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Kyoto University
A parameter bound on distance-biregular graphs

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Abstract

In this note we consider distance-biregular graphs with valencies $(\sigma + 1, \delta + 1)$ with $\sigma < \delta$. We show that bigger valency $\delta$ is bounded by a function of smaller valency $\sigma$.

Definition 0.1 Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a bipartite connected graph with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$. Let $D$ be a diameter of $\Gamma$. Then $\Gamma$ is called a distance-biregular if for any pair of vertices $u, v \in V(\Gamma)$ with $d(u, v) = i$, the numbers $c_i(u, v)$ and $b_i(u, v)$ are depend only on $d(u, v) = i$ and the partition the vertex $u$ belongs to, where

$$
c_i(u, v) := |\Gamma_{i-1}(u) \cap \Gamma_1(v)|,
$$

$$
b_i(u, v) := |\Gamma_{i+1}(u) \cap \Gamma_1(v)|.
$$

Remark 0.2 (i) The vertices in the same partition have the same intersection array. From now on, let $V(\Gamma) = V_\sigma \cup V_\delta$, where $V_\sigma = \{v \in V(\Gamma) | \deg(v) = \sigma + 1\}$ and $V_\delta = \{v \in V(\Gamma) | \deg(v) = \delta + 1\}$. (ii) For $u \in V_\sigma$ and $d(u, v) = i$, define

$$
D_\sigma := \max\{d(u, x) | x \in V(\Gamma)\}
$$

$$
c_i^\sigma := |\Gamma_{i-1}(u) \cap \Gamma_1(v)|
$$

$$
b_i^\sigma := |\Gamma_{i+1}(u) \cap \Gamma_1(v)|.
$$

In the same way, we could define $D_\delta$, $c_i^\delta$ and $b_i^\delta$.

Example 0.3 Bipartite distance-regular graphs are distance-biregular graphs.

Lemma 0.4 (Delorme)
For each $i$,

(i) $c_{2i}^\sigma c_{2i+1}^\sigma = c_{2i+1}^\delta c_{2i+1}^\delta$

(ii) $b_{2i-1}^\delta b_{2i}^\delta = b_{2i-1}^\delta b_{2i}^\delta$.

*This note is based on a joint work with Akira Hiraki(Division of Mathematical Sciences, Osaka Kyoiku University, Japan) and Jack Koolen(Department of Mathematics, Pohang University of Science and Technology, South Korea)
Proof: \(i\) Let \(d(x, y) = 2i + 1\) and \(x \in V_\sigma\). By counting the number of paths between \(x\) and \(y\) in two ways, we have

\[
c_{2i+1}^\sigma c_{2i}^\sigma \cdots c_2^\sigma = c_{2i+1}^\delta c_{2i}^\delta \cdots c_2^\delta.
\]

(ii) Consider \(\{(u, w) \in V_\sigma \times V_\delta \mid d(u, w) = 2i + 1\}\). Then

\[
\frac{v_2 b_2^\sigma b_4^\sigma \cdots b_{2i}^\delta}{c_2^\sigma \cdots c_{2i}^\sigma} = \frac{v_2 b_2^\delta b_4^\delta \cdots b_{2i}^\delta}{c_2^\delta \cdots c_{2i}^\delta}.
\]

**Proposition 0.5** Let \(\Gamma\) be a distance-biregular graph with valencies \((\sigma + 1, \delta + 1)\). If \(\sigma = \delta\) then \(\Gamma\) is distance-regular.

Proof: By Lemma 0.4, for each \(i \geq 1\), we have the following

\[
(\sigma + 1 - c_{2i-1}^\sigma)(\delta + 1 - c_{2i}^\delta) = (\sigma + 1 - c_{2i-1}^\delta)(\delta + 1 - c_{2i}^\sigma);
\]

\[
c_{2i}^\sigma c_{2i+1}^\sigma = c_{2i}^\delta c_{2i+1}^\delta.
\]

As \(\sigma = \delta\) and \(c_1^\sigma = c_1^\delta = 1\) hold, \(c_i^\sigma = c_i^\delta (1 \leq i \leq D)\) and \(b_i^\sigma = b_i^\delta (0 \leq i \leq D - 1)\) must be satisfied.

By the previous proposition, we may assume that \(\sigma < \delta\) in this note.

**Lemma 0.6** (Delorme)

Let \(\sigma < \delta\). Then

\(i\) \(D_\sigma\) is even;

\(ii\) \(D_\delta \leq D_\sigma \leq D_\delta + 1\).

Let \(g = 2n + 2\) be the girth of a distance biregular graph \(\Gamma\). Note that \(c_{n+1}^\sigma > c_n^\sigma = 1\) and \(c_{n+1}^\delta > c_n^\delta = 1\) are satisfied.

**Theorem 0.7** (Delorme)

\[
\max\{D_\sigma, D_\delta\} \leq \frac{g}{2} \min\{\sigma, \delta\} + 1.
\]

Proof:

\[
\max\{D_\sigma, D_\delta\} \leq \min\{(\sigma - c_{n+1}^\sigma + 2)n + 2, (\delta - c_{n+1}^\delta + 2)n + 2\}
\]

\[
\leq \min\{\sigma, \delta\} n + 2
\]

\[
\leq \frac{g}{2} \min\{\sigma, \delta\} + 1.
\]
We denote by $\Sigma$ (resp. $\Delta$) the graph with vertices the vertices of degree $\sigma + 1$ (resp. $\delta + 1$) and two vertices are adjacent if and only if they are at distance 2 in $\Gamma$. Note that $\Sigma$ (resp. $\Delta$) is a distance-regular graph of order $(\delta, \frac{\sigma + 1}{c_\delta^\sigma} - 1)$ (resp. $(\sigma, \frac{\delta + 1}{c_\sigma^\delta} - 1)$). In other words,

$$\Gamma^{\Sigma}_1(x) \simeq \left( \frac{\sigma + 1}{c_\sigma^\sigma} \right) \cdot K_\delta.$$

Let $N$ be a matrix whose rows and columns are indexed by $V(\Delta)$ and $V(\Sigma)$ respectively. Then we obtain

$$c_\delta^\sigma A_\Delta = NN^T - (1+\delta)I; \quad (1)$$
$$c_\sigma^\delta A_\Sigma = N^TN - (1+\sigma)I. \quad (2)$$

**Lemma 0.8 (Hoffman-Delsarte)**

Let $\Gamma$ be a distance-regular graph with diameter $D$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ be the eigenvalues of $\Gamma$. Then the maximal clique in $\Gamma$ has at most

$$\left( 1 - \frac{\theta_0}{\theta_D} \right)$$

vertices.

**Theorem 0.9** Let $D_\sigma = D_\delta$. Then every maximal clique in $\Sigma$ has exactly

$$\left( 1 - \frac{\theta_0^{(\Sigma)}}{\theta_D^{(\Sigma)}} \right)$$

vertices.

**Theorem 0.10** Let $D_\sigma = D_\delta$. Then every maximal clique in $\Sigma$ has exactly

$$\left( 1 - \frac{\theta_0^{(\Sigma)}}{\theta_D^{(\Sigma)}} \right)$$

vertices.

**Proof:** Let $N \in M_{|V_\delta| \times |V_\delta|}(\mathbb{R})$. Then

$$N^TN = c_\delta^\sigma A_\Sigma + (1+\sigma)I \in M_{|V_\delta| \times |V_\delta|}(\mathbb{R})$$
$$NN^T = c_\sigma^\delta A_\Delta + (1+\delta)I \in M_{|V_\delta| \times |V_\delta|}(\mathbb{R}).$$

As $\sigma < \delta$, $|V(\Sigma)| > |V(\Delta)|$. By the fact $\text{rank}(NN^T) = \text{rank}(N) = \text{rank}(N^TN)$ (i.e., the matrix $N^TN$ has an eigenvalue 0) and the equation (2), the graph $\Sigma$ has an eigenvalue

$$-\frac{1+\sigma}{c_\sigma^\delta}.$$
Therefore the following holds:

\[-\frac{1 + \sigma}{c_2^\sigma} \leq \theta_D^{(\Sigma)} \leq -\frac{1 + \sigma}{c_2^\sigma}.

**Theorem 0.11** (Hiraki and Koolen)

Let $\Gamma$ be a distance-regular graph of order $(s, t)$ with $s > 1$. If $-(t + 1)$ is an eigenvalue of $\Gamma$ then

\[ t < 4^{R-1}. \]

**Theorem 0.12**

Let $\Gamma$ be a distance-biregular graph with valencies $\sigma + 1$ and $\delta + 1$. Let $\sigma < \delta$, $D_\sigma = D_\delta$ and the girth $g \geq 8$. Then there is a function $f$ of $\sigma$ such that

\[ \delta \leq f(\sigma). \]

**Proof:** Recall that $\Sigma$ is distance-regular graph of order $(\sigma, \delta)$. By Theorem 0.11

\[ \delta < 4^D_{\Sigma^{-1}}. \]

Now we will show that

\[ \frac{4D_\Delta}{h_\Delta} - 1 \leq F(\sigma). \]

Note that the following hold:

\[ \frac{1}{h_\Delta} \leq \frac{2}{h - 1} = \frac{4}{g - 4} \]

\[ D_\Delta \leq \frac{g\sigma + 2}{4}. \]

Hence

\[ \frac{4D_\Delta}{h_\Delta} - 1 \leq \frac{4(g\sigma + 2) - g + 4}{g - 4} \leq 9\sigma. \]

Therefore $\delta < \sigma^{9\sigma}$ holds.

**References**
