<table>
<thead>
<tr>
<th>Title</th>
<th>On Strongly Closed Subgraphs with Diameter Two and $Q$-Polynomial Property (Algebraic Combinatorics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Suzuki, Hiroshi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1440: 48-57 (2005)</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2005-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/47547">http://hdl.handle.net/2433/47547</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
On Strongly Closed Subgraphs with Diameter Two and $Q$-Polynomial Property

International Christian University

1 Introduction

Let $\Gamma = (X, R)$ be a distance-regular graph (DRG) of diameter $D$ with vertex set $X$ and edge set $R$. For vertices $x$ and $y$, $\partial(x, y)$ denotes the distance between $x$ and $y$, i.e., the length of a shortest path connecting $x$ and $y$. For a vertex $u \in X$ and $j \in \{0, 1, \ldots, D\}$, let

$$\Gamma_j(u) = \{x \in X \mid \partial(u, x) = j\} \text{ and } \Gamma(u) = \Gamma_1(u).$$

For two vertices $u$ and $v \in X$ with $\partial(u, v) = j$ let

$$C(u, v) = \Gamma_{j-1}(u) \cap \Gamma(v),$$
$$A(u, v) = \Gamma_j(u) \cap \Gamma(v), \text{ and}$$
$$B(u, v) = \Gamma_{j+1}(u) \cap \Gamma(v).$$

The cardinalities $c_j = |C(u, v)|$, $a_j = |A(u, v)|$ and $b_j = |B(u, v)|$ depend only on $j = \partial(u, v)$, and they are called the intersection numbers of $\Gamma$. The number $k = b_0 = |\Gamma(u)|$ is called the valency of $\Gamma$.

A subset $Y$ of the vertex set $X$ is said to be strongly closed if

$$C(u, v) \cup A(u, v) \subset Y \text{ for all } u, v \in Y.$$ 

We often identify a subset of $X$ with the induced subgraph on it. In particular, when $Y$ is strongly closed, $Y$ is referred to as a strongly closed subgraph of $\Gamma$. 
A parallelogram of length $j \geq 2$ is a four-vertex configuration $(w, x, y, z)$ such that

$$\partial(w, x) = \partial(y, z) = j - 1 = \partial(x, z),$$

$$\partial(x, y) = \partial(z, w) = 1 \text{ and } \partial(w, y) = j.$$ 

A distance-regular graph $\Gamma$ of diameter $D$ is called a regular near polygon if there is no parallelogram of length 2 and that

$$a_i = c_ia_1 \text{ for } i = 1, 2, \ldots, D - 1.$$ 

In addition, if $a_D = c_Da_1$, then $\Gamma$ is called a regular near $2D$-gon.

Recently, in [7] P. Terwilliger and C. Weng showed that if $\theta_1$ is the second largest eigenvalue of a regular near polygon with diameter $D \geq 3$, valency $k$ and intersection numbers $a_1 > 0$, $c_2 > 1$, then

$$\theta_1 \leq \frac{k - a_1 - c_2}{c_2 - 1}.$$  

(1.1)

Equality is attained above if and only if $\Gamma$ is $Q$-polynomial with classical parameters with respect to $\theta_1$.

Every regular near polygon contains a strongly closed subset $Y$ such that the induced subgraph on $Y$ is strongly regular, i.e., distance-regular of diameter 2. We noticed that the inequality in (1.1) and its equality condition are closely related to the existence of tight vectors that we defined in [4]. In this exposition, we shall explain the relation, apply the theory to parallelogram-free distance-regular graphs, and give a generalization of the results of Terwilliger and Weng above.

## 2 Terwilliger Algebra and Tight Vectors

Let $\Gamma = (X, R)$ be a distance-regular graph of diameter $D$. For $i \in \{0, 1, \ldots, D\}$ let $A_i$ denote the $i$-th adjacency matrix in $\text{Mat}_X(C)$ whose $(x, y)$-entry is defined by

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{otherwise}. \end{cases}$$

Let $E_0, E_1, \ldots, E_D$ be primitive idempotents corresponding to the eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$ of $A$.

Let $Y$ be a nonempty subset of $X$. $E_i^* = E_i^*(Y) \in \text{Mat}_X(C)$ ($i = 0, 1, \ldots, D$) is defined by

$$(E_i^*)_{x,y} = \begin{cases} 1 & \text{if } x = y \text{ and } \partial(x, Y) = i, \\ 0 & \text{otherwise}, \end{cases}$$
and $E^* = E_0^*$. Then the Terwilliger algebra with respect to $Y$ is a semisimple subalgebra of $\text{Mat}_X(C)$ defined by:

$$T = T(Y) = \langle A, E_0^*, E_1^*, \ldots, E_D^* \rangle.$$ 

Let $V = C^X$, and $W = E^*V$. For $x \in X$, let $\hat{x}$ denote the element of $V$ with a 1 in the $x$-coordinate and 0 in all other coordinates. Then $W$ is the vector subspace of $V$ spanned by the set $\{ \hat{y} \mid y \in Y \}$.

Let $w(Y) = \max\{\partial(y, y') \mid y, y' \in Y\}$ denote the width of $Y$. Then we have the following.

**Proposition 1 ([4, Proposition 9.2])** For $0 \neq v \in W$,

$$|\{i \mid i \in \{0,1, \ldots, D\}, E_i v = 0\}| \leq w(Y). \tag{2.2}$$

Now a nonzero vector $v \in W$ is said to be tight (with respect to $Y$), if equality is attained in (2.2), i.e.,

$$|\{i \mid i \in \{0,1, \ldots, D\}, E_i v = 0\}| = w(Y).$$

## 3 Strongly Closed, Strongly Regular Case

In this section, we review a result to guarantee the existence of strongly closed strongly regular subgraph $Y$, and inequalities related to the existence of tight vectors with respect to $Y$.

**Proposition 2 ([10, Theorem 1], [3, Theorem 1.1])** Let $\Gamma = (X, R)$ be a distance-regular graph of diameter $D \geq 3$. Suppose $b_1 > b_2$ and $a_2 \neq 0$. Then the following are equivalent.

(i) For every pair of vertices $x$ and $y$ with $\partial(x, y) = 2$, there is a strongly closed subgraph containing $x$ and $y$ of diameter 2.

(ii) There is no parallelogram of length 2 or 3.

Moreover, if the conditions are satisfied, then strongly closed subgraphs guaranteed to exist are strongly regular.

Let $Y$ be a strongly closed subset of $X$. Suppose the induced subgraph on $Y$ is strongly regular, i.e., $w(Y) = 2$.

Set $\tilde{A} = E^*AE^*$. Then there are three distinct eigenvalues $\eta_0, \eta_1, \eta_2$ of $\tilde{A}$ on $W$, and they satisfy

$$\eta_0 = c_2 + a_2 > \eta_1 > -1 > \eta_2.$$
Let $1_Y$ denote the characteristic vector of $Y$ defined by
$$1_Y = \sum_{y \in Y} \hat{y} \in W.$$ 

Let $W_0$, $W_1$ and $W_2$ be the eigenspaces of $\tilde{A}$ in $W$ corresponding to eigenvalues $\eta_0$, $\eta_1$ and $\eta_2$, respectively.

Then $W_0 = \langle 1_Y \rangle$, and
$$W = W_0 \oplus W_1 \oplus W_2.$$ 

Note that if $v \in W_1 \oplus W_2$, then $E_0 v = 0$. Hence an eigenvector $v$ of $\tilde{A}$ in $W_1 \oplus W_2$ is tight if $E_i v = 0$ for some $i > 0$ as $w(Y) = 2$.

**Proposition 3 ([4, Proposition 11.7])** Let $v \in W_j$ ($j = 1$ or $2$) be an eigenvector of $\tilde{A}$,

1. For $i \in \{0, 1, \ldots, D\}$,
$$\frac{\|E_i v\|^2}{\|v\|^2} = \frac{m_i (k - \theta_i)((1 + \eta_j)(1 + \theta_i) + b_i)}{kb_1 |X|} \geq 0.$$ 

2. The following hold.
$$\theta_1 \leq -1 - \frac{b_1}{1 + \eta_2}, \text{ and } \theta_D \geq -1 - \frac{b_1}{1 + \eta_1}.$$ 

3. The following are equivalent.
   (a) $v$ is tight.
   (b) One of the following holds.
      (i) $\theta_1 = -1 - \frac{b_1}{1 + \eta_2}$, or
      (ii) $\theta_D = -1 - \frac{b_1}{1 + \eta_1}$.

**Proof.** The inequality in Proposition 3 (1) can be obtained by simple computation, and both (2) and (3) follow from (1) as $\theta_1 \geq \eta_1 > -1$ and $\theta_D \leq \eta_2 < -1$. 

Suppose $\Gamma = (X, R)$ is a regular near polygon of diameter $D \geq 3$. Then it is known that $\Gamma$ does not contain parallelograms of any length. In addition, assume that $a_1 > 0$ and $a_2 > 1$. Then by Proposition 2 there is a strongly
closed subset $Y$ such that the induced subgraph on $Y$ is strongly regular. It is called a quad, and it has the following intersection array.

$$
\begin{array}{ccc}
  c_i & * & 1 \\
  a_i & 0 & c_2 \\
  b_i & c_2(a_1 + 1) & (c_2 - 1)(a_1 + 1) & *
\end{array}
$$

Hence in this case the eigenvalues can be expressed in a very simple form.

$$\eta_0 = c_2(a_1 + 1) > \eta_1 = a_1 > \eta_2 = -c_2.$$ 

Now the inequalities of Proposition 3 (2) yield

$$\theta_1 \leq -1 - \frac{b_1}{1 - c_2}, \quad \text{and} \quad \theta_D \geq -1 - \frac{b_1}{1 + a_1}.$$ 

The first inequality can also be expressed as

$$\theta_1 \leq -1 - \frac{b_1}{1 - c_2} = \frac{k - a_1 - c_2}{c_2 - 1}. \quad (3.3)$$

4 A Theorem of Terwilliger and Weng

**Theorem 4 (Terwilliger–Weng [7])** Let $\Gamma$ denote a regular near polygon with diameter $D \geq 3$, valency $k$ and intersection numbers $a_1 > 0$, $c_2 > 1$. Let $\theta_1$ denote the second largest eigenvalue of $\Gamma$. Then

$$\theta_1 \leq \frac{k - a_1 - c_2}{c_2 - 1}. \quad (4.4)$$

Moreover, the following (i) - (iii) are equivalent.

(i) Equality is attained in (4.4).

(ii) $\Gamma$ is $Q$-polynomial with respect to $\theta_1$.

(iii) $\Gamma$ is a dual polar graph or a Hamming graph.

The inequality in (4.4) is nothing but the one in (3.3). Terwilliger and Weng obtained it using a so-called balanced condition and showed that $\Gamma$ satisfies the $Q$-polynomial property if equality is attained.

In view of Proposition 3, the theorem above asserts under the same assumption that the following are equivalent.

(i) There is a tight vector in $W_2$. 
(ii) $\Gamma$ is $Q$-polynomial with respect to $\theta_1$.

The following theorem identifies typical tight vectors in $W_1$ and $W_2$.

**Theorem 5** Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 3$, and an intersection number $a_2 > 0$. Let $Y$ be a strongly closed subset of $X$ of width 2. Then the induced subgraph on $Y$ is strongly regular with eigenvalues $\eta_0 = c_2 + a_2 > \eta_1 > -1 > \eta_2$, and the following are equivalent.

(i) There is a nonzero vector $v \in E^*V$ such that $E_0 v = E_i v = 0$ for some $i \in \{1, 2, \ldots, D\}$.

(ii) Either one of the following holds.

(a) For every $x, y \in Y$ with $\delta(x, y) = 2$, $E_1 u = 0$ and $\theta_1 = -1 - b_1/(1 + \eta_2)$, where

$$u = \sum_{z \in A(y,x)} \hat{z} - \sum_{w \in A(x,y)} \hat{w} - \eta_2 (\hat{x} - \hat{y}),$$

or

(b) For every $x, y \in Y$ with $\delta(x, y) = 2$, $E_D u = 0$ and $\theta_D = -1 - b_1/(1 + \eta_1)$, where

$$u = \sum_{z \in A(y,x)} \hat{z} - \sum_{w \in A(x,y)} \hat{w} - \eta_1 (\hat{x} - \hat{y}).$$

The conditions in (ii) are related to a balanced condition in the following theorem.

**Theorem 6** (Terwilliger [5]) Let $\Gamma = (V, R)$ be a distance-regular graph of diameter $D \geq 3$. Let

$$E_i = \frac{1}{|X|} \sum_{j=0}^{D} q_i(j) A_j$$

be a primitive idempotent such that $q_i(j) \neq q_i(0)$ for every $j = 1, \ldots, D$. Then the following are equivalent.

(i) $\Gamma$ is $Q$-polynomial with respect to $E_i$.

(ii) The following two 'balanced' conditions are satisfied.

(a) For all $x, y \in X$ with $\delta(x, y) = 2$,

$$\sum_{z \in A(y,x)} E_i \hat{z} - \sum_{w \in A(x,y)} E_i \hat{w} \in \langle E_i (\hat{x} - \hat{y}) \rangle.$$
(b) For all \(x, y \in X\) with \(\partial(x, y) = 3\),
\[
\sum_{z \in C(y, x)} E_i \hat{z} - \sum_{w \in C(x, y)} E_i \hat{w} \in (E_i (\hat{x} - \hat{y})).
\]

In view of Theorem 6, there is a tight vector in \(W_2\) if and only if \(\Gamma\) satisfies (ii)(a), the first half of the condition for \(\Gamma\) to be \(Q\)-polynomial.

5 Parallelogram Free DRGs

Recall that every regular near polygon is parallelogram-free. If we assume that \(\Gamma\) is of parallelogram free, we can prove a bit more. Before we state our result, we review the definition of a distance-regular graph with classical parameters. Such graph is always \(Q\)-polynomial. See [1].

Definition 1 Let \(\Gamma\) denote a distance-regular graph with diameter \(D \geq 3\). We say \(\Gamma\) has classical parameters \((D, q, \alpha, \beta)\) whenever the intersection numbers are given by
\[
c_i = \left[\begin{array}{c} i \\ 1 \end{array}\right] \left(1 + \alpha \left[\begin{array}{c} i-1 \\ 1 \end{array}\right]\right) \quad (0 \leq i \leq D),
\]
\[
b_i = \left(\left[\begin{array}{c} D \\ 1 \end{array}\right] - \left[\begin{array}{c} i \\ 1 \end{array}\right]\right) \left(\beta - \alpha \left[\begin{array}{c} i \\ 1 \end{array}\right]\right) \quad (0 \leq i \leq D),
\]
where
\[
\left[\begin{array}{c} j \\ 1 \end{array}\right] := 1 + q + q^2 + \cdots + q^{j-1}.
\]

Now we assume the following.

Hypothesis 1 Let \(\Gamma = (X, R)\) be a parallelogram-free distance-regular graph with diameter \(D \geq 3\). Suppose \(a_2 > 0\) and \(b_1 > b_2\).

Then by Proposition 2, \(\Gamma\) contains a strongly closed subset \(Y\) such that the induced subgraph on \(Y\) is strongly regular. Let
\[
\eta_0 = c_2 + a_2 > \eta_1 > \eta_2
\]
be its distinct eigenvalues.

Theorem 7 Under Hypothesis 1, the following hold.
(i) $\theta_1 \leq -1 - \frac{b_1}{1 + \eta_2}$, and $\theta_D \geq -1 - \frac{b_1}{1 + \eta_1}$.

(ii) Suppose $\theta \in \{\theta_1, \theta_D\}$ attains one of the bounds above. Let $q = b_1/(\theta + 1)$. Then the following hold.

(a) The intersection numbers of $\Gamma$ are such that
\[qc_i - b_i - q(qc_{i-1} - b_{i-1})\]
is independent of $i$ $(1 \leq i \leq D)$.

(b) $c_3 \geq (c_2 - q)(q^2 + q + 1)$.

(c) If $\theta = \theta_1$, then $q + 1 \geq c_2$ and $q^2 + q + 1 \geq c_3$, and if $\theta = \theta_D$, then $q + 1 \leq -a_1$.

(d) The equality holds in (b) if and only if $\Gamma$ is $Q$-polynomial with classical parameters $(D, q, \alpha, \beta)$ with suitable choices of real numbers $\alpha$ and $\beta$.

If $\Gamma$ is a regular near polygon, then $\eta_2 = -c_2$ and $q = c_2 - 1$. Hence by (b), $c_3 \geq q^2 + q + 1$ and by (c), $q^2 + q + 1 \geq c_3$. Therefore $\Gamma$ is $Q$-polynomial with classical parameters by (d).

As a by-product, we obtained the following result as well.

**Proposition 8** Let $\Gamma = (X, R)$ be a parallelogram-free distance-regular graph with diameter $D \geq 3$ and intersection numbers $a_2 = s - 1 > 0$, $b_1 = b_2$. Suppose for all $x, y \in X$ with $\partial(x, y) = 2$,
\[\sum_{z \in A(x, y)} E_i \hat{z} - \sum_{w \in A(x, y)} E_i \hat{w} \in \langle E_i(\hat{x} - \hat{y}) \rangle.\]

Then $\Gamma$ is a regular near $2D$-gon and $c_3 \geq 1 - q^3$, where $q = -s = -(a_1 + 1)$. If equality holds, then $\Gamma$ is a classical distance-regular graph with parameters
\[(D, q, \alpha, \beta) = (D, -s, s, \frac{s}{1-s}, \frac{k(1+s)}{1-(-s)^D}).\]

If $D = 3$, then $\Gamma$ is a generalized hexagon. No examples are known if $D > 3$. 

6 Examples

1. If $\Gamma$ contains a strongly closed subgraph isomorphic to (the collinearity graph of) a generalized quadrangle, $\theta_D$ attains the bound if and only if $\theta_D = -k/(a_1 + 1)$.

2. Dual polar graphs and Hamming graphs are the only $Q$-polynomial regular near polygons of diameter $D \geq 4$ with intersection numbers $c_0 > 1$ and $a_1 > 0$ and these are distance-regular graphs having classical parameters with $\alpha = 0$ and $a_1 \neq 0$. These graphs are $Q$-polynomial with respect to $\theta_1$ and attain both of the bounds.

3. Let $\Gamma$ be a parallelogram-free $Q$-polynomial distance-regular graph of diameter $D \geq 4$ with $a_2 > 0$. Then $\Gamma$ has classical parameters $(D, q, \alpha, \beta)$ and $\Gamma$ is either a regular near polygon or $q < -1$. Distance-regular graphs having classical parameters $(D, q, \alpha, \beta)$ with $q < -1$ are said to be of negative type. These graphs satisfy the bound for $\theta_D$.

Finally we include a table of the list of known parallelogram-free $Q$-polynomial distance-regular graphs taken from [1]. There is a series of excellent articles on parallelogram-free distance-regular graphs by C. Weng and others. See [2, 6, 8, 9, 10, 11]. We hope that our observations may shed light on the classification of this class of distance-regular graphs.

Known Parallelogram-Free $Q$-DRGs

<table>
<thead>
<tr>
<th>Name</th>
<th>Diam.</th>
<th>$b$</th>
<th>$\alpha + 1$</th>
<th>$\beta + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(D, q)$</td>
<td>$D$</td>
<td>1</td>
<td>1</td>
<td>$q$</td>
</tr>
<tr>
<td>$DP(D, q, e)$</td>
<td>$D$</td>
<td>$q$</td>
<td>1</td>
<td>$q^e + 1$</td>
</tr>
<tr>
<td>$U(2D, r)$</td>
<td>$D$</td>
<td>$-r$</td>
<td>$1+r^2$</td>
<td>$1-(-r)^{D+1}$</td>
</tr>
<tr>
<td>$Her_D(r)$</td>
<td>$D$</td>
<td>$-r$</td>
<td>$-r$</td>
<td>$-(-r)^D$</td>
</tr>
<tr>
<td>$GH(q, q^3)$</td>
<td>3</td>
<td>$-q$</td>
<td>$\frac{1}{1-q}$</td>
<td>$q^2 + q + 1$</td>
</tr>
<tr>
<td>$M_{24}$</td>
<td>3</td>
<td>$-2$</td>
<td>$-3$</td>
<td>11</td>
</tr>
<tr>
<td>$M_{23}$</td>
<td>3</td>
<td>$-2$</td>
<td>$-1$</td>
<td>6</td>
</tr>
<tr>
<td>$ExtTGolay$</td>
<td>3</td>
<td>$-2$</td>
<td>$-2$</td>
<td>9</td>
</tr>
</tbody>
</table>
References


The content of this exposition is included in the following.