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The intersection of normal closed subsets of association scheme is not always normal

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Three questions on normal closed subsets of association schemes:

1. The intersection of normal closed subsets of an association scheme is normal?

2. When a simple association scheme is primitive?

3. If N and N' are normal closed subsets of G such that $G//N$ and $G//N'$ are commutative, then $G//N \cap N'$ is commutative?

Motivation

1. Q1 is a natural question for non-commutative schemes.

Association schemes have closed subsets corresponding to subgroups, and normal closed subsets and strongly normal closed subsets corresponding to normal subgroups.

Closed subsets and strongly normal closed subsets are closed on the intersection as well as subgroups and normal subgroups. However we have not known for normal closed subsets.

We have Hanaki's result for group-like schemes but we do not know for general schemes.

I had to calculate normal closed subsets for Q2 and Q3.

2. We want to consider the solvability of association schemes. However we do not have the suitable definition. So I want to know what condition occurs when to be simple is to be primitive.

An association scheme is simple iff the scheme has no non-trivial normal closed subsets.

An association scheme is primitive iff the scheme has no non-trivial closed subsets.

A finite solvable simple group has no non-trivial subgoups. So simple $\Leftrightarrow$ primitive is the one of characteristics of finite solvable groups.

I will verify the validity by considering Burnside's $p^a q^b$ theorem and Feit-Thompson theorem.

3. We want to consider the normal closed subsets corresponding to the derived subgroup.

We have the strongly normal closed subsets corresponding to the derived subgroup. However the factor scheme by a strongly normal closed subsets is essentially a finite group.

Normal Closed Subsets and Character Table (Hanaki)

We can find all normal closed subsets of G from its character table.

Let η be a character of G. We set

$$K(\eta) := \{g \in G \mid \eta(g) = n_g \eta(1)\}.$$

Generally $K(\eta)$ is not always a normal closed subset.

Proposition 1. *For a normal closed subset N of G, there exist a character η of G such that $N = K(\eta)$.*

For a character η of G, we set

$$I(\eta) := \{\chi \in Irr(G) \mid \chi(g) = n_g \chi(1) \text{ for all } g \in K(\eta)\}.$$

Theorem 2. *For a character η of G, $K(\eta) \triangleleft G$ if and only if*

$$\sum_{\chi \in I(\eta)} m_\chi \chi(1) = \frac{n_G}{n_{K(\eta)}}.$$

Theorem 3. *Let χ and η be characters of G.*

$$K(\chi+\eta)=K(\chi)\cap K(\eta).$$

Group-like association scheme (Hanaki)
We define a relation $\sim$ on G.

For $g,h\in G$, we write $g\sim h$ if $n_g^{-1}\chi(g)=n_h^{-1}\chi(h)$ for any $\chi\in Irr(G)$.

Then this relation is an equivalent relation. Let $T_1,\ldots,T_s$ be $\sim$ equivalent classes.

We say (X,G) is group-like if $s=\mid Irr(G)\mid$.

If G is a finite group, then this relation is the conjugacy relation.

Theorem 4. *We set $A_{T_i}=\sum_{g\in T_i}A_g$.*

$$Z(\mathbb{C}G)=\bigoplus_{1\leq i\leq s}\mathbb{C}A_{T_i}.$$

For the question 1, we know the following theorem.

Theorem 5. *Let (X,G) be a group-like association scheme. Then the intersection of normal closed subsets is also normal.*

Q1. The intersection of normal closed subsets of an association scheme is also normal?

From Theorem 5, it is enough that we checked only non-group-like schemes.

I found three couterexamples for this question.
(as16[186], as18[79], as24[452])

as16[186]

	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	m_χ
χ_1	1	1	1	1	1	1	1	1	2	2	2	2	1
χ_2	1	1	1	1	1	1	1	1	−2	−2	−2	−2	1
χ_3	1	1	1	1	−1	−1	−1	−1	2	2	−2	−2	1
χ_4	1	1	1	1	−1	−1	−1	−1	−2	−2	2	2	1
χ_5	2	2	−2	−2	0	0	0	0	0	0	0	0	2
χ_6	1	−1	1	−1	1	−1	1	−1	0	0	0	0	2
χ_7	1	−1	1	−1	−1	1	−1	1	0	0	0	0	2
χ_8	1	−1	−1	1	1	−1	−1	1	0	0	0	0	2
χ_9	1	−1	−1	1	−1	1	1	−1	0	0	0	0	2
$\chi_7 + \chi_8$	2	−2	0	0	0	0	−2	2	0	0	0	0	

$K(\chi_7) = [0, 2, 5, 7], K(\chi_8) = [0, 3, 4, 7], K(\chi_7 + \chi_8) = [0, 7]$
$I(\chi_7) = \{\chi_1, \chi_2, \chi_7\}, I(\chi_8) = \{\chi_1, \chi_2, \chi_8\}, I(\chi_7+\chi_8) = \{\chi_1, \chi_2, \chi_7, \chi_8\}$

$m_{\chi_1}\chi_1(A_0) + m_{\chi_2}\chi_2(A_0) + m_{\chi_7}\chi_7(A_0) = 4 = 16/4 = n_G/n_{K(\chi_7)}$
$m_{\chi_1}\chi_1(A_0) + m_{\chi_2}\chi_2(A_0) + m_{\chi_8}\chi_8(A_0) = 4 = 16/4 = n_G/n_{K(\chi_8)}$
$m_{\chi_1}\chi_1(A_0) + m_{\chi_2}\chi_2(A_0) + m_{\chi_7}\chi_7(A_0) + m_{\chi_8}\chi_8(A_0) = 6.$
$n_G/n_{K(\chi_7+\chi_8)} = 8.$

Calculation

We assume that we have the set G of adjacency matrices of an association scheme.

We set $e_s = n_s^{-1}A_s$ for $s \subseteq G$, where $n_s = \sum_{A_i \in s} n_i, A_s = \sum_{A_i \in s} A_i$.

We use the following facts.

$H \subseteq G$ is a closed subset iff e_H is an idempotent of $\mathbb{C}G$.

$H \subseteq G$ is a normal closed subset iff e_H is a central idempotent of $\mathbb{C}G$.

1. We make the set $S = \{s \subseteq G \mid A_0 \in s\}$.

2. We make the set $C = \{s \in S \mid e_s^2 = e_s\}$.
   Thus C is the set of all closed subsets of G.

3. We make the set $N = \{s \in C \mid A_i e_s = e_s A_i \text{ for } A_i \in G\}$.
   Thus N is the set of all normal closed subsets of G.

We can get relation matrices of all association schemes with $|X| \leq 29$ from Hanaki's HP. This method is very simple but it is enough for assocation schemes with $|X| \leq 29$. To be enough means that the computation time is not very long.

When a simple scheme is primitive?

I searched for association schemes with $|X| \leq 29$.

as20[51], as20[66], as21[19], as28[123] are simple and imprimitive.

All of them are non-group-like association schemes.

We consider under the following conditions:

1. We use the cardinality of X instead of the order of the finite group.
2. We define the solvability of assoication schemes as simple $\not\Leftrightarrow$ primitive.

$\longrightarrow$ Burnside's $p^a q^b$ theorem $\times$

Feit-Thompson theorem $\times$

If we consider only group-like association schemes with $|X| \leq 29$, both of them hold.

Future Tasks

1. Under above conditions, two theorems hold for group-like association schemes?

2. Under above conditions, two theorems hold for p-power order association schemes?

Hirasaka, Ponomarenko and Zieschang approach to this problem by restriction of valencies.

We consider the Frame number of an association scheme.

$$\mathfrak{F}(G) = |X|^{|G|} \frac{\prod_{g \in G} n_g}{\prod_{\chi \in Irr(G)} m_\chi^{\chi(1)^2}}$$

We know the following theorem.

Theorem 6. *Let p be a prime number and F a field of characteristic p.*

$$p \nmid \mathfrak{F}(G) \text{ iff } FG \text{ is semisimple.}$$

We may use the Frame number instead of the order of the finite group.

$$\begin{aligned}
\mathfrak{F}(\text{as20}[51]) &= 2^{11}\, 3^4\, 5^6 \\
\mathfrak{F}(\text{as20}[66]) &= 2^{13}\, 3^4\, 5^7 \\
\mathfrak{F}(\text{as21}[19]) &= 2^{4}\, 3^5\, 7^6 \\
\mathfrak{F}(\text{as28}[123]) &= 2^{12}\, 3^5\, 7^6
\end{aligned}$$

Thus they do not satisfy the assumption of two theorem. However I think that the Frame number is very large for the order of the finite group.

Q3. If $N, N' \triangleleft G$ such that $G//N, G//N'$ are commutative, $G//N \cap N'$ is commutative?

Generally the question holds if N, N' are strongly normal.

Actually we can define a strongly normal closed subset corresponding to the derived subgroup.

Let R be the thin residue of G.

Then $G//R$ is essentially the finite group.

We can obtain the derived subgroup $D(G//R)$ of $G//R$.

By the Homomorphis theorem, we can obtain $D(G) \triangleleft^{\sharp} G$ corresponding to $D(G//R)$.

$D(G)$ is the intersection of strongly normal closed subsets by which the factor schemes is an abelian group.

We can check by using the following fact. We assume that $N \triangleleft G$ and $\chi \in Irr(G)$. Then

$$\chi \in Irr(G//N) \text{ iff } \chi(h) = n_h \chi(1) \text{ for any } h \in N.$$

Result

We do not have the normal closed subset corresponding to the derived subgroup even for group-like association schemes.

(as16[158],as16[170])

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