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A parameter bound on distance-biregular graphs

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Abstract

In this note we consider distance-biregular graphs with valencies $(\sigma+1, \delta+1)$ with $\sigma<\delta$. We show that bigger valency δ is bounded by a function of smaller valency σ.

Definition 0.1 *Let $\Gamma=(V(\Gamma), E(\Gamma))$ be a bipartite connected graph with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$. Let D be a diameter of Γ. Then Γ is called a distance-biregular if for any pair of vertices $u, v \in V(\Gamma)$ with $d(u,v)=i$, the numbers $c_i(u,v)$ and $b_i(u,v)$ are depend only on $d(u,v)=i$ and the partition the vertex u belongs to, where*

$$\begin{aligned} c_i(u,v) &:= |\Gamma_{i-1}(u) \cap \Gamma_1(v)|, \\ b_i(u,v) &:= |\Gamma_{i+1}(u) \cap \Gamma_1(v)|. \end{aligned}$$

Remark 0.2 *(i) The vertices in the same partition have the same intersection array. From now on, let $V(\Gamma)=V_\sigma \cup V_\delta$, where $V_\sigma=\{v \in V(\Gamma) \mid deg(v)=\sigma+1\}$ and $V_\delta=\{v \in V(\Gamma) \mid deg(v)=\delta+1\}$.*
(ii) For $u \in V_\sigma$ and $d(u,v)=i$, define

$$\begin{aligned} D_\sigma &:= \max\{d(u,x) \mid x \in V(\Gamma)\} \\ c_i^\sigma &:= |\Gamma_{i-1}(u) \cap \Gamma_1(v)| \\ b_i^\sigma &:= |\Gamma_{i+1}(u) \cap \Gamma_1(v)|. \end{aligned}$$

In the same way, we could define D_δ, c_i^δ and b_i^δ.

Example 0.3 *Bipartite distance-regular graphs are distance-biregular graphs.*

Lemma 0.4 *(Delorme)*
For each i,
(i) $c_{2i}^\sigma c_{2i+1}^\sigma = c_{2i}^\delta c_{2i+1}^\delta$
(ii) $b_{2i-1}^\sigma b_{2i}^\sigma = b_{2i-1}^\delta b_{2i}^\delta$.

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*This note is based on a joint work with Akira Hiraki(Division of Mathematical Sciences, Osaka Kyoiku University, Japan) and Jack Koolen(Department of Mathematics, Pohang University of Science and Technology, South Korea)

Proof: (i) Let $d(x,y) = 2i+1$ and $x \in V_\sigma$. By counting the number of paths between x and y in two ways, we have

$$c^\sigma_{2i+1}c^\sigma_{2i}\cdots c^\sigma_2 = c^\delta_{2i+1}c^\delta_{2i}\cdots c^\delta_2.$$

(ii) Consider $|\{(u,w) \in V_\sigma \times V_\delta \mid d(u,w) = 2i+1\}|$. Then

$$\frac{v_\sigma b^\sigma_0 b^\sigma_1 \cdots b^\sigma_{2i}}{c^\sigma_2 \cdots c^\sigma_{2i} c^\sigma_{2i+1}} = \frac{v_\delta b^\delta_0 b^\delta_1 \cdots b^\delta_{2i}}{c^\delta_2 \cdots c^\delta_{2i} c^\delta_{2i+1}}.$$

∎

Proposition 0.5 *Let Γ be a distance-biregular graph with valencies $(\sigma+1, \delta+1)$. If $\sigma = \delta$ then Γ is distance-regular.*

Proof: By Lemma 0.4, for each $i \geq 1$, we have the following

$$\begin{aligned} (\delta + 1 - c^\sigma_{2i-1})(\sigma + 1 - c^\sigma_{2i}) &= (\sigma + 1 - c^\delta_{2i-1})(\delta + 1 - c^\delta_{2i}); \\ c^\sigma_{2i} c^\sigma_{2i+1} &= c^\delta_{2i} c^\delta_{2i+1}. \end{aligned}$$

As $\sigma = \delta$ and $c^\sigma_1 = 1 = c^\delta_1$ hold, $c^\delta_i = c^\sigma_i$ $(1 \leq i \leq D)$ and $b^\delta_i = b^\sigma_i$ $(0 \leq i \leq D-1)$ must be satisfied.∎

By the previous proposition, we may assume that $\sigma < \delta$ in this note.

Lemma 0.6 *(Delorme)*
Let $\sigma < \delta$. Then
(i) D_σ is even ;
(ii) $D_\delta \leq D_\sigma \leq D_\delta + 1$.

Let g $= 2n+2$ be the girth of a distance biregular graph Γ. Note that $c^\sigma_{n+1} > c^\delta_n = 1$ and $c^\delta_{n+1} > c^\sigma_n = 1$ are satisfied.

Theorem 0.7 *(Delorme)*

$$\max\{D_\sigma, D_\delta\} \leq \frac{g}{2}\min\{\sigma,\delta\} + 1.$$

Proof:

$$\begin{aligned} \max\{D_\sigma, D_\delta\} &\leq \min\{(\sigma - c^\sigma_{n+1} + 2)n + 2, (\delta - c^\delta_{n+1} + 2)n + 2\} \\ &\leq \min\{\sigma,\delta\}n + 2 \\ &\leq \frac{g}{2}\min\{\sigma,\delta\} + 1. \end{aligned}$$

∎

We denote by Σ (resp. Δ) the graph with vertices the vertices of degree $\sigma+1$ (resp. $\delta+1$) and two vertices are adjacent if and only if they are at distance 2 in Γ. Note that Σ (resp. Δ) is a distance-regular graph of order $(\delta, \frac{\sigma+1}{c_2^\sigma}-1)$ (resp. $(\sigma, \frac{\delta+1}{c_2^\delta}-1)$). In other words,

$$\Gamma_1^{(\Sigma)}(x) \simeq \left(\frac{\sigma+1}{c_2^\sigma}\right) * K_\delta.$$

Let N be a matrix whose rows and columns are indexed by $V(\Delta)$ and $V(\Sigma)$ respectively. Then we obtain

$$\begin{aligned} c_2^\delta A_\Delta &= NN^T-(1+\delta)I\ ; & (1)\\ c_2^\sigma A_\Sigma &= N^TN-(1+\sigma)I. & (2) \end{aligned}$$

Lemma 0.8 *(Hoffman-Delsarte)*
Let Γ be a distance-regular graph with diameter D. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ be the eigenvalues of Γ. Then the maximal clique in Γ has at most

$$\left(1-\frac{\theta_0}{\theta_D}\right)$$

vertices.

Theorem 0.9 *Let $D_\sigma = D_\delta$. Then every maximal clique in Σ has exactly*

$$\left(1-\frac{\theta_0^{(\Sigma)}}{\theta_{D_\sigma/2}^{(\Sigma)}}\right)$$

vertices.

Theorem 0.10 *Let $D_\sigma = D_\delta$. Then every maximal clique in Σ has exactly*

$$\left(1-\frac{\theta_0^{(\Sigma)}}{\theta_{D_\Sigma}^{(\Sigma)}}\right)$$

vertices.

Proof: Let $N \in M_{|V_\delta|\times|V_\sigma|}(\mathbb{R})$. Then

$$\begin{aligned} N^TN &= c_2^\sigma A_{(\Sigma)} + (1+\sigma)I \ \in M_{|V_\sigma|\times|V_\sigma|}(\mathbb{R})\\ NN^T &= c_2^\delta A_{(\Delta)} + (1+\delta)I \ \in M_{|V_\delta|\times|V_\delta|}(\mathbb{R}). \end{aligned}$$

As $\sigma<\delta$, $|V(\Sigma)| > |V(\Delta)|$. By the fact $rank(NN^T) = rank(N) = rank(N^TN)$ (i.e., the matrix N^TN has an eigenvalue 0) and the equation (2), the graph Σ has an eigenvalue

$$-\frac{1+\sigma}{c_2^\sigma}.$$

Therefore the following holds:

$$-\frac{1+\sigma}{c_2^\sigma} \leq \theta_{D_\Sigma}^{(\Sigma)} \leq -\frac{1+\sigma}{c_2^\sigma}.$$

∎

Theorem 0.11 *(Hiraki and Koolen)*
Let Γ be a distance-regular graph of order (s,t) with $s > 1$.
If $-(t+1)$ is an eigenvalue of Γ then

$$t < s^{\frac{4D}{h}-1}.$$

Theorem 0.12 *Let Γ be a distance-biregular graph with valencies $\sigma+1$ and $\delta+1$. Let $\sigma < \delta$, $D_\sigma = D_\delta$ and the girth $g \geq 8$. Then there is a function f of σ such that*

$$\delta \leq f(\sigma).$$

Proof: Recall that Σ is distance-regular graph of order (σ, δ). By Theorem 0.11

$$\delta < \sigma^{\frac{4D_\Delta}{h_\Delta}-1}.$$

Now we will show that

$$\frac{4D_\Delta}{h_\Delta} - 1 \leq F(\sigma).$$

Note that the following hold:

$$\begin{aligned} \frac{1}{h_\Delta} &\leq \frac{2}{h-1} = \frac{4}{g-4} \\ D_\Delta &\leq \frac{g\sigma+2}{4}. \end{aligned}$$

Hence

$$\frac{4D_\Delta}{h_\Delta} - 1 \leq \frac{4(g\sigma+2)-g+4}{g-4} \leq 9\sigma.$$

Therefore $\delta < \sigma^{9\sigma}$ holds. ∎

References

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[2] Akira Hiraki and Jack Koolen, A Higman-Haemers Inequality for Thick Regular Near Polygons, *Journal of algebraic combinatorics* 20 (2004), 87-92