数理解析研究所講究録 1440 巻 2005 年 41-47

Tridiagonal pairs and the quantum affine algebra $U_q(\widehat{sl_2})$

Tatsuro Ito

Kanazawa University
Kanazawa Japan

Paul Terwilliger

University of Wisconsin
Madison Wisconsin

Overview

In this talk, I will first recall the notion of a **Leonard pair** and discuss how these objects are related to certain classical orthogonal polynomials.

I will then define a generalization of a Leonard pair called a **Tridiagonal pair**.

I will then show how certain tridiagonal pairs are related to finite dimensional modules for the quantum affine algebra $U_q(\widehat{sl_2})$.

Leonard pairs

We recall the notion of a Leonard pair. To do this, we first recall what it means for a matrix to be **tridiagonal**.

The following matrices are tridiagonal.

$$\begin{pmatrix} 2 & 3 & 0 & 0 \\ 1 & 4 & 2 & 0 \\ 0 & 5 & 3 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

Tridiagonal means each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal.

The tridiagonal matrix on the left is **irreducible**. This means each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

The Definition of a Leonard Pair

We now define a Leonard pair. From now on $\mathbb{K}$ will denote a field.

Definition Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a **Leonard pair** on V, we mean a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ which satisfy both conditions below.

1. There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.

2. There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Example of a Leonard pair

For any nonnegative integer d the pair

$$A = \begin{pmatrix} 0 & d & 0 & & & 0 \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ 0 & & & & d & 0 \end{pmatrix},$$

$$A^* = \text{diag}(d, d-2, d-4, \ldots, -d)$$

is a Leonard pair on the vector space $\mathbb{K}^{d+1}$, provided the characteristic of $\mathbb{K}$ is 0 or an odd prime greater than d.

Reason: There exists an invertible matrix P such that $P^{-1}AP = A^*$ and $P^2 = 2^d I$.

Why Leonard pairs are of interest

There is a natural correspondence between Leonard pairs and a family of orthogonal polynomials consisting of the following types:

q-Racah,
q-Hahn,
dual q-Hahn,
q-Krawtchouk,
dual q-Krawtchouk,
quantum q-Krawtchouk,
affine q-Krawtchouk,
Racah,
Hahn,
dual-Hahn,
Krawtchouk,
Bannai/Ito,
orphans (char($\mathbb{K}$) = 2 only).

This family coincides with the terminating branch of the Askey scheme of orthogonal polynomials.

Tridiagonal pairs

We now consider a generalization of a Leonard pair called a **tridiagonal pair**.

A tridiagonal pair is defined as follows.

As before, V will denote a vector space over $\mathbb{K}$ with finite positive dimension.

As before, we consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$.

Definition of a Tridiagonal pair

We say the pair A, A^* is a **Tridiagonal pair** on V whenever (1)–(4) hold below.

1. Each of A, A^* is diagonalizable on V.
2. There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of A such that
   $A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \le i \le d)$,
   where $V_{-1} = 0$, $V_{d+1} = 0$.
3. There exists an ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of A^* such that
   $A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \le i \le \delta)$,
   where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.
4. There is no subspace $W \subseteq V$ such that $AW \subseteq W$ and $A^*W \subseteq W$ and $W \neq 0$ and $W \neq V$.

A comment

Referring to our definition of a tridiagonal pair,

it turns out $d = \delta$; we call this the **diameter** of the pair.

Leonard pairs and Tridiagonal pairs

We mentioned a tridiagonal pair is a generalization of a Leonard pair.

A Leonard pair is the same thing as a tridiagonal pair for which the eigenspaces V_i and V_i^* all have dimension 1.

An open problem

Problem Classify the tridiagonal pairs.

For the rest of this talk we focus on a special case of tridiagonal pair said to have **geometric type.**

We will show these tridiagonal pairs are related to $U_q(\widehat{sl_2})$.

We hope this will lead to a classification of the tridiagonal pairs of geometric type.

Tridiagonal pairs of geometric type

Let A, A^* denote a tridiagonal pair on V with diameter d.

Let the eigenspaces V_i, V_i^* $(0 \leq i \leq d)$ be as in the definition.

Let q denote a nonzero scalar in $\mathbb{K}$ which is not a root of unity.

We say A, A^* has q-**geometric type** whenever for $0 \leq i \leq d$, the eigenvalue of A for V_i is q^{2i-d} and the eigenvalue of A^* for V_i^* is q^{d-2i}.

Tridiagonal pairs of geometric type and $U_q(\widehat{sl_2})$

For the rest of this talk, A, A^* denotes a tridiagonal pair on V of diameter d and q-geometric type.

Using A, A^* we will construct two actions of $U_q(\widehat{sl_2})$ on V.

We use the following notation.

Decompositions of V

By a **decomposition of** V, we mean a sequence $U_0, U_1, \ldots, U_d$ consisting of nonzero subspaces of V such that

$$V = U_0 + U_1 + \cdots + U_d \qquad \text{(direct sum).}$$

We do not assume each of $U_0, U_1, \ldots, U_d$ has dimension 1.

Six Decompositions of V

We are about to define six decompositions of V.

In order to keep track of these decompositions we will give each of them a name.

Our naming scheme is as follows.

Let Ω denote the set consisting of the four symbols $0, D, 0^*, D^*$.

Each of the six decompositions will get a name $[u]$ where u is a two-element subset of Ω.

We now define the six decompositions.

Six Decompositions of V, cont.

Lemma For each of the six rows in the table below, and for $0 \le i \le d$, let U_i denote the ith subspace described in that row. Then the sequence $U_0, U_1, \ldots, U_d$ is a decomposition of V.

name	ith subspace of the decomposition
$[0D]$	V_i
$[0^*D^*]$	V_i^*
$[0^*D]$	$(V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d)$
$[0^*0]$	$(V_0^* + \cdots + V_i^*) \cap (V_0 + \cdots + V_{d-i})$
$[D^*0]$	$(V_{d-i}^* + \cdots + V_d^*) \cap (V_0 + \cdots + V_{d-i})$
$[D^*D]$	$(V_{d-i}^* + \cdots + V_d^*) \cap (V_i + \cdots + V_d)$

How the six decompositions are related

Let $U_0, U_1, \ldots, U_d$ denote any one of the six decompositions of V. Then for $0 \leq i \leq d$ the sums $U_0 + \cdots + U_i$ and $U_i + \cdots + U_d$ are given as follows.

name	$U_0 + \cdots + U_i$	$U_i + \cdots + U_d$
[0D]	$V_0 + \cdots + V_i$	$V_i + \cdots + V_d$
[0*D*]	$V_0^* + \cdots + V_i^*$	$V_i^* + \cdots + V_d^*$
[0*D]	$V_0^* + \cdots + V_i^*$	$V_i + \cdots + V_d$
[0*0]	$V_0^* + \cdots + V_i^*$	$V_0 + \cdots + V_{d-i}$
[D*0]	$V_{d-i}^* + \cdots + V_d^*$	$V_0 + \cdots + V_{d-i}$
[D*D]	$V_{d-i}^* + \cdots + V_d^*$	$V_i + \cdots + V_d$

The linear transformations B, B^*, K, K^*

We now define four linear transformations from V to V.

We call these B, B^*, K, K^*.

Each of these transformations is diagonalizable.

To define them, we list their eigenspaces and the corresponding eigenvalues as follows.

name	ith sbspace is eigspace	Corresp. eigval
B	[0*0]	q^{2i-d}
B^*	[D*D]	q^{d-2i}
K	[0*D]	q^{2i-d}
K^*	[D*0]	q^{2i-d}

The q-Weyl Relations

We have

$$\begin{aligned} \frac{qAB - q^{-1}BA}{q - q^{-1}} &= I, \\ \frac{qBA^* - q^{-1}A^*B}{q - q^{-1}} &= I, \\ \frac{qA^*B^* - q^{-1}B^*A^*}{q - q^{-1}} &= I, \\ \frac{qB^*A - q^{-1}AB^*}{q - q^{-1}} &= I. \end{aligned}$$

The q-Weyl Relations, cont.

We have

$$\begin{aligned} \frac{qK^{-1}A - q^{-1}AK^{-1}}{q - q^{-1}} &= I, \\ \frac{qBK^{-1} - q^{-1}K^{-1}B}{q - q^{-1}} &= I, \\ \frac{qKA^* - q^{-1}A^*K}{q - q^{-1}} &= I, \\ \frac{qB^*K - q^{-1}KB^*}{q - q^{-1}} &= I. \end{aligned}$$

q-Weyl Relations cont.

We have

$$
\begin{aligned}
\frac{qAK^* - q^{-1}K^*A}{q-q^{-1}} &= I, \\
\frac{qK^{*-1}B - q^{-1}BK^{*-1}}{q-q^{-1}} &= I, \\
\frac{qA^*K^{*-1} - q^{-1}K^{*-1}A^*}{q-q^{-1}} &= I, \\
\frac{qK^*B^* - q^{-1}B^*K^*}{q-q^{-1}} &= I.
\end{aligned}
$$

The q-Serre relations

We have

$$
\begin{aligned}
A^3A^* - [3]_qA^2A^*A + [3]_qAA^*A^2 - A^*A^3 &= 0, \\
A^{*3}A - [3]_qA^{*2}AA^* + [3]_qA^*AA^{*2} - AA^{*3} &= 0, \\
B^3B^* - [3]_qB^2B^*B + [3]_qBB^*B^2 - B^*B^3 &= 0, \\
B^{*3}B - [3]_qB^{*2}BB^* + [3]_qB^*BB^{*2} - BB^{*3} &= 0,
\end{aligned}
$$

where

$$
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad n = 0, 1, \ldots
$$

Tridiagonal pairs of geometric type and $U_q(\widehat{sl_2})$

We now use A, A^*, B, B^*, K, K^* to get two actions of $U_q(\widehat{sl_2})$ on V.

Before proceeding we recall the definition of $U_q(\widehat{sl_2})$.

The algebra $U_q(\widehat{sl_2})$

Definition The quantum affine algebra $U_q(\widehat{sl_2})$ is the unital associative $\mathbb{K}$-algebra with generators $e_i^{\pm}$, $K_i^{\pm 1}$, $i \in \{0, 1\}$ and the following relations:

$$
\begin{aligned}
K_iK_i^{-1} &= K_i^{-1}K_i = 1, \\
K_0K_1 &= K_1K_0, \\
K_ie_i^{\pm}K_i^{-1} &= q^{\pm 2}e_i^{\pm}, \\
K_ie_j^{\pm}K_i^{-1} &= q^{\mp 2}e_j^{\pm}, \qquad i \neq j, \\
[e_i^+, e_i^-] &= \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
[e_0^{\pm}, e_1^{\mp}] &= 0,
\end{aligned}
$$

$$
(e_i^{\pm})^3e_j^{\pm} - [3]_q(e_i^{\pm})^2e_j^{\pm}e_i^{\pm} + [3]_qe_i^{\pm}e_j^{\pm}(e_i^{\pm})^2 - e_j^{\pm}(e_i^{\pm})^3 = 0, \qquad i \neq j.
$$

We call $e_i^{\pm}$, $K_i^{\pm 1}$, $i \in \{0, 1\}$ the **Chevalley generators** for $U_q(\widehat{sl_2})$.

Another presentation of $U_q(\widehat{sl_2})$

In order to state our main results we introduce a second presentation of $U_q(\widehat{sl_2})$.

Alternate presentation of $U_q(\widehat{sl_2})$

theorem The quantum affine algebra $U_q(\widehat{sl_2})$ is isomorphic to the unital associative $\mathbb{K}$-algebra with generators $y_i^{\pm}$, $k_i^{\pm 1}$, $i \in \{0,1\}$ and the following relations:

$$
\begin{aligned}
k_i k_i^{-1} = k_i^{-1} k_i &= 1, \\
k_0 k_1 \text{ is central,} & \\
\frac{q y_i^+ k_i - q^{-1} k_i y_i^+}{q - q^{-1}} &= 1, \\
\frac{q k_i y_i^- - q^{-1} y_i^- k_i}{q - q^{-1}} &= 1, \\
\frac{q y_i^- y_i^+ - q^{-1} y_i^+ y_i^-}{q - q^{-1}} &= 1, \\
\frac{q y_i^+ y_j^- - q^{-1} y_j^- y_i^+}{q - q^{-1}} &= k_0^{-1} k_1^{-1}, \qquad i \neq j,
\end{aligned}
$$

$$
(y_i^{\pm})^3 y_j^{\pm} - [3]_q (y_i^{\pm})^2 y_j^{\pm} y_i^{\pm} + [3]_q y_i^{\pm} y_j^{\pm} (y_i^{\pm})^2 - y_j^{\pm} (y_i^{\pm})^3 = 0, \qquad i \neq j.
$$

We call $y_i^{\pm}$, $k_i^{\pm 1}$, $i \in \{0,1\}$ the **alternate generators** of $U_q(\widehat{sl_2})$.

Two actions for $U_q(\widehat{sl_2})$

Theorem Let A, A^* denote a tridiagonal pair on V of geometric type. Let the maps B, B^*, K, K^* be as above.

Then V is an irreducible $U_q(\widehat{sl_2})$-module on which the alternate generators act as follows.

generator	y_0^+	y_1^+	y_0^-	y_1^-	k_0	k_1	k_0^{-1}	k_1^{-1}
action on V	B^*	B	A^*	A	K	K^{-1}	K^{-1}	K

Also, V is an irreducible $U_q(\widehat{sl_2})$-module on which the alternate generators act as follows.

generator	y_0^+	y_1^+	y_0^-	y_1^-	k_0	k_1	k_0^{-1}	k_1^{-1}
action on V	A	A^*	B^*	B	K^*	K^{*-1}	K^{*-1}	K^*

In conclusion

From the previous theorem we readily obtain the following.

Corollary Let V denote a vector space over $\mathbb{K}$ with finite positive dimension.

Let A, A^* denote a tridiagonal pair on V of geometric type.

Then there exists a unique $U_q(\widehat{sl_2})$-module structure on V such that y_1^- acts as A and y_0^- acts as A^*.

Moreover there exists a unique $U_q(\widehat{sl_2})$-module structure on V such that y_0^+ acts as A and y_1^+ acts as A^*.

Both $U_q(\widehat{sl_2})$-module structures are irreducible.

THE END