma+1/c_2^{\Sigma} - 1)$ (resp. $(\sigma, \delta+1/c_2^{\Delta} - 1)$). In other words,

$$\Gamma^{(\Sigma)}_1(x) \cong \left( \frac{\sigma+1}{c_2^{\Sigma}} \right) * K_\delta.$$ 

Let $N$ be a matrix whose rows and columns are indexed by $V(\Delta)$ and $V(\Sigma)$ respectively. Then we obtain

$$c_2^{\delta} A_{\Delta} = NN^T - (1 + \delta)I;$$

$$c_2^{\sigma} A_{\Sigma} = N^T N - (1 + \sigma)I. \quad (1)$$

$$c_2^{\sigma} A_{\Sigma} = N^T N - (1 + \sigma)I. \quad (2)$$

**Lemma 0.8 (Hoffman-Delsarte)**

Let $\Gamma$ be a distance-regular graph with diameter $D$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ be the eigenvalues of $\Gamma$. Then the maximal clique in $\Gamma$ has at most

$$\left( 1 - \frac{\theta_0}{\theta_D} \right)$$

vertices.

**Theorem 0.9** Let $D_\sigma = D_\delta$. Then every maximal clique in $\Sigma$ has exactly

$$\left( 1 - \frac{\theta_0^{(\Sigma)}}{\theta_D^{(\Sigma)}} \right)$$

vertices.

**Theorem 0.10** Let $D_\sigma = D_\delta$. Then every maximal clique in $\Sigma$ has exactly

$$\left( 1 - \frac{\theta_0^{(\Sigma)}}{\theta_D^{(\Sigma)}} \right)$$

vertices.

**Proof:** Let $N \in M_{|V_\delta| \times |V_\delta|}(\mathbb{R})$. Then

$$N^T N = c_2^{\delta} A_{(\Sigma)} + (1 + \delta)I \in M_{|V_\delta| \times |V_\delta|}(\mathbb{R})$$

$$NN^T = c_2^{\sigma} A_{(\Delta)} + (1 + \delta)I \in M_{|V_\delta| \times |V_\delta|}(\mathbb{R}).$$

As $\sigma < \delta$, $|V(\Sigma)| > |V(\Delta)|$. By the fact $\text{rank}(NN^T) = \text{rank}(N) = \text{rank}(N^T N)$ (i.e., the matrix $N^T N$ has an eigenvalue 0) and the equation (2), the graph $\Sigma$ has an eigenvalue

$$-\frac{1 + \sigma}{c_2^{\sigma}}.$$
Therefore the following holds:

\[-\frac{1 + \sigma}{c_2^2} \leq \theta_n^{(\Sigma)} \leq -\frac{1 + \sigma}{c_2^2}.

**Theorem 0.11 (Hiraki and Koolen)**

Let \( \Gamma \) be a distance-regular graph of order \((s, t)\) with \(s > 1\). If \(-(t + 1)\) is an eigenvalue of \( \Gamma \) then

\[ t < \frac{4D}{h} - 1. \]

**Theorem 0.12**

Let \( \Gamma \) be a distance-biregular graph with valencies \( \sigma + 1 \) and \( \delta + 1 \). Let \( \sigma < \delta \), \( D_\sigma = D_\delta \) and the girth \( g \geq 8 \). Then there is a function \( f \) of \( \sigma \) such that

\[ \delta \leq f(\sigma). \]

**Proof:** Recall that \( \Sigma \) is distance-regular graph of order \((\sigma, \delta)\). By Theorem 0.11

\[ \delta < \frac{4D_\Delta}{h_\Delta} - 1. \]

Now we will show that

\[ \frac{4D_\Delta}{h_\Delta} - 1 \leq F(\sigma). \]

Note that the following hold:

\[ \frac{1}{h_\Delta} \leq \frac{2}{h - 1} = \frac{4}{g - 4}, \]

\[ D_\Delta \leq \frac{g\sigma + 2}{4}. \]

Hence

\[ \frac{4D_\Delta}{h_\Delta} - 1 \leq \frac{4(g\sigma + 2) - g + 4}{g - 4} \leq 9\sigma. \]

Therefore \( \delta < \sigma^{9\sigma} \) holds.

**References**
