On Strongly Closed Subgraphs with Diameter Two and $Q$-Polynomial Property

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1 Introduction

Let $\Gamma = (X, R)$ be a distance-regular graph (DRG) of diameter $D$ with vertex set $X$ and edge set $R$. For vertices $x$ and $y$, $\vartheta(x, y)$ denotes the distance between $x$ and $y$, i.e., the length of a shortest path connecting $x$ and $y$. For a vertex $u \in X$ and $j \in \{0, 1, \ldots, D\}$, let

$$\Gamma_j(u) = \{x \in X \mid \vartheta(u, x) = j\} \text{ and } \Gamma(u) = \Gamma_1(u).$$

For two vertices $u$ and $v \in X$ with $\vartheta(u, v) = j$ let

- $C(u, v) = \Gamma_{j-1}(u) \cap \Gamma(v)$,
- $A(u, v) = \Gamma_j(u) \cap \Gamma(v)$, and
- $B(u, v) = \Gamma_{j+1}(u) \cap \Gamma(v)$.

The cardinalities $c_j = |C(u, v)|$, $a_j = |A(u, v)|$ and $b_j = |B(u, v)|$ depend only on $j = \vartheta(u, v)$, and they are called the intersection numbers of $\Gamma$. The number $k = b_0 = |\Gamma(u)|$ is called the valency of $\Gamma$.

A subset $Y$ of the vertex set $X$ is said to be strongly closed if

$$C(u, v) \cup A(u, v) \subset Y \quad \text{for all } u, v \in Y.$$  

We often identify a subset of $X$ with the induced subgraph on it. In particular, when $Y$ is strongly closed, $Y$ is referred to as a strongly closed subgraph of $\Gamma$. 

A parallelogram of length \( j \geq 2 \) is a four-vertex configuration \((w, x, y, z)\) such that

\[
\partial(w, x) = \partial(y, z) = j - 1 = \partial(x, z), \\
\partial(x, y) = \partial(z, w) = 1 \text{ and } \partial(w, y) = j.
\]

A distance-regular graph \( \Gamma \) of diameter \( D \) is called a regular near polygon if there is no parallelogram of length 2 and that

\[ a_i = c_i a_1 \text{ for } i = 1, 2, \ldots, D - 1. \]

In addition, if \( a_D = c_D a_1 \), then \( \Gamma \) is called a regular near 2D-gon.

Recently, in [7] P. Terwilliger and C. Weng showed that if \( \theta_1 \) is the second largest eigenvalue of a regular near polygon with diameter \( D \geq 3 \), valency \( k \) and intersection numbers \( a_1 > 0, c_2 > 1 \), then

\[
\theta_1 \leq \frac{k - a_1 - c_2}{c_2 - 1}. \tag{1.1}
\]

Equality is attained above if and only if \( \Gamma \) is \( Q \)-polynomial with classical parameters with respect to \( \theta_1 \).

Every regular near polygon contains a strongly closed subset \( Y \) such that the induced subgraph on \( Y \) is strongly regular, i.e., distance-regular of diameter 2. We noticed that the inequality in (1.1) and its equality condition are closely related to the existence of tight vectors that we defined in [4]. In this exposition, we shall explain the relation, apply the theory to parallelogram-free distance-regular graphs, and give a generalization of the results of Terwilliger and Weng above.

2 Terwilliger Algebra and Tight Vectors

Let \( \Gamma = (X, R) \) be a distance-regular graph of diameter \( D \). For \( i \in \{0, 1, \ldots, D\} \) let \( A_i \) denote the \( i \)-th adjacency matrix in \( \text{Mat}_X(C) \) whose \((x, y)\)-entry is defined by

\[
(A_i)_{x,y} = \begin{cases} 
1 & \text{if } \partial(x, y) = i, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( E_0, E_1, \ldots, E_D \) be primitive idempotents corresponding to the eigenvalues \( \theta_0 > \theta_1 > \cdots > \theta_D \) of \( A \).

Let \( Y \) be a nonempty subset of \( X \). \( E_i^* = E_i^*(Y) \in \text{Mat}_X(C) \) \((i = 0, 1, \ldots, D)\) is defined by

\[
(E_i^*)_{x,y} = \begin{cases} 
1 & \text{if } x = y \text{ and } \partial(x, Y) = i, \\
0 & \text{otherwise},
\end{cases}
\]
and $E^* = E_0^*$. Then the Terwilliger algebra with respect to $Y$ is a semisimple subalgebra of $\text{Mat}_X(C)$ defined by:

$$T = T(Y) = \langle A, E_0^*, E_1^*, \ldots, E_D^* \rangle.$$ 

Let $V = C^X$, and $W = E^*V$. For $x \in X$, let $\hat{x}$ denote the element of $V$ with a 1 in the $x$-coordinate and 0 in all other coordinates. Then $W$ is the vector subspace of $V$ spanned by the set $\{\hat{y} \mid y \in Y\}$.

Let $w(Y) = \max\{\partial(y, y') \mid y, y' \in Y\}$ denote the width of $Y$. Then we have the following.

**Proposition 1 ([4, Proposition 9.2])** For $0 \neq v \in W$,

$$|\{i \mid i \in \{0, 1, \ldots, D\}, E_i v = 0\}| \leq w(Y). \quad (2.2)$$

Now a nonzero vector $v \in W$ is said to be **tight** (with respect to $Y$), if equality is attained in (2.2), i.e.,

$$|\{i \mid i \in \{0, 1, \ldots, D\}, E_i v = 0\}| = w(Y).$$

## 3 Strongly Closed, Strongly Regular Case

In this section, we review a result to guarantee the existence of strongly closed strongly regular subgraph $Y$, and inequalities related to the existence of tight vectors with respect to $Y$.

**Proposition 2 ([10, Theorem 1], [3, Theorem 1.1])** Let $\Gamma = (X, R)$ be a distance-regular graph of diameter $D \geq 3$. Suppose $b_1 > b_2$ and $a_2 \neq 0$. Then the following are equivalent.

(i) For every pair of vertices $x$ and $y$ with $\partial(x, y) = 2$, there is a strongly closed subgraph containing $x$ and $y$ of diameter 2.

(ii) There is no parallelogram of length 2 or 3.

Moreover, if the conditions are satisfied, then strongly closed subgraphs guaranteed to exist are strongly regular.

Let $Y$ be a strongly closed subset of $X$. Suppose the induced subgraph on $Y$ is strongly regular, i.e., $w(Y) = 2$.

Set $\bar{A} = E^*AE^*$. Then there are three distinct eigenvalues $\eta_0, \eta_1, \eta_2$ of $\bar{A}$ on $W$, and they satisfy

$$\eta_0 = c_2 + a_2 > \eta_1 > -1 > \eta_2.$$
Let $1_Y$ denote the characteristic vector of $Y$ defined by

$1_Y = \sum_{y \in Y} \hat{y} \in W$.\[1\]

Let $W_0$, $W_1$, and $W_2$ be the eigenspaces of $\tilde{A}$ in $W$ corresponding to eigenvalues $\eta_0$, $\eta_1$, and $\eta_2$, respectively.

Then $W_0 = \langle 1_Y \rangle$, and

$W = W_0 \oplus W_1 \oplus W_2$.\[2\]

Note that if $v \in W_1 \oplus W_2$, then $E_i v = 0$. Hence an eigenvector $v$ of $\tilde{A}$ in $W_1 \oplus W_2$ is tight if $E_i v = 0$ for some $i > 0$ as $w(Y) = 2$.

**Proposition 3 ([4, Proposition 11.7])** Let $v \in W_j$ ($j = 1$ or 2) be an eigenvector of $\tilde{A}$,

1. For $i \in \{0, 1, \ldots, D\}$,

$$\frac{\|E_i v\|^2}{\|v\|^2} = \frac{m_i(k - \theta_i)((1 + \eta_j)(1 + \theta_i) + b_1)}{kb_1|X|} \geq 0.$$\[3\]

2. The following hold.

$$\theta_1 \leq -1 - \frac{b_1}{1 + \eta_2}, \text{ and } \theta_D \geq -1 - \frac{b_1}{1 + \eta_1}.$$\[4\]

3. The following are equivalent.

(a) $v$ is tight.

(b) One of the following holds.

(i) $\theta_1 = -1 - \frac{b_1}{1 + \eta_2}$, or

(ii) $\theta_D = -1 - \frac{b_1}{1 + \eta_1}$.\[5\]

**Proof.** The inequality in Proposition 3 (1) can be obtained by simple computation, and both (2) and (3) follow from (1) as $\theta_2 \geq \eta_1 > 1$ and $\theta_D \leq \eta_2 < -1$.

Suppose $\Gamma = (X, R)$ is a regular near polygon of diameter $D \geq 3$. Then it is known that $\Gamma$ does not contain parallelograms of any length. In addition, assume that $\alpha_1 > 0$ and $\alpha_2 > 1$. Then by Proposition 2 there is a strongly
closed subset $Y$ such that the induced subgraph on $Y$ is strongly regular. It is called a quad, and it has the following intersection array.

$$
\begin{array}{ccc}
c_i & a_i & b_i \\
\hline
* & 1 & c_2 \\
0 & a_1 & c_2 a_1 \\
c_2(a_1+1) & (c_2-1)(a_1+1) & * \\
\end{array}
$$

Hence in this case the eigenvalues can be expressed in a very simple form.

$$\eta_0 = c_2(a_1+1) > \eta_1 = a_1 > \eta_2 = -c_2.$$

Now the inequalities of Proposition 3 (2) yield

$$\theta_1 \leq -1 - \frac{b_1}{1-c_2}, \text{ and } \theta_D \geq -1 - \frac{b_1}{1+a_1}.$$  

The first inequality can also be expressed as

$$\theta_1 \leq -1 - \frac{b_1}{1-c_2} = \frac{k-a_1-c_2}{c_2-1}. \quad (3.3)$$

4 A Theorem of Terwilliger and Weng

**Theorem 4 (Terwilliger–Weng [7])** Let $\Gamma$ denote a regular near polygon with diameter $D \geq 3$, valency $k$ and intersection numbers $a_1 > 0$, $c_2 > 1$. Let $\theta_1$ denote the second largest eigenvalue of $\Gamma$. Then

$$\theta_1 \leq \frac{k-a_1-c_2}{c_2-1}. \quad (4.4)$$

Moreover, the following (i) - (iii) are equivalent.

(i) Equality is attained in (4.4).

(ii) $\Gamma$ is $Q$-polynomial with respect to $\theta_1$.

(iii) $\Gamma$ is a dual polar graph or a Hamming graph.

The inequality in (4.4) is nothing but the one in (3.3). Terwilliger and Weng obtained it using a so-called balanced condition and showed that $\Gamma$ satisfies the $Q$-polynomial property if equality is attained.

In view of Proposition 3, the theorem above asserts under the same assumption that the following are equivalent.

(i) There is a tight vector in $W_2$. 

A
(ii) $\Gamma$ is $Q$-polynomial with respect to $\theta_1$.

The following theorem identifies typical tight vectors in $W_1$ and $W_2$.

**Theorem 5** Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 3$, and an intersection number $a_2 > 0$. Let $Y$ be a strongly closed subset of $X$ of width 2. Then the induced subgraph on $Y$ is strongly regular with eigenvalues $\eta_0 = c_2 + a_2 > \eta_1 > -1 > \eta_2$, and the following are equivalent.

(i) There is a nonzero vector $v \in E^*V$ such that $E_0v = E_i v = 0$ for some $i \in \{1, 2, \ldots, D\}$.

(ii) Either one of the following holds.

(a) For every $x, y \in Y$ with $\partial(x, y) = 2$, $E_1u = 0$ and $\theta_1 = -1 - b_1/(1 + \eta_2)$, where

$$u = \sum_{z \in A(y, x)} \hat{z} - \sum_{w \in A(x, y)} \hat{w} - \eta_2(\hat{x} - \hat{y}),$$

or

(b) For every $x, y \in Y$ with $\partial(x, y) = 2$, $E_Du = 0$ and $\theta_D = -1 - b_1/(1 + \eta_1)$, where

$$u = \sum_{z \in A(y, x)} \hat{z} - \sum_{w \in A(x, y)} \hat{w} - \eta_1(\hat{x} - \hat{y}).$$

The conditions in (ii) are related to a balanced condition in the following theorem.

**Theorem 6** (Terwilliger [5]) Let $\Gamma = (V, R)$ be a distance-regular graph of diameter $D \geq 3$. Let

$$E_i = \frac{1}{|X|} \sum_{j=0}^{D} q_i(j) A_j$$

be a primitive idempotent such that $q_i(j) \neq q_i(0)$ for every $j = 1, \ldots, D$. Then the following are equivalent.

(i) $\Gamma$ is $Q$-polynomial with respect to $E_i$.

(ii) The following two 'balanced' conditions are satisfied.

(a) For all $x, y \in X$ with $\partial(x, y) = 2$,

$$\sum_{z \in A(y, x)} E_i \hat{z} - \sum_{w \in A(x, y)} E_i \hat{w} \in \langle E_i (\hat{x} - \hat{y}) \rangle.$$
(b) For all \( x, y \in X \) with \( \partial(x, y) = 3 \),
\[
\sum_{z \in C(y, x)} E_i \hat{z} - \sum_{w \in C(x, y)} E_i \hat{w} \in (E_i (\hat{x} - \hat{y}))
\]

In view of Theorem 6, there is a tight vector in \( W_2 \) if and only if \( \Gamma \) satisfies (ii)(a), the first half of the condition for \( \Gamma \) to be \( Q \)-polynomial.

5 Parallelogram Free DRGs

Recall that every regular near polygon is parallelogram-free. If we assume that \( \Gamma \) is of parallelogram free, we can prove a bit more. Before we state our result, we review the definition of a distance-regular graph with classical parameters. Such graph is always \( Q \)-polynomial. See [1].

**Definition 1** Let \( \Gamma \) denote a distance-regular graph with diameter \( D \geq 3 \). We say \( \Gamma \) has classical parameters \( (D, q, \alpha, \beta) \) whenever the intersection numbers are given by
\[
c_i = \begin{bmatrix} i \end{bmatrix} \left( 1 + \alpha \begin{bmatrix} i - 1 \end{bmatrix} \right) \quad (0 \leq i \leq D),
\]
\[
b_i = \left( \left[ D \right] - \begin{bmatrix} i \end{bmatrix} \right) \left( \beta - \alpha \begin{bmatrix} i \end{bmatrix} \right) \quad (0 \leq i \leq D),
\]
where
\[
\begin{bmatrix} j \end{bmatrix} := 1 + q + q^2 + \cdots + q^{j-1}.
\]

Now we assume the following.

**Hypothesis 1** Let \( \Gamma = (X, R) \) be a parallelogram-free distance-regular graph with diameter \( D \geq 3 \). Suppose \( a_2 > 0 \) and \( b_1 > b_2 \).

Then by Proposition 2, \( \Gamma \) contains a strongly closed subset \( Y \) such that the induced subgraph on \( Y \) is strongly regular. Let
\[
\eta_0 = c_2 + a_2 > \eta_1 > \eta_2
\]
be its distinct eigenvalues.

**Theorem 7** Under Hypothesis 1, the following hold.
(i) \( \theta_1 \leq -1 - \frac{b_1}{1 + \eta_2} \), and \( \theta_D \geq -1 - \frac{b_1}{1 + \eta_1} \).

(ii) Suppose \( \theta \in \{\theta_1, \theta_D\} \) attains one of the bounds above. Let \( q = b_1/(\theta + 1) \). Then the following hold.

(a) The intersection numbers of \( \Gamma \) are such that
\[
qc_i - b_i - q(qc_i - b_i) - 1
\]
is independent of \( i \) (\( 1 \leq i \leq D \)).

(b) \( c_3 \geq (c_2 - q)(q^2 + q + 1) \).

(c) If \( \theta = \theta_1 \), then \( q + 1 \geq c_2 \) and \( q^2 + q + 1 \geq c_3 \), and if \( \theta = \theta_D \), then
\[
q + 1 \leq -a_1.
\]

(d) The equality holds in (b) if and only if \( \Gamma \) is \( Q \)-polynomial with classical parameters \((D, q, \alpha, \beta)\) with suitable choices of real numbers \( \alpha \) and \( \beta \).

If \( \Gamma \) is a regular near polygon, then \( \eta_2 = -c_2 \) and \( q = c_2 - 1 \). Hence by (b), \( c_3 \geq q^2 + q + 1 \) and by (c), \( q^3 + q + 1 \geq c_3 \). Therefore \( \Gamma \) is \( Q \)-polynomial with classical parameters by (d).

As a by-product, we obtained the following result as well.

**Proposition 8** Let \( \Gamma = (X, R) \) be a parallelogram-free distance-regular graph with diameter \( D \geq 3 \) and intersection numbers \( a_2 = s - 1 > 0 \), \( b_1 = b_2 \). Suppose for all \( x, y \in X \) with \( \partial(x, y) = 2 \),
\[
\sum_{z \in A(x, y)} E_i z - \sum_{w \in A(x, y)} E_i w \in \langle E_i(x - y) \rangle.
\]
Then \( \Gamma \) is a regular near \( 2D \)-gon and \( c_3 \geq 1 - q^3 \), where \( q = -s = -(a_1 + 1) \). If equality holds, then \( \Gamma \) is a classical distance-regular graph with parameters
\[
(D, q, \alpha, \beta) = (D, -s, \frac{s}{1 - s}, \frac{k(1 + s)}{1 - (-s)^D}).
\]
If \( D = 3 \), then \( \Gamma \) is a generalized hexagon. No examples are known if \( D > 3 \).
6 Examples

1. If $\Gamma$ contains a strongly closed subgraph isomorphic to (the collinearity graph of) a generalized quadrangle, $\theta_D$ attains the bound if and only if $\theta_D = -k/(a_1 + 1)$.

2. Dual polar graphs and Hamming graphs are the only $Q$-polynomial regular near polygons of diameter $D \geq 4$ with intersection numbers $c_2 > 1$ and $a_1 > 0$ and these are distance-regular graphs having classical parameters with $\alpha = 0$ and $a_1 \neq 0$. These graphs are $Q$-polynomial with respect to $\theta_1$ and attain both of the bounds.

3. Let $\Gamma$ be a parallelogram-free $Q$-polynomial distance-regular graph of diameter $D \geq 4$ with $a_2 > 0$. Then $\Gamma$ has classical parameters $(D, q, \alpha, \beta)$ and $\Gamma$ is either a regular near polygon or $q < -1$. Distance-regular graphs having classical parameters $(D, q, \alpha, \beta)$ with $q < -1$ are said to be of negative type. These graphs satisfy the bound for $\theta_D$.

Finally we include a table of the list of known parallelogram-free $Q$-polynomial distance-regular graphs taken from [1]. There is a series of excellent articles on parallelogram-free distance-regular graphs by C. Weng and others. See [2, 6, 8, 9, 10, 11]. We hope that our observations may shed light on the classification of this class of distance-regular graphs.

Known Parallelogram-Free $Q$-DRGs

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<th>Name</th>
<th>Diam.</th>
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<th>$\alpha + 1$</th>
<th>$\beta + 1$</th>
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<td>1</td>
<td>$q$</td>
</tr>
<tr>
<td>$DP(D, q, e)$</td>
<td>$D$</td>
<td>$q$</td>
<td>1</td>
<td>$q^e + 1$</td>
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<td>$-r$</td>
<td>$\frac{1+r^2}{1-r}$</td>
<td>$\frac{1-(-r)^{D+1}}{1-r}$</td>
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<td>$-(-r)^D$</td>
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<td>$-q$</td>
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References


The content of this exposition is included in the following.