Numerical verification by infinite dimensional Newton’s method for stationary solutions of the Navier-Stokes problems

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1 Introduction

We consider the following Navier-Stokes equations

\[
\begin{cases}
-\Delta u + R \cdot (u \cdot \nabla)u + \nabla p = f \quad \text{in } \Omega, \\
\text{div } u = 0 \quad \text{in } \Omega, \\
u = g \quad \text{on } \partial \Omega,
\end{cases}
\]

where \(u, p\) and \(R\) are the velocity vector, pressure and the Reynolds number, respectively and the flow region \(\Omega\) is a convex polygonal domain in \(\mathbb{R}^2\). In what follows, for each rational number \(m\), let \(H^m(\Omega)\) denote the \(L^2\)-Sobolev space of order \(m\) on \(\Omega\). The function \(f = (f_1, f_2)\) means a density of body forces with \(f \in (H^1(\Omega))\) and \(g = (g_1, g_2) \in H^{1/2}(\partial \Omega)\), where we assume that there exists a function \(\varphi \in H^2(\Omega)\) satisfying \((\varphi_y, -\varphi_x) = g\) on \(\partial \Omega\).
The above problem was discussed by Wiener [7] for low Reynolds numbers. The method proposed in it is based on Newton-Kantorovich theorem but it would not be able to apply to high Reynolds numbers, because the estimation for the inverse of the linearized operator directly depends on the Reynolds number. We also use Newton type verification condition, but the method which verifies the invertibility of linearized operator is different from the Wiener's formulation. Our method has an advantage which enables us to verify the invertibility of the linearized operator, even for high Reynolds numbers, provided that the approximation space is sufficiently accurate and that the inverse operator actually exists in the rigorous sense.

In Section 2 we introduce a stream function formulation of our problem and consider the linearized operator which is needed in the infinite dimensional Newton method. In Section 3, we formulate the verification method by computer to verify the invertibility of the linearized operator. We then derive the infinite dimensional Newton method to enclose the solution in Section 4 and we show some enclosure results in Section 5.

2 Stream function and the linearized operator

We first introduce a stream function \( \psi \) satisfying \( u = (\psi_y, -\psi_x) \) by the incompressibility condition in (1.1), where subscripts \( x \) and \( y \) denote the partial derivative for \( x \) and \( y \) respectively. Using this function we can rewrite the equations (1.1) as

\[
\begin{aligned}
\Delta^2 \psi + R \cdot J(\psi, \Delta \psi) &= (f_2)_x - (f_1)_y \quad \text{in} \Omega, \\
\psi &= \varphi \quad \text{on} \partial \Omega, \\
\frac{\partial \psi}{\partial n} &= \frac{\partial \varphi}{\partial n} \quad \text{on} \partial \Omega,
\end{aligned}
\]

(2.1)

where \( J \) is a bilinear form defined by \( J(u,v) = u_x v_y - u_y v_x \) and \( \frac{\partial}{\partial n} \) stands for the normal derivative. Newly denoting \( u \) as \( \psi - \varphi \) we have

\[
\begin{aligned}
\Delta^2 u + \Delta^2 \varphi + R \cdot J(u + \varphi, \Delta(u + \varphi)) &= (f_2)_x - (f_1)_y \quad \text{in} \Omega, \\
u = \frac{\partial u}{\partial n} &= 0 \quad \text{on} \partial \Omega.
\end{aligned}
\]

(2.2)

Our aim is to verify the existence of a weak solution \( u \in H^2_0(\Omega) \) of (2.2), where \( H^2_0(\Omega) \equiv \{ v \in H^2(\Omega) \mid v = \frac{\partial v}{\partial n} = 0 \text{ on } \partial \Omega \} \) with the inner product \( \langle u, v \rangle_{H^2_0} \equiv (\Delta u, \Delta v)_{L^2} \) for \( u, v \in H^2_0(\Omega) \).
Let $S_h$ be a finite dimensional subspace of $H_0^2(\Omega)$ that depends on $h$ ($0 < h < 1$). Usually $S_h$ is taken to be a finite element subspace with mesh size $h$.

We calculate an approximate solution $u_h \in C^1(\Omega)$ of (2.2) in the finite dimensional space, satisfying for all $v_h \in S_h$

$$(\Delta u_h + \Delta \varphi, \Delta v_h)_{L^2} + (R \cdot J(u_h + \varphi, \Delta(u_h + \varphi)), v_h)_{L^2} = ((f_2)_x - (f_1)_y, v_h)_{L^2},$$

(2.3)

and calculate $u_s \in C^2(\Omega)$ by smoothing of $u_h$. Then the linearized operator at $u_s$ is represented as

$$L u \equiv \Delta^2 u + R \cdot \{ J(u_s + \varphi, \Delta u) + J(u, \Delta(u_s + \varphi)) \},$$

and $L$ is considered as the operator from $H_0^2(\Omega)$ to $H^{-2}(\Omega)$ in weak sense. We will verify the existence of the inverse $L^{-1} : H^{-2}(\Omega) \rightarrow H_0^2(\Omega)$ and formulate the infinite dimensional Newton's method.

### 3 Invertibility of the linearized operator

By direct computations, we find that for any $q \in H^{-2}(\Omega)$ there exists a unique solution $v \in H_0^2(\Omega)$ satisfying

$$\begin{cases}
\Delta^2 v = q \text{ in } \Omega, \\
v = \frac{\partial v}{\partial n} = 0 \text{ on } \partial \Omega.
\end{cases} \quad (3.1)$$

For $q \in H^{-2}(\Omega)$, let $Kq$ be the unique solution $v \in H_0^2(\Omega)$ of the equation (3.1) then $K$ is a compact operator from $H^{-1}(\Omega)$ to $H_0^2(\Omega)$. Using the following compact operator on $H_0^2(\Omega)$

$$F_1(u) \equiv -R \cdot K \{ J(u_s + \varphi, \Delta u) + J(u, \Delta(u_s + \varphi)) \},$$

the equation $Lu = 0$ is equivalent to the fixed point equation

$$u = F_1(u). \quad (3.2)$$
In order to show the invertibility of the linearized operator $\mathcal{L}$, by the Fredholm alternative, we only have to show the uniqueness of the solution of the equation $\mathcal{L}u = 0$.

Now let $P_h : H^3_0(\Omega) \rightarrow S_h$ denote the $H^3_0$-projection defined by

\[ (\Delta(u - P_h u), \Delta v_h)_{L^2} = 0 \quad \text{for all } v_h \in S_h, \]

and we derive some error estimations for $P_h$. In what follows, we restrict ourselves to that the domain $\Omega$ is a unit square $(0, 1) \times (0, 1)$, and that $S_h$ is the set of piecewise bicubic Hermite functions with uniform mesh on $\Omega$ (e.g., [5]). However, our verification principle can also be applied to more general domains and approximation subspaces, when the appropriate a priori error estimates are obtained.

At first we derive the following interpolation error estimation.

**Lemma 1.** Let $I_\Omega$ denote the cubic Hermite interpolation on $\Omega = (0, 1)^2$. For $u \in H^4(\Omega) \cap H^3_0(\Omega)$ we have

\[ \|u - I_\Omega u\|_{H^2_0} \leq \frac{h^2}{\pi^2} \|\Delta^2 u\|_{L^2}. \] (3.3)

**Proof.** At first, we have

\[ |u|_{H^4(\Omega)} = \|\Delta^2 u\|_{L^2} \quad \text{for } u \in H^4(\Omega) \cap H^3_0(\Omega) \]

where $|\cdot|_{H^4(\Omega)}$ denotes the $H^4$ seminorm on $\Omega$ defined by

\[ |u|_{H^4}^2 = \sum_{n_1+n_2=4, n_1,n_2 \in \mathbb{N} \cup \{0\}} \left\| \frac{\partial^4 u}{\partial x^{n_1} \partial y^{n_2}} \right\|_{L^2}^2. \]

Actually, by expanding as $u = \sum_{m,n=1}^{\infty} a_{mn} \sin m\pi x \sin n\pi y$, noting the convergence of the Fourier series, we have

\[ |u|_{H^4}^2 = \frac{1}{4} \sum_{m,n=1}^{\infty} \left( (m\pi)^2 + (n\pi)^2 \right)^2 a_{mn}^2 = \|\Delta^2 u\|_{L^2}^2. \]

Denoting $I_H$ the Hermite interpolation on $I = (0, 1)$, we have by [5]

\[
\|u - I_H u\|_{H^3_0} \leq \|u''\|_{L^2} \quad \text{for } u \in H^3_0(I),
\]

\[
\|u - I_H u\|_{H^3_0} \leq \frac{h^2}{\pi^2} \|u^{(4)}\|_{L^2} \quad \text{for } u \in H^4(I) \cap H^3_0(I).
\]
Representing $I_H$ as $I_{H_x}$ and $I_{H_y}$ for $x$ and $y$ direction respectively, we have

$$
\|u - I_\Omega u\|_{H^2_0} = \|u - I_{I_x} u + I_{I_x} u - I_{H_x} I_{H_y} u\|_{H^2_0}
\leq \|u - I_{H_x} u\|_{H^2_0} + \|I_{H_x} (u - I_{H_y} u)\|_{H^2_0}
\leq \|u - I_{H_x} u\|_{H^2_0} + \|I_{H_x} (u - I_{H_y} u) - (u - I_{H_x} u)\|_{H^2_0} + \|u - I_{H_y} u\|_{H^2_0}
\leq \frac{h^2}{\pi^2} \|\Delta^2 u\|_{L^2} + \frac{h^2}{\pi^2} \|\Delta^2 u\|_{L^2} + \frac{h^2}{\pi^2} \|\Delta^2 u\|_{L^2}
\leq \frac{h^2}{\pi^2} \|\Delta^2 u\|_{L^2} + \frac{h^2}{\pi^2} \|\Delta^2 u\|_{L^2} + \frac{h^2}{\pi^2} \|\Delta^2 u\|_{L^2}
\leq \frac{h^2}{\pi^2} |u|_{H^4} = \frac{h^2}{\pi^2} \|\Delta^2 u\|_{L^2}
\]
For $e \equiv u - P_h u$ we denote $\phi \in H^4(\Omega) \cap H_0^2(\Omega)$ as the solution of the following equations:

$$
\begin{cases}
\Delta^2 \phi = e & \text{in } \Omega \\
\phi = \frac{\partial \phi}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(3.7)

Then we have

$$
\|u - P_h u\|_{L^2}^2 = (u - P_h u, u - P_h u)_{L^2}
= (e, \Delta^2 \phi)_{L^2}
= (\Delta e, \Delta \phi)_{L^2}
= (\Delta e, \Delta (\phi - P_h \phi)_{L^2}
\leq \|\Delta e\|_{L^2} \|\Delta (\phi - P_h \phi)\|_{L^2}
\leq \|\Delta e\|_{L^2} \cdot \frac{h^2}{\pi^2} \|\Delta^2 \phi\|_{L^2}
\leq \|\Delta e\|_{L^2} \cdot \frac{h^2}{\pi^2} \|e\|_{L^2}.
$$

Therefore $\|u - P_h u\|_{L^2} \leq \frac{h^2}{\pi^2} \|u - P_h u\|_{H_0^2}$ holds and the third estimation (3.6) is proved. Finally using (3.4) and (3.6) we can prove the second estimation (3.5) as follows:

$$
\|u - P_h u\|_{H_0^2}^2 = (\nabla (u - P_h u), \nabla (u - P_h u))_{L^2}
= (-\Delta (u - P_h u), u - P_h u)_{L^2}
\leq \|\Delta (u - P_h u)\|_{L^2} \|u - P_h u\|_{L^2}
\leq \frac{h^2}{\pi^2} \|\Delta^2 u\|_{L^2} \cdot \frac{h^4}{\pi^4} \|\Delta^2 u\|_{L^2}.
$$

Now, as in [1] or [3], we decompose (3.2) into the finite and infinite dimensional parts:

$$
\begin{cases}
P_h u = P_h F_1(u), \\
(I - P_h) u = (I - P_h) F_1(u).
\end{cases}
$$

(3.8)

Since we apply a Newton-like method only for the former part of (3.8), we define the following operator:

$$
\mathcal{N}_h^1(u) \equiv P_h u - [I - F_1]_{h}^{-1}(P_h u - P_h F_1(u)),
$$
where \( I \) is the identity map on \( H_0^2(\Omega) \). And we assume that the restriction to \( S_h \) of the operator \( P_h[I - F_1] : S_h \rightarrow S_h \) has the inverse \([I - F_1]_h^{-1}\). The validity of this assumption can be numerically confirmed in actual computations.

We next define the operator \( T_1 : H_0^2(\Omega) \rightarrow H_0^2(\Omega) \) by

\[
T_1(u) \equiv \mathcal{N}_h^1(u) + (I - P_h)F_1(u).
\]

Then \( T_1 \) becomes a compact map on \( H_0^2(\Omega) \) and we have the following equivalence relation

\[
u = T_1(u) \iff u = F_1(u).
\]

Our purpose is to find a unique fixed point of \( T_1 \) in a certain set \( U \subset H_0^2(\Omega) \), which is called a 'candidate set'. Given positive real numbers \( \gamma \) and \( \alpha \) we define the corresponding candidate set \( U \) by

\[
U \equiv U_h \oplus [\alpha], \tag{3.9}
\]

where \( U_h \equiv \{\phi_h \in S_h \mid \|\phi_h\|_{H_0^2} \leq \gamma\}, [\alpha] \equiv \{\phi \perp \in S_\perp \mid \|\phi \perp\|_{H_0^2} \leq \alpha\} \) and \( S_\perp \) means the orthogonal complement of \( S_h \) in \( H_0^2(\Omega) \). If the relation

\[
\overline{T_1(U)} \subset \text{int}(U) \tag{3.10}
\]

holds, by Schauder's fixed point theorem and the linearity of \( T_1 \), there exists a fixed point \( u \) of \( T_1 \) in \( U \) and the fixed point is unique, i.e., \( u = 0 \), which implies that the operator \( \mathcal{L} \) is invertible. Decomposing (3.10) into finite and infinite dimensional parts we have a sufficient condition for (3.10) as follows:

\[
\begin{align*}
\sup_{u \in U} \|\mathcal{N}_h^1(u)\|_{H_0^2} &< \gamma \\
\sup_{u \in U} \|(I - P_h)F_1(u)\|_{H_0^2} &< \alpha.
\end{align*} \tag{3.11}
\]

We now derive the following theorem in which the verification condition (3.11) is numerically and simply described.

**Theorem 1.** Let \( \{\phi_i\} \) be the basis of \( S_h \) and define the following constants:

\[
C_0 = \frac{h}{\pi}, \quad C_1^s = \|\nabla(u_s + \varphi)\|_\infty, \quad C_2^s = \|\nabla \frac{\partial(u_s + \varphi)}{\partial x}\|_\infty + \|\nabla \frac{\partial(u_s + \varphi)}{\partial y}\|_\infty.
\]
\[ C_3^s = \|\nabla\Delta(u_s + \varphi)\|_\infty, \quad C_p = \frac{1}{\pi\sqrt{2}}, \quad M_1 = \|L^TG^{-1}L\|_E, \]
\[ K_1 = C_1^s + C_0C_3^s, \quad K_2 = C_1^s + C_0C_3^sC_p, \quad K_3 = \sqrt{2}C_1^s + C_p(C_2^s + C_0C_3^s), \]

where \( \|\nabla v\|_\infty \equiv (\|\nabla v_x\|_\infty^2 + \|\nabla v_y\|_\infty^2)^{\frac{1}{2}} \), \( \|\cdot\|_E \) denotes the matrix norm corresponding to the Euclidean vector norm, \( C_p \) is the Poincaré constant, the matrix \( G = (G_{ij}) \) is defined by

\[ G_{ji} \equiv R(J(u_s + \varphi, \Delta\phi_i) + J(\phi_i, \Delta(u_s + \varphi)), \phi_j)_{L^2} + (\Delta\phi_i, \Delta\phi_j)_{L^2}, \]

and \( D = LL^T \) is a Cholesky decomposition for the matrix \( D = (D_{ij}) \) defined by

\[ D_{ij} \equiv (\Delta\phi_i, \Delta\phi_j)_{L^2}. \]

For these constants, if the inequality

\[ RC_0(K_1 + K_2K_3M_1RC_0) < 1 \quad (3.12) \]

holds then the operator \( \mathcal{L} \) is invertible.

**Proof.** We show sufficient conditions for (3.11). Denoting \( u = u_1 + u_2, \ u_1 \in U_h, \ u_2 \in [\alpha], \) by some simple calculations we have \( N_h^2(u) = [I - F_1]^{-1}P_hF_1(u_2), \) and thus

\[ \|N_h^2(u)\|_{H^2} \leq M_1\|P_hF_1(u_2)\|_{H^2} \quad (3.13) \]

holds. (See [1] or [3] for details to such estimation.) Using error estimation in Lemma 2, we can estimate \( \|P_hF_1(u_2)\|_{H^2} \) as follows:

\[ \|P_hF_1(u_2)\|_{H^2} \leq RC_0K_3\alpha. \quad (3.14) \]

Thus we derive a sufficient condition for the first inequality in (3.11) as

\[ M_1RC_0K_3\alpha < \gamma. \quad (3.15) \]
Now we estimate the left hand side of the second inequality in (3.11). Noting that

\[
\|(I - P_h)F_1(u)\|_{H^2_0} \leq R\|(I - P_h)KJ(u_s + \varphi, \Delta u)\|_{H^2_0} + R\|(I - P_h)KJ(u, \Delta(u_s + \varphi))\|_{H^2_0} \\
\leq RC_0K_2\gamma + RC_0K_1\alpha,
\]

we obtain the sufficient condition for the second inequality in (3.11) as

\[
RC_0(K_1\alpha + K_2\gamma) < \alpha. \tag{3.16}
\]

Combining the conditions (3.15) and (3.16) we finally obtain the sufficient condition for (3.11) as

\[
RC_0(K_1 + K_2K_3M_1RC_0) < 1.
\]

4 Verification procedure for nonlinear problem

In what follows we assume that the invertibility of the linearized operator $L$ is confirmed by the method described in the previous section. We will verify the existence of solutions for (2.2) in the neighborhood of $u_X \in C^1(\Omega)$ satisfying for all $v_h \in S_h$

\[
(\Delta u_X + \Delta \varphi, \Delta v_h)_{L^2} + (R \cdot J(u_s + \varphi, \Delta(u_s + \varphi)), v_h)_{L^2} = ((f_2)_x - (f_1)_y, v_h)_{L^2}. \tag{4.1}
\]

Considering the function $\bar{u}$ satisfying

\[
\begin{cases}
\Delta^2 \bar{u} = -\Delta^2 \varphi - R \cdot J(u_s + \varphi, \Delta(u_s + \varphi)) + (f_2)_x - (f_1)_y & \text{in } \Omega, \\
\frac{\partial \bar{u}}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases} \tag{4.2}
\]

and writing $w \equiv u - \bar{u}$, $v_0 \equiv \bar{u} - u_X$, $u - u_X$ can be represented as $w + v_0$.

Noting that $u_X = P_h \bar{u}$, we see that $v_0 \in S_\perp$ and, by Lemma 2 and its proof, the error estimates for $v_0$ can be derived:

\[
\|v_0\|_{H^2_0} \leq \frac{h^2}{\pi^2} \| - \Delta^2 \varphi - R \cdot J(u_s + \varphi, \Delta(u_s + \varphi)) + (f_2)_x - (f_1)_y \|_{L^2},
\]
\[ \frac{h}{\pi} \|v_0\|_{H_0^2} \leq \frac{h^2}{\pi^2} \|v_0\|_{H_0^2}, \]

\[ \frac{h}{\pi} \|v_0\|_{L^2} \leq \frac{h^2}{\pi^2} \|v_0\|_{H_0^2}. \]

Now we can rewrite (2.2) as

\[ \begin{cases} \Delta^2 w = -R \cdot J(w + u_X + v_0 + \varphi, \Delta(w + u_X + v_0 + \varphi)) \\ + R \cdot J(u_s + \varphi, \Delta(u_s + \varphi)) \text{ in } \Omega, \\ w = \frac{\partial w}{\partial n} = 0 \text{ on } \partial \Omega. \end{cases} \] (4.3)

Thus defining the following compact map on \( H_0^2(\Omega) \)

\[ F_2(w) \equiv RK \{ J(u_s + \varphi, \Delta(u_s + \varphi)) - J(w + u_X + v_0 + \varphi, \Delta(w + u_X + v_0 + \varphi)) \}, \] (4.4)

we have the fixed point equation

\[ w = F_2(w). \] (4.5)

Now we formulate the infinite dimensional Newton method for the equation (4.5). Note that \( w - [I - F_2'(-v_0 - u_X + u_s)]^{-1}(I - F_2)(w) \) can be equivalently represented as \( \mathcal{L}^{-1}q(w) \), where \( F_2'(-v_0 - u_X + u_s) \) stands for Fréchet derivative of \( F_2 \) at \( -v_0 - u_X + u_s \) and

\[ q(w) \equiv R\{ J(u_s + \varphi, \Delta(u_s + \varphi)) - J(w + u_X + v_0 + \varphi, \Delta(w + u_X + v_0 + \varphi)) \\ + J(u_s + \varphi, \Delta w) + J(w, \Delta(u_s + \varphi)) \}. \]

Then we have the relation

\[ w = F_2(w) \iff w = T_2(w), \] (4.6)

where \( T_2(w) \equiv \mathcal{L}^{-1}q(w) \) is a compact map on \( H_0^2(\Omega) \).

We intend to find a fixed point of \( T_2 \) in a set \( W \) defined by

\[ W = \{ w \in H_0^2(\Omega) \mid \|w\|_{H_0^2} \leq \alpha \}, \] (4.7)
where $\alpha$ is a positive number. If the relation

$$T_2(W) \subset W \tag{4.8}$$

holds, by Schauder's fixed point theorem there exists a fixed point of $T_2$ in $W$. Since a sufficient condition for (4.8) is

$$\sup_{w \in W} ||T_2(w)||_{H^2_0} \leq \alpha, \tag{4.9}$$

by estimating the left hand side of (4.8), we can derive the following theorem.

**Theorem 2.** Assume that the invertibility condition (3.12) holds. Using the same constants in Theorem 1, we define the following constants:

$$\kappa \equiv C_0 R (K_1 + K_2 K_3 M_1 C_0 R),$$

$$\tau_1 = -\frac{C_0 R M_1 K_2}{1 - \kappa}, \quad \tau_2 = \frac{1}{1 - \kappa},$$

$$\tau_3 = M_1 (C_0 R K_3 \tau_1 + 1), \quad \tau_4 = M_1 C_0 R K_3 \tau_2,$$

$$b = ||u_0||_{H^2_0}, \quad C_4 = \frac{1}{\pi},$$

where $C_4$ is an embedding constant satisfying $\|\nabla u\|_{L^4} \leq C_4 \|\Delta u\|_{L^2}$ for $u \in H^2_0(\Omega)$ and we have used the optimal embedding estimates $C_4 = \frac{1}{\pi}$ which can be derived by the result in [6]. Moreover for a matrix $S = \begin{pmatrix} \tau_1^2 + \tau_3^2 \tau_1 \tau_2 + \tau_3 \tau_4 \tau_1 \tau_2 + \tau_3 \tau_4 \tau_2^2 + \tau_4^2 \end{pmatrix}$ and $M_2 = \|S\|^{\frac{1}{2}}$, define the following constants:

$$C_1^X = \|\nabla (u_X + \varphi)\|_{\infty}, \quad C_2^X = \left\|\frac{\partial (u_X + \varphi)}{\partial x}\right\|_{\infty} \quad + \left\|\frac{\partial (u_X + \varphi)}{\partial y}\right\|_{\infty},$$

$$C_3^X = \|\nabla \Delta (u_X + \varphi)\|_{\infty},$$

$$D_1^\delta = \|\nabla (u_X - u_s)\|_{L^2},$$

$$D_2^\delta = \|J(u_X - u_s, \Delta (u_s + \varphi))\|_{L^2},$$

If there exists a real number $\alpha > 0$ satisfying

$$M_2 R \{C_4^2 (\alpha + b)^2 + C_2^2 \alpha D_3^\delta + C_3^X C_\rho b + \alpha D_1^\delta C_\rho$$

$$+ C_0 b (\sqrt{2} C_1^X + C_\rho C_3^X) + C_2^D D_2^\delta + C_\rho C_1^X D_3^\delta\} \leq \alpha, \tag{4.10}$$

$$\rho = \sqrt{C_0 R (K_1 + K_2 K_3 M_1 C_0 R)}.$$
then there exists a fixed point of \( T_2 \) in \( W \).

**Proof.** For \( q(w) \in H^{-2}(\Omega) \) consider the solution \( \phi \in H_0^2(\Omega) \) of the problem

\[
\begin{align*}
\mathcal{L} \phi &= q(w) \quad \text{in } \Omega, \\
\phi &= \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(4.11)

Then writing \( \phi = \phi_h + \phi_\perp, \phi_h \in S_h, \phi_\perp \in S_\perp \), we have

\[
\begin{align*}
\|\phi_h\|_{H_0^2} &\leq M_1 R C_0 K_3 \|\phi_\perp\|_{H_0^2} + M_1 \|P_h K q(w)\|_{H_0^2}, \\
\|\phi_\perp\|_{H_0^2} &\leq R C_0 (K_1 \|\phi_\perp\|_{H_0^2} + K_2 \|P_h K q(w)\|_{H_0^2}) + \|(I - P_h) K q(w)\|_{H_0^2}.
\end{align*}
\]

(4.12)

Noting that \( \kappa < 1 \) holds because of the invertibility of \( \mathcal{L} \), we have

\[
\begin{align*}
\|\phi_h\|_{H_0^2} &\leq \tau_3 \|P_h K q(w)\|_{H_0^2} + \tau_4 \|(I - P_h) K q(w)\|_{H_0^2}, \\
\|\phi_\perp\|_{H_0^2} &\leq \tau_1 \|P_h K q(w)\|_{H_0^2} + \tau_2 \|(I - P_h) K q(w)\|_{H_0^2}.
\end{align*}
\]

(4.13)

Therefore we obtain

\[
\|\phi\|_{H_0^2} \leq M_2 \|K q(w)\|_{H_0^2} \leq M_2 \|q(w)\|_{H^{-2}}.
\]

(4.14)

Furthermore, we have the estimations

\[
\|q(w)\|_{H^{-2}} = \sup_{\theta \in H_0^2, \|\theta\|_{H_0^2} = 1} |<q(w), \theta>_{H^{-2}, H_0^2}| \leq R \{ C_4^2 (\alpha + b)^2 + C_4^2 \alpha D_3^\delta + C_3^X C_\rho b + \alpha D_1^\delta C_\rho \\
+ C_0 b (\sqrt{2}C_1^X + C_\rho C_3^X) + C_2^2 D_2^\delta + C_\rho C_1^X D_3^\delta \}
\]

Thus we obtain

\[
\|\mathcal{L}^{-1} q(w)\|_{H_0^2} \leq \\
M_2 R \{ C_4^2 (\alpha + b)^2 + C_4^2 \alpha D_3^\delta + C_3^X C_\rho C_0 b + \alpha D_1^\delta C_\rho
\]
\[ + C_0 b (\sqrt{2} C_1^X + C_\rho C_3^X) + C_\rho^2 D_2^\delta + C_\rho C_1^X D_3^\delta \].

This means that the inequality (4.10) is a sufficient condition for \( T_2(W) \subseteq W \) and the desired assertion is proved.

5 Numerical examples

Particularly, we consider the two dimensional driven cavity problem with \( f = 0 \) and \( g = (\varphi_y, -\varphi_x) \) in (1.1), where \( \varphi(x, y) = x^2(1-x)^2y^2(1-y) \).

In calculations, we used interval arithmetic in order to avoid the effects of rounding errors in the floating-point computations. The computations were carried out on the DELL Precision WorkStation 650 (Intel Xeon 3.2GHz) using MATLAB (Ver. 6.5.1) and the interval arithmetic toolbox INTLAB (Ver. 4.2.1) coded by Prof. Rump in TU Hamburg-Harburg ([4]). The verification results are shown in Table 1, in which 'smallest \( \alpha \)' means the smallest bound \( \alpha \) satisfying the verification condition (4.9) and the solution \( u \) in (2.2) is enclosed as \( \|u - u_X\|_{H_{0}^{2}(\Omega)} \leq \|v_0\|_{H_{0}^{2}(\Omega)} + \alpha \).

<table>
<thead>
<tr>
<th>( R )</th>
<th>( M_1 )</th>
<th>( M_2 )</th>
<th>( |v_0|<em>{H</em>{0}^{2}(\Omega)} )</th>
<th>( D_3^\delta )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1.0183</td>
<td>1.4511</td>
<td>9.2855e-4</td>
<td>1.2940e-4</td>
<td>1.1199e-3</td>
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<tr>
<td>200</td>
<td>1.0355</td>
<td>2.4945</td>
<td>1.0094e-3</td>
<td>1.3666e-4</td>
<td>7.5815e-3</td>
</tr>
</tbody>
</table>

It seems that Wieners' method would not be able to apply to the Reynolds number higher than 20 in [7]. On the other hand, we enclosed the stationary solution for the Reynolds number over 100, and our method can be applied, in principle, to more higher Reynolds numbers by using more accurate approximation subspaces, i.e., smaller mesh sizes.
参考文献


