A parameter bound on distance-biregular graphs

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Abstract
In this note we consider distance-biregular graphs with valencies $(\sigma+1, \delta+1)$ with $\sigma < \delta$. We show that bigger valency $\delta$ is bounded by a function of smaller valency $\sigma$.

Definition 0.1 Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a bipartite connected graph with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$. Let $D$ be a diameter of $\Gamma$. Then $\Gamma$ is called a distance-biregular if for any pair of vertices $u, v \in V(\Gamma)$ with $d(u, v) = i$, the numbers $c_i(u,v)$ and $b_i(u,v)$ are depend only on $d(u,v) = i$ and the partition the vertex $u$ belongs to, where
\[
c_i(u,v) := |\Gamma_{i-1}(u) \cap \Gamma_1(v)|,
b_i(u,v) := |\Gamma_{i+1}(u) \cap \Gamma_1(v)|.
\]

Remark 0.2 (i) The vertices in the same partition have the same intersection array. From now on, let $V(\Gamma) = V_\sigma \cup V_\delta$, where $V_\sigma = \{ v \in V(\Gamma) \mid \deg(v) = \sigma + 1 \}$ and $V_\delta = \{ v \in V(\Gamma) \mid \deg(v) = \delta + 1 \}$.
(ii) For $u \in V_\sigma$ and $d(u,v) = i$, define
\[
D_\sigma := \max\{d(u,x) \mid x \in V(\Gamma)\},
c_i^\sigma := |\Gamma_{i-1}(u) \cap \Gamma_1(v)|,
b_i^\sigma := |\Gamma_{i+1}(u) \cap \Gamma_1(v)|.
\]

In the same way, we could define $D_\delta, c_i^\delta$ and $b_i^\delta$.

Example 0.3 Bipartite distance-regular graphs are distance-biregular graphs.

Lemma 0.4 (Delorme)
For each $i$,
(i) $c_{2i}^\sigma c_{2i+1}^\sigma = c_{2i+1}^\delta b_{2i+1}^\delta$
(ii) $b_{2i-1}^\delta b_{2i}^\delta = b_{2i-1}^\delta b_{2i}^\delta$.

*This note is based on a joint work with Akira Hiraki(Division of Mathematical Sciences, Osaka Kyoku University, Japan) and Jack Koolen(Department of Mathematics, Pohang University of Science and Technology, South Korea)
Proof: (i) Let $d(x, y) = 2i + 1$ and $x \in V_{\sigma}$. By counting the number of paths between $x$ and $y$ in two ways, we have
\[ c_{2i+1}^{\sigma}c_{2i}^{\sigma} \cdots c_{2}^{\sigma} = c_{2i+1}^{\delta}c_{2i}^{\delta} \cdots c_{2}^{\delta}. \]
(ii) Consider $|\{(u, w) \in V_{\sigma} \times V_{\delta} \mid d(u, w) = 2i + 1\}|$. Then
\[ \frac{v_{\sigma}b_{0}^{\sigma}b_{1}^{\sigma} \cdots b_{2i}^{\sigma}}{c_{2}^{\sigma} \cdots c_{2i}^{\sigma}c_{2i+1}^{\sigma}} = \frac{v_{\delta}b_{0}^{\delta}b_{1}^{\delta} \cdots b_{2i}^{\delta}}{c_{2}^{\delta} \cdots c_{2i}^{\delta}c_{2i+1}^{\delta}}. \]

**Proposition 0.5** Let $\Gamma$ be a distance-biregular graph with valencies $(\sigma + 1, \delta + 1)$. If $\sigma = \delta$ then $\Gamma$ is distance-regular.

**Proof:** By Lemma 0.4, for each $i \geq 1$, we have the following
\[ (\delta + 1 - c_{2i-1}^{\sigma})(\sigma + 1 - c_{2i}^{\sigma}) = (\sigma + 1 - c_{2i-1}^{\delta})(\delta + 1 - c_{2i}^{\delta}); \]
\[ c_{2i}^{\sigma}c_{2i+1}^{\sigma} = c_{2i}^{\delta}c_{2i+1}^{\delta}. \]
As $\sigma = \delta$ and $c_{1}^{\sigma} = 1 = c_{1}^{\delta}$ hold, $c_{i}^{\sigma} = c_{i}^{\delta} (1 \leq i \leq D)$ and $b_{i}^{\sigma} = b_{i}^{\delta} (0 \leq i \leq D - 1)$ must be satisfied. By the previous proposition, we may assume that $\sigma < \delta$ in this note.

**Lemma 0.6** (Delorme)
Let $\sigma < \delta$. Then
(i) $D_{\sigma}$ is even;
(ii) $D_{\delta} \leq D_{\sigma} \leq D_{\delta} + 1$.

Let $g = 2n + 2$ be the girth of a distance biregular graph $\Gamma$. Note that $c_{n+1}^{\sigma} > c_{n}^{\sigma} = 1$ and $c_{n+1}^{\delta} > c_{n}^{\delta} = 1$ are satisfied.

**Theorem 0.7** (Delorme)
\[ \max\{D_{\sigma}, D_{\delta}\} \leq \frac{g}{2} \min\{\sigma, \delta\} + 1. \]

**Proof:**
\[ \max\{D_{\sigma}, D_{\delta}\} \leq \min\{(\sigma - c_{n+1}^{\sigma} + 2)n + 2, (\delta - c_{n+1}^{\delta} + 2)n + 2\} \]
\[ \leq \min\{\sigma, \delta\}n + 2 \]
\[ \leq \frac{g}{2} \min\{\sigma, \delta\} + 1. \]
We denote by $\Sigma$ (resp. $\Delta$) the graph with vertices the vertices of degree $\sigma + 1$ (resp. $\delta + 1$) and two vertices are adjacent if and only if they are at distance 2 in $\Gamma$. Note that $\Sigma$ (resp. $\Delta$) is a distance-regular graph of order $(\delta, \frac{\sigma + 1}{c_{\sigma}^2} - 1)$ (resp. $(\sigma, \frac{\delta + 1}{c_{\delta}^2} - 1)$). In other words,

$$\Gamma_1^{(\Sigma)}(x) \simeq \left( \frac{\sigma + 1}{c_{\sigma}^2} \right) * K_\delta.$$

Let $N$ be a matrix whose rows and columns are indexed by $V(\Delta)$ and $V(\Sigma)$ respectively. Then we obtain

$$c_{\delta}^2 A_{\Delta} = NN^T - (1 + \delta)I; \quad (1)$$
$$c_{\sigma}^2 A_{\Sigma} = N^T N - (1 + \sigma)I. \quad (2)$$

**Lemma 0.8 (Hoffman-Delsarte)**

Let $\Gamma$ be a distance-regular graph with diameter $D$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ be the eigenvalues of $\Gamma$. Then the maximal clique in $\Gamma$ has at most

$$\left(1 - \frac{\theta_0}{\theta_D}\right)$$
vertices.

**Theorem 0.9** Let $D_\sigma = D_\delta$. Then every maximal clique in $\Sigma$ has exactly

$$\left(1 - \frac{\theta_0^{(\Sigma)}}{\theta_D^{(\Sigma)}}\right)$$
vertices.

**Theorem 0.10** Let $D_\sigma = D_\delta$. Then every maximal clique in $\Sigma$ has exactly

$$\left(1 - \frac{\theta_0^{(\Sigma)}}{\theta_D^{(\Sigma)}}\right)$$
vertices.

**Proof:** Let $N \in M_{|V_\delta| \times |V_\delta|}(\mathbb{R})$. Then

$$N^T N = c_{\delta}^2 A_{\Delta} + (1 + \delta)I \in M_{|V_\delta| \times |V_\delta|}(\mathbb{R})$$
$$NN^T = c_{\sigma}^2 A_{\Sigma} + (1 + \delta)I \in M_{|V_\delta| \times |V_\delta|}(\mathbb{R}).$$

As $\sigma < \delta$, $|V(\Sigma)| > |V(\Delta)|$. By the fact $\text{rank}(NN^T) = \text{rank}(N) = \text{rank}(N^T N)$ (i.e., the matrix $N^T N$ has an eigenvalue 0) and the equation (2), the graph $\Sigma$ has an eigenvalue

$$-\frac{1 + \sigma}{c_{\sigma}^2}.$$
Therefore the following holds:

\[-\frac{1 + \sigma}{c_2^2} \leq \theta_{\Sigma}^{(\Sigma)} \leq -\frac{1 + \sigma}{c_2^2}.

**Theorem 0.11** (Hiraki and Koolen)

Let $\Gamma$ be a distance-regular graph of order $(s, t)$ with $s > 1$. If $-(t + 1)$ is an eigenvalue of $\Gamma$ then

\[ t < \frac{4D}{h_{\Delta}} - 1. \]

**Theorem 0.12** Let $\Gamma$ be a distance-biregular graph with valencies $\sigma + 1$ and $\delta + 1$. Let $\sigma < \delta$, $D_\sigma = D_\delta$ and the girth $g \geq 8$. Then there is a function $f$ of $\sigma$ such that

\[ \delta \leq f(\sigma). \]

**Proof:** Recall that $\Sigma$ is distance-regular graph of order $(\sigma, \delta)$. By Theorem 0.11

\[ \delta < \frac{4D_{\Delta}}{h_{\Delta}} - 1. \]

Now we will show that

\[ \frac{4D_{\Delta}}{h_{\Delta}} - 1 \leq F(\sigma). \]

Note that the following hold:

\[ \frac{1}{h_{\Delta}} \leq \frac{2}{h - 1} = \frac{4}{g - 4}, \]

\[ D_{\Delta} \leq \frac{g\sigma + 2}{4}. \]

Hence

\[ \frac{4D_{\Delta}}{h_{\Delta}} - 1 \leq \frac{4(g\sigma + 2) - g + 4}{g - 4} \leq 9\sigma. \]

Therefore $\delta < \sigma^{9\sigma}$ holds.

**References**
