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Differential Calculus in Second Order Arithmetic

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1 Introduction

In this paper, we develop basic part of differential calculus within some weak subsystems of second order arithmetic. Our work is motivated by the program of Reverse Mathematics, whose ultimate goal is to determine which set existence axioms are needed to prove ordinary mathematical theorems.

The Reverse Mathematics program was initiated by Friedman and carried forward by Friedman, Simpson, Tanaka, and others. They proved that many of theorems of analysis, algebra and other branches of mathematics are either proved in RCA_0 or equivalent over RCA_0 to particular set existence axioms such as WKL_0, ACA_0, e.g., [6, 7]. Here, RCA_0, WKL_0 and ACA_0 are relatively weak, important subsystems of second order arithmetic.

For differential calculus in the second order arithmetic, Hardin and Velleman [5] showed that the mean value theorem is provable in RCA_0. Though various fields of mathematics have been developed in subsystems of second order arithmetic, differential calculus has not been studied very much in the program of Reverse Mathematics. In this paper, we carry out basic differential calculus and prove basic theorems such as the termwise differentiation theorem and the inverse function theorem in these systems in RCA_0.

To develop differential calculus, we define C^1-functions. Here, we consider the following two versions of C^1-functions in RCA_0. By a weak C^1-function, we mean a continuous function which is continuously differentiable, and by a strong C^1-function, a pair of a continuous function and its continuous derivative. There is a serious difference between them in RCA_0, since we may not construct the derivative of a weak C^1-function in RCA_0. In fact, most of simple properties of weak C^1-functions require ACA_0, in other words, RCA_0 is too weak to deal with weak C^1-functions. To avoid this difficulty, we adopt the notion of strong C^1-functions. Fortunately, usual C^1-functions constructed in terms of polynomials, power series and other concrete manners can be shown to be strong C^1-functions in RCA_0. From now on, we use the word 'C^1-functions' for strong C^1-functions.

Using the strong version of C^1-functions, we can construct a very useful function to develop differential calculus (Theorem 3.11). It expresses the continuous differentiability at each point of a C^1-function, and so we call it 'a differentiable condition function for a C^1-function.' By this function, we can check differentiabilities of uncountably many points at once, which allows us to imitate or modify some usual methods of basic differential calculus in RCA_0. For example, the termwise differentiation and integration theorems (Theorems 3.17 and 3.21) can be proved in RCA_0. We can also prove the inverse function theorem in RCA_0. We remark that if we simply imitate the usual proofs, we need WKL_0 or ACA_0 to construct the inverse continuous function.

Based on the above, we can develop complex analysis in second order arithmetic [10].

2 Preliminaries

2.1 Subsystems of second-order arithmetic

The language $\mathcal{L}_2$ of second-order arithmetic is a two-sorted language with number variables $x, y, z, \ldots$ and set variables $X, Y, Z, \ldots$. Numerical terms are built up from numerical variables and constant symbols $0, 1$ by means of binary operations $+$ and $\cdot$. Atomic formulas are $s = t$, $s < t$ and $s \in X$, where s and t are numerical terms. *Bounded* (Σ^0_0 or Π^0_0) formulas are constructed from atomic formulas by propositional connectives and bounded numerical quantifiers $(\forall x < t)$ and $(\exists x < t)$, where t does not contain x. A Σ^0_n formula is of the form $\exists x_1 \forall x_2 \ldots x_n \theta$ with θ bounded, and a Π^0_n formula is of the form $\forall x_1 \exists x_2 \ldots x_n \theta$ with θ bounded. All the Σ^0_n and Π^0_n formulas are the *arithmetical* (Σ^1_0 or Π^1_0) formulas. A Σ^1_n formula is of the form $\exists X_1 \forall X_2 \ldots X_n \varphi$ with φ arithmetical, and a Π^1_n formula is of the form $\forall X_1 \exists X_2 \ldots X_n \varphi$ with φ arithmetical.

Definition 2.1. *The system of* RCA_0 *consists of*

(1) *the ordered semiring axioms for* $(\omega, +, \cdot, 0, 1, <)$,

(2) Δ^0_1-CA :

$$\forall x(\varphi(x) \leftrightarrow \psi(x)) \rightarrow \exists X \forall x(x \in X \leftrightarrow \varphi(x)),$$

where $\varphi(x)$ *is* Σ^0_1, $\psi(x)$ *is* Π^0_1, *and* X *does not occur freely in* $\varphi(x)$,

(3) Σ^0_1 *induction scheme*:

$$\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \varphi(x),$$

where $\varphi(x)$ *is a* Σ^0_1 *formula.*

The acronym RCA stands for recursive comprehension axiom. Roughly speaking, the set existence axioms of RCA_0 are strong enough to prove the existence of recursive sets.

Definition 2.2. ACA_0 *is the system which consists of* RCA_0 *plus* ACA (*arithmetical comprehension axioms*) :

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n)),$$

where $\varphi(x)$ *is arithmetical and* X *does not occur freely in* $\varphi(x)$.

If X and Y are set variables, we use $X \subseteq Y$ and $X = Y$ as abbreviations for the formulas $\forall n(n \in X \to n \in Y)$ and $\forall n(n \in X \leftrightarrow n \in Y)$. We define $\mathbb{N}$ to be the unique set X such that $\forall n(n \in X)$.

Within RCA_0, we define a *pairing map* $(m, n) = (m+n)^2 + m$. We can prove within RCA_0 that for all m, n, i, j in $\mathbb{N}$, $(m, n) = (i, j)$ if and only if $m = i$ and $n = j$. Moreover, using Δ^0_1-CA, we can prove that for any X and Y, there exists a set $X \times Y \subseteq \mathbb{N}$ such that

$$\forall n(n \in X \times Y \leftrightarrow \exists x \leq n \exists y \leq n (x \in X \wedge y \in Y \wedge (x, y) = n)).$$

For X and Y, a *function* $f : X \to Y$ is defined to be a set $F \subseteq X \times Y$ such that $\forall x \forall y_0 \forall y_1 ((x, y_0) \in F \wedge (x, y_1) \in F \to y_0 = y_1)$ and $\forall x \in X \exists y \in Y (x, y) \in F$. We write $f(x) = y$ for $(x, y) \in F$.

Within RCA_0, the universe of functions is closed under composition, primitive recursion (i.e., given $f : X \to Y$ and $g : \mathbb{N} \times X \times Y \to Y$, there exists a unique $h : \mathbb{N} \times X \to Y$ defined by $h(0, m) = f(m)$, $h(n+1, m) = g(n, m, h(n, m))$ and the least number operator (i.e., given $f : \mathbb{N} \times X \to \mathbb{N}$ such that for all $m \in X$ there exists $n \in \mathbb{N}$ such that $f(n, m) = 1$, there exists a unique $g : X \to \mathbb{N}$ defined by $g(m)$ =the least n such that $f(n, m) = 1$). Especially, if (M, S) is an ω-model of RCA_0, then (M, S) contains all recursive functions on ω.

Theorem 2.1. *The following is provable in* RCA_0. *If* $\varphi(x, y)$ *is* Σ^0_1 *and* $\forall n \exists m \varphi(n, m)$ *holds, then there exists a function from* $\mathbb{N}$ *to* $\mathbb{N}$ *such that* $\forall n \varphi(n, f(n))$ *holds.*

Proof. We reason within RCA_0. Write

$$\varphi(x, y) \equiv \exists z \theta(x, y, z)$$

where θ is Σ^0_0. By Δ^0_1 comprehension, we define *projection functions* p_1 and p_2 as follows: $p_i((n_1, n_2)) = n_i$ for all $n_1, n_2 \in \mathbb{N}$. Again using Δ^0_1 comprehension, there exists a function g from $\mathbb{N}^2$ to $\mathbb{N}$ such that

$$\theta(n, p_1(m), p_2(m)) \leftrightarrow g(n, m) = 1.$$

Then $\forall n \exists m g(n, m) = 1$, hence by the least number operator there exists a function h from $\mathbb{N}$ to $\mathbb{N}$ such that $g(n, h(n)) = 1$. Define a function f as $f(n) = p_1(g(n))$, then $\forall n \varphi(n, f(n))$ holds. This completes the proof. □

The following theorem will be useful in showing that ACA is needed in order to prove various theorems of ordinary mathematics.

Theorem 2.2 ([8] Theorem III.1.3). *The following assertions are pairwise equivalent over* RCA_0*.*

1. *For all one-to-one function f from $\mathbb{N}$ to $\mathbb{N}$, there exists a set $X \subseteq \mathbb{N}$ such that X is the range of f.*

2. ACA_0.

For details of the definitions of these three subsystems, see [8] I.

2.2 Real number system and Euclidian space

Next, we construct the real number system. We first define $\mathbb{Z}$ and $\mathbb{Q}$. Define an equivalence relation $=_{\mathbb{Z}}$ on $\mathbb{N}^2$ as $(m,n) =_{\mathbb{Z}} (p,q) \leftrightarrow m+q = n+p$, and by Δ^0_1 comprehension, define $\mathbb{Z}$, a set of integers, as $(m,n) \in \mathbb{Z} \leftrightarrow \forall k < (m,n)\ (p_1(k), p_2(k)) \neq_{\mathbb{Z}} (m,n)$, *i.e.*, $\mathbb{Z}$ is a set of least number elements of equivalence classes of $=_{\mathbb{Z}}$. We define $+_{\mathbb{Z}}$ as $(l_1,l_2) +_{\mathbb{Z}} (m_1,m_2) := (n_1,n_2) \leftrightarrow (l_1,l_2),(m_1,m_2),(n_1,n_2) \in \mathbb{Z} \wedge (l_1+m_1, l_2+m_2) =_{\mathbb{Z}} (n_1,n_2)$, and define $\cdot_{\mathbb{Z}}$ similarly. We can also define $|\cdot|_{\mathbb{Z}}$ and $\leq_{\mathbb{Z}}$ naturally. Similarly, we can define $\mathbb{Q}$, $+_{\mathbb{Q}}$, $\cdot_{\mathbb{Q}}$, etc.

Definition 2.3 (Real number system). *The following definitions are made in* RCA_0*. A real number is an infinite sequence of rational numbers $\alpha = \{q_n\}_{n\in\mathbb{N}}$ (i.e. a function from $\mathbb{N}$ to $\mathbb{Q}$) which satisfies $|q_k - q_l|_{\mathbb{Q}} \leq_{\mathbb{Q}} 2^{-k}$ for all $l \geq k$. Here, each q_n is said to be n-th approximation of α. Define $\{p_n\}_{n\in\mathbb{N}} =_{\mathbb{R}} \{q_n\}_{n\in\mathbb{N}}$ as $\forall k\ |p_k - q_k|_{\mathbb{Q}} \leq_{\mathbb{Q}} 2^{-k+1}$. We can also define $+_{\mathbb{R}}$, $\cdot_{\mathbb{R}}$, $|\cdot|_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ naturally. We usually write $\alpha \in \mathbb{R}$ if α is a real number. For details of the definition of the real number system, see [8] II or [9].*

Imitating the definition of $\mathbb{R}$, we define Euclidean space $\mathbb{R}^n$. We define $\mathbb{Q}^n$ as a set of rational numbers of length n, *i.e.* $\mathbf{q} \in \mathbb{Q}^n$ if and only if $\mathbf{q} = (q_1, \dots, q_n)$ and each q_i is a rational number. We define addition and scalar multiplication naturally, and see $\mathbb{Q}^n$ as a (countable) vector space. We also define $\|\cdot\|_{\mathbb{Q}^n}$ as

$$\|\mathbf{q}\|_{\mathbb{Q}^n} = \sqrt{{q_1}^2 + \cdots + {q_n}^2}.$$

Definition 2.4 (Euclidian space). *The following definitions are made in* RCA_0*. An element of $\mathbb{R}^n$ is an infinite sequence of elements of $\mathbb{Q}^m$ $\mathbf{a} = \{\mathbf{q}_k\}_{k\in\mathbb{N}}$ which satisfies*

$\|\mathbf{q}_k - \mathbf{q}_l\| \leq_{\mathbb{Q}^n} 2^{-k}$ *for all* $l \geq k$. *Then, each* $a_i = \{q_{ki}\}_{k \in \mathbb{N}}$ *is a real number. (Here,* $\mathbf{q}_k = (q_{k1}, \cdots, q_{kn})$*.) We define* $\| \cdot \|_{\mathbb{R}^n}$*, the norm of* $\mathbb{R}^n$ *as the following:*

$$\|\mathbf{a}\|_{\mathbb{R}^n} = \sqrt{a_1{}^2 + \cdots + a_n{}^2}.$$

Here, of course the real number field $\mathbb{R}$ *is the* 1*-dimensional Euclidean space* $\mathbb{R}^1$.

Remark 2.3. *In this paper, to avoid too many subscript, we use the intuitive expression such as* $\mathbf{q} = (q_1, \cdots, q_n)$ *even if the dimension of Euclidean space* n *may be nonstandard.*

A *sequence of sets of natural numbers* is defined to be a set $X \subseteq \mathbb{N} \times \mathbb{N}$. By Δ^0_1 comprehension, we define X_k as $m \in X_k \leftrightarrow (k, m) \in X$ and write $X = \{X_k\}_{k \in \mathbb{N}}$. If $X_k = \mathbf{a}_k \in \mathbb{R}^n$, *i.e.*, each X_k is formed an element of $\mathbb{R}^n$, then $X = \{\mathbf{a}_k\}_{k \in \mathbb{N}}$ is said to be a *sequence of points of* $\mathbb{R}^n$. We say that a sequence $\{\mathbf{a}_k\}_{k \in \mathbb{N}}$ converges to $\mathbf{b}$, written $\mathbf{b} = \lim_{k \to \infty} \mathbf{a}_k$, if

$$\forall \varepsilon > 0 \; \exists k \; \forall i \; \|\mathbf{b} - \mathbf{a}_{k+i}\| < \varepsilon.$$

The next theorem show that $\mathbb{R}^n$ is 'weakly' complete.

Theorem 2.4. *The following is provable in* RCA_0. *Let* $\{\mathbf{a}_k\}_{k \in \mathbb{N}}$ *be a sequence of points of* $\mathbb{R}^n$. *If there exists a sequence of real numbers* $\{r_k\}_{k \in \mathbb{N}}$ *such that* $\lim_{k \to \infty} r_k = 0$ *and* $\forall k \forall i \; \|\mathbf{a}_k - \mathbf{a}_{k+i}\| < r_k$*, then* $\{\mathbf{a}_k\}_{k \in \mathbb{N}}$ *is convergent, i.e., there exists* $\mathbf{b}$ *such that* $\mathbf{b} = \lim_{k \to \infty} \mathbf{a}_k$.

Proof. This theorem is a generalization of nested interval completeness [8, Theorem II.4.8], and modifying its proof, we can easily prove this theorem. □

Next, we define an open or closed set. It is coded by the countable open basis of $\mathbb{R}^n$.

Definition 2.5 (open and closed sets). *The following definitions are made in* RCA_0.

1. *A (code for an) open set* U *in* $\mathbb{R}^n$ *is a set* $U \subseteq \mathbb{N} \times \mathbb{Q}^n \times \mathbb{Q}$. *A point* $\mathbf{x} \in \mathbb{R}^n$ *is said to belong to* U *(abbreviated* $\mathbf{x} \in U$*) if*

$$\exists n \; \exists \mathbf{a} \; \exists r \; (\|\mathbf{x} - \mathbf{a}\| < r \wedge (n, \mathbf{a}, r) \in U).$$

2. *A (code for a) closed set* C *in* $\mathbb{R}^n$ *is a set* $C \subseteq \mathbb{N} \times \mathbb{Q}^n \times \mathbb{Q}$. *A point* $\mathbf{x} \in \mathbb{R}^n$ *is said to belong to* C *(abbreviated* $\mathbf{x} \in C$*) if*

$$\forall n \; \forall \mathbf{a} \; \forall r \; ((n, \mathbf{a}, r) \in C \to \|\mathbf{x} - \mathbf{a}\| < r).$$

The following lemma is very useful to construct open or closed sets.

Lemma 2.5 ([8] Lemma II.5.7). *For any* Σ^0_1 *(or* Π^0_1*) formula* $\varphi(X)$*, the following is provable in* RCA_0. *Assume that for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} = \mathbf{y}$ *and* $\varphi(\mathbf{x})$ *imply* $\varphi(\mathbf{y})$. *Then there exists an open (or closed) set* $U \subseteq \mathbb{R}^n$ *such that for all* $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \in U$ *if and only if* $\varphi(\mathbf{x})$.

3 Differential calculus

3.1 Continuous functions

In this section, we define continuous functions and show some basic results for continuous functions. We first define continuous functions as a certain code given by the countable open basis of $\mathbb{R}^n$.

Definition 3.1 (continuous functions). *The following definition is made in* RCA_0. *A (code for a) continuous partial function f from $\mathbb{R}^n$ to $\mathbb{R}$ is a set of quintuples $F \subseteq \mathbb{N} \times \mathbb{Q}^n \times \mathbb{Q}^+ \times \mathbb{Q} \times \mathbb{Q}^+$ which is required to have certain properties. We write $(\mathbf{a}, r)F(b, s)$ as an abbreviation for $\exists m((m, \mathbf{a}, r, b, s) \in F)$. The property which we require are:*

1. *if $(\mathbf{a}, r)F(b, s)$ and $(\mathbf{a}, r)F(b', s')$, then $|b - b'| \leq s + s'$;*
2. *if $(\mathbf{a}, r)F(b, s)$ and $\|\mathbf{a}' - \mathbf{a}\| + r' < r$, then $(\mathbf{a}', r')F(b, s)$;*
3. *if $(\mathbf{a}, r)F(b, s)$ and $|b - b'| + s < s'$, then $(\mathbf{a}, r)F(b', s')$.*

A point $\mathbf{x} \in \mathbb{R}^n$ is said to belong to the domain of f, abbreviated $\mathbf{x} \in \mathrm{dom}(f)$, if and only if for all $\varepsilon > 0$ there exists $(\mathbf{a}, r)F(b, s)$ such that $\|\mathbf{x} - \mathbf{a}\| < r$ and $s < \varepsilon$. If $\mathbf{x} \in \mathrm{dom}(f)$, we define the value $f(\mathbf{x})$ to be the unique $y \in \mathbb{R}$ such that $|y - b| < s$ for all $(\mathbf{a}, r)F(b, s)$ with $\|\mathbf{x} - \mathbf{a}\| < r$. The existence of $f(\mathbf{x})$ is provable in RCA_0.

Let U be an open or closed subset of $\mathbb{R}^n$, and V be an open or closed subset of $\mathbb{R}$. Then f is said to be a continuous function from U to V if and only if for all $\mathbf{x} \in U$, $\mathbf{x} \in \mathrm{dom}(f)$ and $f(\mathbf{x}) \in V$.

Definition 3.2. *The following definition is made in* RCA_0. *A continuous partial function from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a (code for a) finite sequence of continuous partial functions $\mathbf{f} = (f_1, \dots, f_m)$ such that $f_1, \dots, f_m$ are continuous partial functions from $\mathbb{R}^n$ to $\mathbb{R}$.*

Let U be an open or closed subset of $\mathbb{R}^n$, and V be an open or closed subset of $\mathbb{R}^m$. Then $\mathbf{f}$ is said to be a continuous function from U to V if and only if for all $\mathbf{x} \in U$ and for all $1 \leq i \leq n$, $\mathbf{x} \in \mathrm{dom}(f_i)$ and $\mathbf{y} = (f_1(\mathbf{x}) \dots f_m(\mathbf{x})) \in V$.

Remark 3.1. *Imitating definition 3.1, we can define another code for a continuous partial function from $\mathbb{R}^n$ to $\mathbb{R}^m$. A (code for a) continuous partial function $\mathbf{f}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a set of quintuples $\mathbf{F} \subseteq \mathbb{N} \times \mathbb{Q}^n \times \mathbb{Q}^+ \times \mathbb{Q} \times \mathbb{Q}^+$ which is required:*

1. *if $(\mathbf{a}, r)\mathbf{F}(\mathbf{b}, s)$ and $(\mathbf{a}, r)\mathbf{F}(\mathbf{b}', s')$, then $\|\mathbf{b} - \mathbf{b}'\| \leq s + s'$;*
2. *if $(\mathbf{a}, r)\mathbf{F}(\mathbf{b}, s)$ and $\|\mathbf{a}' - \mathbf{a}\| + r' < r$, then $(\mathbf{a}', r')\mathbf{F}(\mathbf{b}, s)$;*

3. *if* $(\mathbf{a}, r)\mathbf{F}(\mathbf{b}, s)$ *and* $\|\mathbf{b} - \mathbf{b}'\| + s < s'$, *then* $(\mathbf{a}, r)\mathbf{F}(\mathbf{b}', s')$.

We can easily and effectively construct a code for $\mathbf{f}$ *from codes for* $f_1, \dots, f_m$. *Conversely we can easily and effectively construct codes for* $f_1, \dots, f_m$ *from a code for* $\mathbf{f}$.

First, there exist a code for an identity function, a constant function, a norm function, and so on. We can construct other elementary continuous functions by next theorem.

Theorem 3.2 (**[8] II.6.3 and II.6.4.**). *The following is provable in* RCA_0. *There exists a (code for a) continuous function of sum, product and quotient of two* $\mathbb{R}$*-valued continuous functions. Also there exists a (code for a) continuous function of a composition of two continuous functions.*

The next two theorems show the basic properties of continuous functions.

Theorem 3.3. *The following assertions are provable in* RCA_0.

1. *Let* U *be an open subset of* $\mathbb{R}^n$, V *be an open subset of* $\mathbb{R}^m$ *and* f *be a continuous function from* U *to* $\mathbb{R}^m$. *Then we can effectively construct an open set* $W = f^{-1}(V) \cap U$, *the inverse image of* V.

2. *Let* C *be a closed subset of* $\mathbb{R}^n$, V *be an open subset of* $\mathbb{R}^m$ *and* f *be a continuous function from* C *to* $\mathbb{R}^m$. *Then we can effectively construct an open set* $W \subseteq \mathbb{R}^n$ *such that* $W \cap C = f^{-1}(V) \cap C$.

We write such W *as* $W = \tilde{f}^{-1}(V)$.

Proof. Immediate from Lemma 2.5. □

The next theorem is very useful to show that constructing some continuous functions requires ACA_0.

Theorem 3.4. *The following assertions are pairwise equivalent over* RCA_0.

1. ACA_0.

2. *If* f *is a continuous function from* $(0,1)$ *to* $\mathbb{R}$ *such that* $\lim_{x \to +0} f(x) = 0$, *then there exists a (code for a) continuous function* $\bar{f}$ *from* $[0,1)$ *to* $\mathbb{R}$ *such that*

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in (0,1), \\ 0 & \text{if } x = 0. \end{cases}$$

Proof. We reason within RCA_0. $1 \to 2$ is obvious. We show $2 \to 1$. By Theorem 2.2, we show that for all one-to-one function h from $\mathbb{N}$ to $\mathbb{N}$, there exists a set X such that X is the range of h. Let h be a one-to-one function from $\mathbb{N}$ to $\mathbb{N}$. Then $\lim_{n\to\infty} h(n) = \infty$. Define $\{a_n\}_{n\in\mathbb{N}}$ as

$$a_n \quad := \quad \frac{\frac{1}{h(n)+1} - \frac{1}{h(n+1)+1}}{\frac{1}{n+1} - \frac{1}{n+2}}.$$

Then we define a continuous function f from $(0,1)$ to $\mathbb{R}$ such that

$$f(x) = a_n \left(x + \frac{1}{n+1} \right) + \frac{1}{(h(n)+1)}$$

for each n and $x \in \left[\frac{1}{n+2}, \frac{1}{n+1}\right]$. Then, $f(1/(n+1)) = 1/(h(n)+1)$ for all $n \in \mathbb{N}$, and $\lim_{x\to 0} f(x) = 0$. Hence by 2, we can expand f into $\bar{f}$ such that

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in (0,1), \\ 0 & \text{if } x = 0. \end{cases}$$

Now we construct the range of h. Let $\bar{F}$ be a code for $\bar{f}$, and let $\varphi(k,l)$ be a Σ^0_1 formula which expresses that there exist (a,r,b,s) such that $(a,r)\bar{F}(b,s)$, $|a| + 1/(l+1) < r$ and $|b| + s < 1/(k+1)$. Then by conditions of a code for a continuous function, $\forall k \exists l \varphi(k,l)$ holds. Hence, there exists a function h_0 from $\mathbb{N}$ to $\mathbb{N}$ such that $\forall k \varphi(k, h_0(k))$ holds. This implies

$$\forall m \in \mathbb{N}\ m \geq h_0(n) \to n < h(m).$$

By Δ^0_1 comprehension, define a set $X \subseteq \mathbb{N}$ as $n \in X \leftrightarrow \exists m < h_0(n)\ n = h(m)$. Then clearly, X is the range of h. This completes the proof of $2 \to 1$. □

3.2 C^1-functions

We first define a weak C^1-functions as a continuously differentiable continuous function.

Definition 3.3 (weak C^1-functions). *The following definition is made in* RCA_0. *Let U be an open subset of $\mathbb{R}$, and let f be a continuous functions from U to $\mathbb{R}$. Then f is said to be weak C^1 if and only if*

$$\forall x \in U\ \exists \alpha \in \mathbb{R}\ \alpha = \lim_{x'\to x} \frac{f(x') - f(x)}{x' - x}$$

and

$$\forall x \in U\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in U\ |x - y| < \delta \to |\alpha_x - \alpha_y| < \varepsilon$$

holds. Here, $\alpha_x = \lim_{x'\to x} \frac{f(x')-f(x)}{x'-x}$.

Theorem 3.5. *The following assertions are pairwise equivalent over* RCA_0.

1. ACA_0.
2. *If f is a weak* C^1*-function from* $(-1,1)$ *to* $\mathbb{R}$*, then there exists a (code for a) continuous function* f' *which is the derivative of* f.

Proof. We reason within RCA_0. We can easily prove $1 \to 2$ by arithmetical comprehension. For the converse, we assume 2. By Theorem 2.2, we show that for all one-to-one function h from $\mathbb{N}$ to $\mathbb{N}$, there exists a set X such that X is the range of h. Let h be a one-to-one function from $\mathbb{N}$ to $\mathbb{N}$. Then $\lim_{n\to\infty} h(n) = \infty$. Define $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that

$$\begin{aligned} a_n &:= \frac{\frac{1}{h(n)+1} - \frac{1}{h(n+1)+1}}{\frac{1}{n+1} - \frac{1}{n+2}}; \\ b_n &:= \frac{1}{2}\left(\frac{1}{h(n)+1} + \frac{1}{h(n+1)+1}\right)\left(\frac{1}{n+1} - \frac{1}{n+2}\right). \end{aligned}$$

Then $b_n \le 1/(n+1) - 1/(n+2)$, hence by Theorem 3.16.1, $\sum_{k=n}^{\infty} b_k$ is convergent for all $n \in \mathbb{N}$. Using these, we define a continuously differentiable function from $(-1,1)$ to $\mathbb{R}$. Define a continuous function f_0 from $(-1,0)\cup(0,1)$ such that

$$f_0(x) = \begin{cases} -\frac{a_n}{2}\left(x + \frac{1}{n+1}\right)^2 + \frac{x(n+1)+1}{(n+1)(h(n)+1)} - \sum_{k=n}^{\infty} b_k & \text{if } x \in \left[\frac{-1}{n+1}, \frac{-1}{n+2}\right], \\ \frac{a_n}{2}\left(x - \frac{1}{n+1}\right)^2 + \frac{x(n+1)-1}{(n+1)(h(n)+1)} + \sum_{k=n}^{\infty} b_k & \text{if } x \in \left[\frac{1}{n+2}, \frac{1}{n+1}\right] \end{cases}$$

for each n. Here, if $|x| < 1/(n+1)$, then $|f_0(x)| < 1/(n+1)$. Hence, we can extend f_0 into f from $(-1,1)$ to $\mathbb{R}$ such that

$$f(x) = \begin{cases} f_0(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

To extend f_0 into f, we need to construct a code for f. Let F_0 be a code for f_0 and let $\varphi(a,r,b,s)$ be a Σ^0_1 formula which expresses $(a,r)F_0(b,s) \vee \exists m \in \mathbb{N}\ |a| + r < 1/(m+1) < |b| - s$. Write

$$\varphi(a,r,b,s) \equiv \exists m \theta(m,a,r,b,s)$$

where θ is Σ^0_0. By Δ^0_1 comprehension, define F as $(m,a,r,b,s) \in F \leftrightarrow \theta(m,a,r,b,s)$. Then clearly f is coded by F.

Next, we show that f is weak C^1. Define α_x as above, then

$$\alpha_x = \begin{cases} -a_n\left(x + \frac{1}{n+1}\right) + \frac{1}{(h(n)+1)} & \text{if } x \in \left[\frac{-1}{n+1}, \frac{-1}{n+2}\right], \\ 0 & \text{if } x = 0, \\ a_n\left(x + \frac{1}{n+1}\right) + \frac{1}{(h(n)+1)} & \text{if } x \in \left[\frac{1}{n+2}, \frac{1}{n+1}\right]. \end{cases}$$

We can easily check the condition of continuously differentiability, hence f is weak C^1. By 2, there exists a continuous function g from $(-1,1)$ to $\mathbb{R}$ such that $g(x) = \alpha_x$. Note that this continuous function g is similar to the continuous function we constructed in the proof of Theorem 3.4. Hence, we can construct the range of h as in the proof of Theorem 3.4. This completes the proof of $2 \to 1$. □

Theorem 3.5 pointed out the difficulty of constructing the derivative of a weak C^1-function. To avoid this difficulty, we mainly consider the following (strong) C^1-functions to develop differential calculus. We first define C^1-function in $\mathbb{R}$, and similarly we define C^r and C^∞-function in $\mathbb{R}$.

Definition 3.4 (C^1,C^r,C^∞-functions). *The following definitions are made in* RCA_0.

1. *Let U be an open subset of $\mathbb{R}$, and let f , f' be continuous functions from U to $\mathbb{R}$. Then a pair (f, f') is said to be C^1 if and only if*

$$\forall x \in U \ \lim_{x' \to x} \frac{f(x') - f(x)}{x' - x} = f'(x).$$

2. *Let U be an open subset of $\mathbb{R}$, and let $\{f^{(n)}\}_{n \le r}$ be a finite sequence of continuous functions from U to $\mathbb{R}$. Then $\{f^{(n)}\}_{n \le r}$ is said to be C^r if and only if for all n less than r, $(f^{(n)}, f^{(n+1)})$ is C^1.*

3. *Let U be an open subset of $\mathbb{R}$, and let $\{f^{(n)}\}_{n \in \mathbb{N}}$ be an infinite sequence of continuous functions from U to $\mathbb{R}$. Then $\{f^{(n)}\}_{n \in \mathbb{N}}$ is said to be C^∞ if and only if for all $r \in \mathbb{N}$, $\{f^{(n)}\}_{n \le r}$ is C^r.*

We usually write f_0 as f when $\{f^{(n)}\}_{n \le r}$ is C^r or $\{f^{(n)}\}_{n \in \mathbb{N}}$ is C^∞, and if (f, f') is C^1, $\{f^{(n)}\}_{n \le r}$ is C^r or $\{f^{(n)}\}_{n \in \mathbb{N}}$ is C^∞, f is said to be C^1, C^r or C^∞.

The next lemma shows that the uniqueness of the derivative is provable in RCA_0.

Lemma 3.6. *The following is provable in* RCA_0. *Let U be an open subset of $\mathbb{R}$, and let f, g be C^r or C^∞-functions from U to $\mathbb{R}$. If $\forall x \in U \ f(x) = g(x)$, then for all $k \le r$ or $k \in \mathbb{N}$ $\forall x \in U \ f^{(k)}(x) = g^{(k)}(x)$.*

Proof. Immediate from Π^0_1-induction. □

To develop differential calculus, we have to begin with the mean value theorem. Fortunately, the mean value theorem for C^1-functions is easily provable in RCA_0 using the intermediate value theorem([8] Theorem II.6.6).

Lemma 3.7. *The following is provable in* RCA_0. *Let* U *be an open subset of* $\mathbb{R}$, *and let* f *be a* C^1*-function from* U *to* $\mathbb{R}$. *Let* K *be a positive real number. If* $[a,b] \subseteq U$ *and for all* $x \in [a,b]$ $|f'(x)| \leq K$, *then*

$$\left|\frac{f(b)-f(a)}{b-a}\right| \leq K.$$

Theorem 3.8 (mean value theorem). *The following is provable in* RCA_0. *Let* $[a,b]$ *be an interval of* $\mathbb{R}$ *and let* f *be a continuous function from* $[a,b]$ *to* $\mathbb{R}$. *If* f *is* C^1 *on* (a,b), *i.e. there exists a continuous function from* (a,b) *to* $\mathbb{R}$ *such that* (f,f') *is* C^1, *then there exists* $c \in (a,b)$ *such that*

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Proof. The proof is an easy direction from Lemma 3.7 and the intermediate value theorem.

□

Remark 3.9. *We can prove stronger version of Theorem 3.8. In fact, mean value theorem for a differentiable function (a continuous function which is differentiable at each point) can be proved in* RCA_0. *See Hardin and Velleman [5].*

Next, we define C^r and C^∞-function in $\mathbb{R}^n$.

Definition 3.5 (C^r and C^∞-functions from $U \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$). *The following definitions are made in* RCA_0. *Let* U *be an open subset of* $\mathbb{R}^n$. *The notation* $\alpha = (a_1, \dots, a_n) \in \mathbb{N}^n$ *is a multi-index and* $|\alpha| = a_1 + \cdots + a_n$.

1. *A* C^r*-function from* U *to* $\mathbb{R}$ *is a finite sequence of continuous functions* $\{f_\alpha\}_{|\alpha| \leq r}$ *from* U *to* $\mathbb{R}$ *which satisfies the following: for all* $\alpha = (a_1, \dots, a_n)$ *such that* $|\alpha| \leq r-1$, $(f_{(a_1,\dots,a_i,\dots,a_n)}, f_{(a_1,\dots,a_i+1,\dots,a_n)})$ *is* C^1 *as a function of* x_i, *i.e.,*

$$\forall \mathbf{x} \in U \ f_{(a_1,\dots,a_i+1,\dots,a_n)}(\mathbf{x}) = \lim_{t\to 0} \frac{f_{(a_1,\dots,a_i,\dots,a_n)}(\mathbf{x}+t\mathbf{e}_i) - f_{(a_1,\dots,a_i,\dots,a_n)}(\mathbf{x})}{t}$$

   where $\mathbf{e}_i$ *is the unit vector along* x_i.

2. *A* C^∞*-function from* U *to* $\mathbb{R}$ *is an infinite sequence of continuous functions* $\{f_\alpha\}_{\alpha\in\mathbb{N}^n}$ *from* U *to* $\mathbb{R}$ *such that for all* $r \in \mathbb{N}$, $\{f_\alpha\}_{|\alpha|\leq r}$ *is a* C^r*-function.*

3. *A* C^r *or* C^∞*-function from* U *to* $\mathbb{R}^m$ *is a finite sequence of* C^r *or* C^∞ *functions* $\mathbf{f} = (f_1, \dots, f_m)$ *from* U *to* $\mathbb{R}$.

If $\{f_\alpha\}_{|\alpha|\leq r}$ *is* C^r *or* $\{f_\alpha\}_{|\alpha|\in\mathbb{N}^n}$ *is* C^∞, *then* f *is said to be* C^r *or* C^∞. *As usual, we write*

$$f_{(a_1,\dots,a_n)} = \frac{\partial^{a_1+\cdots+a_n} f}{\partial^{a_1}x_1 \dots \partial^{a_n}x_n}.$$

Theorem 3.10. *The following is provable in* RCA_0. *Let* U *be an open subset of* $\mathbb{R}^n$, *and let* f *be a* C^1*-function from* U *to* $\mathbb{R}$. *If its derivatives* f_{x_i} *and* f_{x_j} *are also* C^1, *i.e., there exist finite sequences* $\{(f_{x_i})_\alpha\}_{|\alpha|\le 1}$ *and* $\{(f_{x_j})_\alpha\}_{|\alpha|\le 1}$ *which satisfy the condition for* C^1, *then*

$$\frac{\partial f_{x_i}}{\partial x_j} = \frac{\partial f_{x_j}}{\partial x_i}.$$

Proof. Straightforward imitation of the usual proof. □

To prove basic properties of C^1-functions in RCA_0, we construct following differentiable condition functions. A differentiable condition function for a C^1-function f expresses the condition of differentiability at each point of $\mathrm{dom}(f)$. It also expresses the continuity of the derivative f'. Hence using a differentiable condition function, we can easily prove basic properties of C^1-functions in RCA_0.

Theorem 3.11. *The following is provable in* RCA_0. *Let* U *be an open subset of* $\mathbb{R}^n$, *and let* f *be a* C^1*-function from* U *to* $\mathbb{R}$. *Then there exists a continuous function* e_f *from* $U \times U$ *to* $\mathbb{R}$ *such that*

(1) $$\forall \mathbf{x} \in U \ e_f(\mathbf{x},\mathbf{x}) = 0;$$

(2) $$\forall \mathbf{x},\mathbf{y} \in U \ f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^{n} f_{x_i}(\mathbf{x})(y_i - x_i) + e_f(\mathbf{x},\mathbf{y})\|\mathbf{y}-\mathbf{x}\|.$$

(Here, $f_{x_i} = \frac{\partial f}{\partial x_i}$.*) Moreover, we can find a code for* e_f *effectively. We call this* e_f *differentiable condition function for* f.

Remark 3.12. *Theorem 3.11 is not trivial. Actually, for 3.11.??, we want to define* e_f *as*

(3) $$e_f(x,y) = \begin{cases} \frac{f(y)-f(x)}{y-x} - f'(x) & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

and of course this e_f *is a continuous function in the usual sense. However, Theorem 3.4 points out that* RCA_0 *cannot guarantee the existence of a code for a continuous function which is defined like as above, hence it is not easy to construct (a code for)* e_f.

Proof of Theorem 3.11. We reason within RCA_0. Define a (code for a) closed set $\Delta \subseteq \mathbb{R}^{2n}$ as $\Delta = \{(\mathbf{x},\mathbf{x}) \mid \mathbf{x} \in U\}$. By Theorem 3.2, we can construct a continuous function g from U to $\mathbb{R}$ and a continuous function e_f^0 from $U \times U \setminus \Delta$ to $\mathbb{R}$ such that

$$\begin{aligned} g(\mathbf{x}) &= \sum_{i=1}^{n} |f_{x_i}(\mathbf{x})|; \\ e_f^0(\mathbf{x},\mathbf{y}) &= \frac{f(\mathbf{y}) - f(\mathbf{x}) - \sum_{i=1}^{n} f_{x_i}(\mathbf{x})(y_i - x_i)}{\|\mathbf{y}-\mathbf{x}\|}. \end{aligned}$$

Let E_f^0 be a code for e_f^0, and let G be a code for g. Let $\varphi(\mathbf{a}, r, b, s)$ be a Σ_1^0 formula which expresses the following (i) or (ii) holds:

(i) $(\mathbf{a}, r)E_f^0(b, s)$;

(ii) $b = 0$ and there exists $(m_0, \mathbf{a}_0, r_0, b_0, s_0) \in G$ such that $\|\mathbf{a} - (\mathbf{a}_0, \mathbf{a}_0)\| + r < r_0$ and $s > 2ns_0$.

Write

$$\varphi(\mathbf{a}, r, b, s) \equiv \exists m \theta(m, \mathbf{a}, r, b, s)$$

where θ is Σ_0^0. By Δ_1^0 comprehension, define E_f as $(m, \mathbf{a}, r, b, s) \in E_f \leftrightarrow \theta(m, \mathbf{a}, r, b, s)$, *i.e.*, $(\mathbf{a}, r)E_f(b, s)$ holds if and only if (i) or (ii) holds. Then E_f is a code for a continuous (partial) function. To show this, we have to check the conditions of a code for a continuous function. It is clear that E_f satisfies conditions 2 and 3 of definition 3.1. We must check condition 1. Assume $(\mathbf{a}, r)E_f(b, s)$ and $(\mathbf{a}, r)E_f(b', s')$. If $(\mathbf{a}, r, b, s)$ and $(\mathbf{a}, r, b', s')$ satisfy (i), then clearly condition 1 holds. If $(\mathbf{a}, r, b, s)$ and $(\mathbf{a}, r, b', s')$ satisfy (ii), then we can show condition 1 holds easily by G holding condition 1. Now we consider the case $(\mathbf{a}, r, b, s)$ satisfies (i) and $(\mathbf{a}, r, b', s')$ satisfies (ii). By condition 2, it is sufficient that we only check the case $\{(\mathbf{x}', \mathbf{y}') \mid \|(\mathbf{x}', \mathbf{y}') - \mathbf{a}\| < r\} \subseteq U \times U \setminus \Delta$ holds. Let $(m_0, \mathbf{a}_0, r_0, b_0, s_0)$ be an element of G such that $\|\mathbf{a} - (\mathbf{a}_0, \mathbf{a}_0)\| + r < r_0$ and $s' > 2ns_0$. Here, $(m_0, \mathbf{a}_0, r_0, b_0, s_0) \in G$ implies

$$\forall \mathbf{z} \in U \ \|\mathbf{z} - \mathbf{a}_0\| < r_0 \to \left| \sum_{i=1}^{n} |f_{x_i}(\mathbf{x})| - b_0 \right| \leq s_0. \tag{4}$$

Write

$$\begin{aligned} \mathbf{a} &= (\mathbf{a}^x, \mathbf{a}^y)(\in \mathbb{R}^n \times \mathbb{R}^n); \\ \mathbf{a}^x &= (a_1^x, \dots, a_n^x); \\ \mathbf{a}^y &= (a_1^y, \dots, a_n^y); \\ \mathbf{z}_i &= (a_1^y, \dots, a_i^y, a_{i+1}^x, \dots, a_n^x). \end{aligned}$$

Here $\mathbf{a}^x \neq \mathbf{a}^y$, $\mathbf{z}_0 = \mathbf{a}^x$, $\mathbf{z}_n = \mathbf{a}^y$ and each $\mathbf{z}_i$ satisfies $\|\mathbf{z}_i - \mathbf{a}_0\| < r_0$. Then,

$$|e_f^0(\mathbf{a}) - b| \leq s; \tag{5}$$

$$\begin{aligned} |e_f^0(\mathbf{a}) - b'| &= |e_f^0((\mathbf{a}^x, \mathbf{a}^y))| \\ &\leq \sum_{i=1}^{n} \frac{|f(\mathbf{z}_i) - f(\mathbf{z}_{i-1}) - f_{x_i}(\mathbf{a}^x)(a_i^y - a_i^x)|}{\|\mathbf{a}^y - \mathbf{a}^x\|}. \end{aligned} \tag{6}$$

On the other hand, using Theorem 3.8, for all $1 \le i \le n$, if $a_i^x \ne a_i^y$, there exists $0 < \theta < 1$ such that

$$\frac{f(\mathbf{z}_i) - f(\mathbf{z}_{i-1})}{a_i^y - a_i^x} = f_{x_i}(\mathbf{z}_{i-1} + \theta(\mathbf{z}_i - \mathbf{z}_{i-1})).$$

(Here, $\|(\mathbf{z}_{i-1} + \theta(\mathbf{z}_i - \mathbf{z}_{i-1})) - \mathbf{a}_0\| < r_0$.) Then,

$$(7) \quad \begin{aligned} & \frac{|f(\mathbf{z}_i) - f(\mathbf{z}_{i-1}) - f_{x_i}(\mathbf{a}^x)(a_i^y - a_i^x)|}{\|\mathbf{a}^y - \mathbf{a}^x\|} \\ \le\ & \left|\frac{f(\mathbf{z}_i) - f(\mathbf{z}_{i-1})}{a_i^y - a_i^x} - f_{x_i}(\mathbf{a}^x)\right| \\ =\ & |f_{x_i}(\mathbf{z}_{i-1} + \theta(\mathbf{z}_i - \mathbf{z}_{i-1})) - f_{x_i}(\mathbf{a}^x)| \\ \le\ & |f_{x_i}(\mathbf{z}_{i-1} + \theta(\mathbf{z}_i - \mathbf{z}_{i-1})) - b_0| + |f_{x_i}(\mathbf{a}^x) - b_0|. \end{aligned}$$

Hence by (4) and (7), for all $1 \le i \le n$,

$$(8) \quad \frac{|f(\mathbf{z}_i) - f(\mathbf{z}_{i-1}) - f_{x_i}(\mathbf{a}^x)(a_i^y - a_i^x)|}{\|\mathbf{a}^y - \mathbf{a}^x\|} \le 2s_0.$$

(If $a_i^x = a_i^y$, then clearly (8) holds.) From (6) and (8),

$$(9) \quad \begin{aligned} |e_f^0(\mathbf{a}) - b'| &\le \sum_{i=1}^{n} \frac{|f(\mathbf{z}_i) - f(\mathbf{z}_{i-1}) - f_{x_i}(\mathbf{a}^x)(a_i^y - a_i^x)|}{\|\mathbf{a}^y - \mathbf{a}^x\|} \\ &\le \sum_{i=1}^{n} 2s_0 \\ &\le s'. \end{aligned}$$

By (5) and (9), $|b - b'| \le s + s'$ holds. This means E_f satisfies condition 1.

Let e_f be a continuous function which is coded by E_f. Then, (i) provides $U \times U \setminus \Delta \subseteq \mathrm{dom}(e_f)$ and (ii) provides $\Delta \subseteq \mathrm{dom}(e_f)$, hence $U \times U \subseteq \mathrm{dom}(e_f)$. Clearly e_f holds (1) and (2), and this completes the proof. □

Remark 3.13. *If U is an open subset of $\mathbb{R}^n$ and $\mathbf{f} = (f_1, \dots, f_m)$ is a C^1-function from U to $\mathbb{R}^m$, then we define the differentiable condition function for f as $\mathbf{e_f} = (e_{f_1}, \dots, e_{f_m})$. Then*

$$\forall \mathbf{x} \in U\ \mathbf{e_f}(\mathbf{x}, \mathbf{x}) = \mathbf{0};$$
$$\forall \mathbf{x}, \mathbf{y} \in U\ \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{f}_{x_i}(\mathbf{x})(y_i - x_i) + \mathbf{e_f}(\mathbf{x}, \mathbf{y})\|\mathbf{y} - \mathbf{x}\|.$$

(Here, $\mathbf{f}_{x_i} = (f_{1\,x_i}, \dots f_{m\,x_i})$.)

Remark 3.14. *Conversely, let U be an open subset of $\mathbb{R}^n$, f, f' be continuous function from U to $\mathbb{R}$ and e_f be a continuous function from $U \times U$ to $\mathbb{R}$. If f, f', e_f satisfy (1) and (2), then clearly (f, f') is C^1.*

Corollary 3.15. *The following assertions are provable in* RCA_0.

1. *Let U be an open subset of $\mathbb{R}$ and let k be a real number. If f and g are C^r or C^∞ functions from U to $\mathbb{R}$, then $kf, f+g, fg, 1/f$ are all C^r or C^∞ functions from U to $\mathbb{R}$. Moreover, $(kf)' = kf'$, $(f+g)' = f'+g'$, $(fg)' = f'g+fg'$ and $(1/f)' = -f'/(f^2)$ hold.*

2. *(chain rule) Let U be an open subset of $\mathbb{R}^n$ and let V be an open subset of $\mathbb{R}^m$. If $\mathbf{f} = (f_1, \dots, f_m)$ is a continuous function from U to V, g is a continuous function from V to $\mathbb{R}$ and both f and g are C^r or C^∞, then $g \circ f$ is a C^r or C^∞ function from U to $\mathbb{R}$ and satisfies*

$$\frac{\partial(g \circ \mathbf{f})}{\partial x_i}(\mathbf{x}) = \sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(\mathbf{f}(\mathbf{x}))\frac{\partial f_j}{\partial x_i}(\mathbf{x}).$$

(Here $\frac{\partial g}{\partial y_j} = g_{(\delta_{1j}, \dots, \delta_{mj})}$.)

Proof. We reason within RCA_0. We only prove 2. (We can prove 1 easily.) For all $\mathbf{x} \in U$, $1 \leq i \leq n$ and $\Delta x \in \mathbb{R} \setminus \{0\}$, define Δy_j $(1 \leq j \leq m)$ as

$$\begin{aligned} \Delta y_j &:= f_j(\mathbf{x} + \Delta x \mathbf{e}_i) - f_j(\mathbf{x}) \\ &= \Delta x \frac{\partial f_j}{\partial x_i}(\mathbf{x}) + |\Delta x| e_{f_j}(\mathbf{x}, \mathbf{x} + \Delta x \mathbf{e}_i). \end{aligned}$$

where $\mathbf{e}_i$ is the unit vector along x_i and each e_{f_j} is the differentiable condition function for f_j. Then

$$\begin{aligned} \|\Delta \mathbf{y}\| &:= \sqrt{\sum_{j=1}^{m} (\Delta y_j)^2} \\ &= |\Delta x| \sqrt{\sum_{j=1}^{m} \left(\frac{\Delta x}{|\Delta x|} \frac{\partial f_j}{\partial x_i}(\mathbf{x}) + e_{f_j}(\mathbf{x}, \mathbf{x} + \Delta x \mathbf{e}_i) \right)^2}. \end{aligned}$$

Define $e^i_{g \circ \mathbf{f}}$ as

$$\begin{aligned} e^i_{g\circ \mathbf{f}}(\Delta x) &:= \sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(\mathbf{f}(\mathbf{x})) e_{f_j}(\mathbf{x}, \mathbf{x} + \Delta x \mathbf{e}_i) \\ &\quad + e_g(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x} + \Delta x \mathbf{e}_i)) \sqrt{\sum_{j=1}^{m} \left(\frac{\Delta x}{|\Delta x|} \frac{\partial f_j}{\partial x_i}(\mathbf{x}) + e_{f_j}(\mathbf{x}, \mathbf{x} + \Delta x \mathbf{e}_i) \right)^2} \end{aligned}$$

where e_g is the differentiable condition function for g. Then

$$\lim_{\Delta x \to 0} e^i_{g \circ \mathbf{f}}(\Delta x) = 0, \tag{10}$$

$$(11) \quad g \circ \mathbf{f}(\mathbf{x}+\Delta x\mathbf{e}_i) - g \circ \mathbf{f}(\mathbf{x}) = \sum_{j=1}^{m} \Delta y_j \frac{\partial g}{\partial y_j}(\mathbf{f}(\mathbf{x})) + \|\Delta \mathbf{y}\| e_g(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x}+\Delta x\mathbf{e}_i))$$
$$= \Delta x \sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(\mathbf{f}(\mathbf{x})) \frac{\partial f_j}{\partial x_i}(\mathbf{x}) + |\Delta x| e^i_{g\circ\mathbf{f}}(\Delta x).$$

(10) and (11) show that $\sum_{j=1}^{m} \frac{\partial g}{\partial y_j}\frac{\partial f_j}{\partial x_i}$ is the first derivative of $g \circ \mathbf{f}$ along x_i, and this completes the proof. □

3.3 Series

In this section, we prove the termwise differentiation and integration theorems. We also construct some C^r or C^∞-functions by series in RCA_0. Especially, we construct power series, which are elementary examples of analytic functions. The next theorem is the core of this section.

Theorem 3.16 ([8] Theorem II.6.5). *Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a (code for a) sequence of non-negative real numbers whose series $\sum_{n=0}^{\infty} \alpha_n$ is convergent. Then the following is provable in RCA_0. Let U be an open subset of $\mathbb{R}^l$, and let $\{f_n\}_{n\in\mathbb{N}}$ be a (code for a) sequence of continuous functions from U to $\mathbb{R}$ which satisfies the following:*

$$\forall \mathbf{x} \in U \ \forall n \in \mathbb{N} \ |f_n(\mathbf{x})| \leq \alpha_n.$$

Then there exists a (code for a) continuous function f from U to $\mathbb{R}$ such that

$$\forall \mathbf{x} \in U \ f(\mathbf{x}) = \sum_{n=0}^{\infty} f_n(\mathbf{x}).$$

We prove the termwise differentiation theorem, and construct a power series, an elementary example of analytic functions.

Theorem 3.17 (termwise differentiation). *The following is provable in RCA_0. Let U be an open interval of $\mathbb{R}$, and let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be nonnegative convergent series. Let $\{(f_n, f'_n)\}_{n\in\mathbb{N}}$ be a sequence of C^1-functions from U to $\mathbb{R}$ which satisfies the following conditions:*

$$\forall x \in U \ \forall n \in \mathbb{N} \ |f_n(x)| \leq a_n,$$
$$\forall x \in U \ \forall n \in \mathbb{N} \ |f'_n(x)| \leq b_n.$$

Then there exists a C^1-function (f, f') from U to $\mathbb{R}$ such that

$$f = \sum_{n=0}^{\infty} f_n, \ f' = \sum_{n=0}^{\infty} f'_n.$$

Proof. We reason within RCA_0. By Theorem 3.16.??, there exist continuous functions f and f' from U to $\mathbb{R}$ which satisfy the following condition:

$$f = \sum_{n=0}^{\infty} f_n,\ f' = \sum_{n=0}^{\infty} f'_n.$$

Let e_{f_n} be a differentiable condition function for (f_n, f'_n). By Theorem 3.8, for all n and for all $x \neq y$ in U, there exists $z \in U$ such that

$$\frac{f_n(y) - f_n(x)}{y - x} = f'_n(z).$$

Hence, for all $n \in \mathbb{N}$, if $x \neq y$, then there exists z and

$$\begin{aligned} |e_{f_n}(x,y)| &= \left| \frac{f_n(y) - f_n(x)}{y - x} - f'_n(x) \right| \\ &= |f'_n(z) - f'_n(x)|. \end{aligned}$$

Then for all $n \in \mathbb{N}$,

$$|e_{f_n}(x,y)| \leq 2b_n. \tag{12}$$

(Clearly, (12) holds if $x = y$.) Then by Theorem 3.16.??, $e_f = \sum_{n=0}^{\infty} e_{f_n}$ exists and e_f holds

$$\begin{aligned} &\forall x \in U\ e_f(x,x) = 0; \\ &\forall x, y \in U\ f(y) - f(x) = (y - x)f'(x) + |y - x| e_f(x,y). \end{aligned}$$

This means (f, f') is C^1 and this completes the proof. □

Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers, and let r be a positive real number. If the series $\sum_{n=0}^{\infty} |a_n| r^n$ is convergent, then for all $a \in \mathbb{R}$ and for all x such that $|x - a| < r$, $\sum_{n=0}^{\infty} a_n (x-a)^n$ is absolutely convergent and $|a_n(x-a)^n| < |a_n| r^n$. Define an open set U and a sequence of continuous functions $\{f_n\}_{n \in \mathbb{N}}$ from U to $\mathbb{R}$ as $U = \{x \mid |x - a| < r\}$ and $f_n(x) = a_n(x-a)^n$. Then by Theorem 3.16.?? there exists a continuous function f from U to $\mathbb{R}$ such that

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} f_n(x) \\ &= \sum_{n=0}^{\infty} a_n (x-a)^n. \end{aligned}$$

Definition 3.6 (analytic functions). *The following definition is made in* RCA_0. *Let* U *be an open subset of* $\mathbb{R}$, *and let* $\{f^{(n)}\}_{n\in\mathbb{N}}$ *be a* C^∞*-function from* U *to* $\mathbb{R}$. *Then* $\{f^{(n)}\}_{n\in\mathbb{N}}$ *is said to be analytic if and only if* $\{f^{(n)}\}_{n\in\mathbb{N}}$ *satisfies the following condition:*

$$\forall x \in U\ \exists \delta > 0\ \forall y \in U\ |x-y| < \delta \to f(y) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(y-x)^n.$$

If $\{f^{(n)}\}_{n\in\mathbb{N}}$ *is analytic, then* f *is said to be analytic.*

Theorem 3.18. *The following is provable in* RCA_0. *Let* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence of real numbers, and let* r *be a positive real number such that* $\sum_{n=0}^{\infty}|a_n|r^n$ *is convergent. Define an open set* U *as* $U = \{x \mid |x-a| < r\}$ *and define a continuous function* f *from* U *to* $\mathbb{R}$ *as* $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$. *Then we can construct a sequence of continuous functions* $\{f^{(n)}\}_{n\in\mathbb{N}}$ *to expand* f *into an analytic function* $\{f^{(n)}\}_{n\in\mathbb{N}}$.

Proof. Obvious. □

The next lemma is very useful to construct continuous, C^r, C^∞ or analytic functions.

Lemma 3.19. *The following is provable in* RCA_0. *Let* $\{U_n\}_{n\in\mathbb{N}}$ *be a (code for a) sequence of open subsets of* $\mathbb{R}^l$, *and let* $\{f_n\}_{n\in\mathbb{N}}$ *be a (code for a) sequence of continuous,* C^r *or* C^∞*-functions. Here, each* f_n *is from* U_n *to* $\mathbb{R}$. *If* $\{f_n\}_{n\in\mathbb{N}}$ *satisfies*

$$\forall \mathbf{x} \in \mathbb{R}^l\ \forall i,j \in \mathbb{N}\ (\mathbf{x} \in U_i \cap U_j \to f_i(\mathbf{x}) = f_j(\mathbf{x})),$$

then there exists a continuous, C^r *or* C^∞*-function* f *from* $U = \bigcup_{n=0}^{\infty} U_n$ *to* $\mathbb{R}$ *such that*

$$\forall \mathbf{x} \in U\ \forall n \in \mathbb{N}\ (\mathbf{x} \in U_n \to f_n(\mathbf{x}) = f(\mathbf{x})).$$

(We usually write $f = \bigcup_{n=0}^{\infty} f_n$.*) Moreover, if* $l = 1$ *and each* f_n *is analytic, then* f *is analytic.*

Proof. We reason within RCA_0. We first prove the continuous case. Let F_n be a code for f_n. Let $\varphi(\mathbf{a}, r, b, s)$ be a Σ^0_1 formula which express there exists n such that $\exists(m', \mathbf{a}', r') \in U_n\ \|\mathbf{a}-\mathbf{a}'\| + r < r'$ and $(\mathbf{a}, r)F_n(b, s)$ holds. Write

$$\varphi(\mathbf{a}, r, b, s) \equiv \exists m \theta(m, \mathbf{a}, r, b, s)$$

where θ is Σ^0_0. By Δ^0_1 comprehension, define F as $(m, \mathbf{a}, r, b, s) \in F \leftrightarrow \theta(m, \mathbf{a}, r, b, s)$. Then clearly F is a code for a continuous (partial) function and f is from U to $\mathbb{R}$ which satisfies

$$\forall \mathbf{x} \in U\ \forall n \in \mathbb{N}\ (\mathbf{x} \in U_n \to f_n(\mathbf{x}) = f(\mathbf{x})).$$

This completes the proof of the continuous case.

To prove the C^r or C^∞ case, by Lemma 3.6, for all $\alpha = (a_1, \dots, a_n)$,

$$\forall \mathbf{x} \in \mathbb{R}^l \ \forall i, j \in \mathbb{N} \left(\mathbf{x} \in U_i \cap U_j \to \frac{\partial^{a_1+\cdots+a_n} f_i}{\partial^{a_1} x_1 \dots \partial^{a_n} x_n}(\mathbf{x}) = \frac{\partial^{a_1+\cdots+a_n} f_j}{\partial^{a_1} x_1 \dots \partial^{a_n} x_n}(\mathbf{x})\right).$$

Then we can use the continuous case to construct

$$\frac{\partial^{a_1+\cdots+a_n} f}{\partial^{a_1} x_1 \dots \partial^{a_n} x_n} = \bigcup_{n=0}^{\infty} \frac{\partial^{a_1+\cdots+a_n} f_n}{\partial^{a_1} x_1 \dots \partial^{a_n} x_n}.$$

We can easily check the condition for C^r or C^∞.

For the analytic case, we can also check the condition for analytic easily, and this completes the proof. □

Example 3.7. *The following analytic functions can be constructed in* RCA_0.

1. *Define* $s(n)$ *as*

$$s(n) = \begin{cases} (-1)^{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

and define $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ *and* $\{c_n\}_{n\in\mathbb{N}}$ *as*

$$a_n = \frac{1}{n!},\ b_n = \frac{s(n+3)}{n!},\ c_n = \frac{s(n)}{n!}.$$

Then for all $m \in \mathbb{N}$, $\sum_{n=0}^{\infty} |a_n| m^n$, $\sum_{n=0}^{\infty} |b_n| m^n$ *and* $\sum_{n=0}^{\infty} |c_n| m^n$ *are convergent. Define* $U_m = \{x \mid |x| < m\}$. *On* U_m, *define* $\exp_m(x) = \sum_{n=0}^{\infty} a_n x^n$, $\sin_m(x) = \sum_{n=0}^{\infty} b_n x^n$ *and* $\cos_m(x) = \sum_{n=0}^{\infty} c_n x^n$. *Then by Corollary 3.18,* $\exp_m(x)$, $\sin_m(x)$ *and* $\cos_m(x)$ *are analytic functions from* U_m *to* $\mathbb{R}$. *Hence by Lemma 3.19, analytic functions* $\exp = \bigcup_{m\in\mathbb{N}} \exp_m$, $\sin = \bigcup_{m\in\mathbb{N}} \sin_m$ *and* $\cos = \bigcup_{m\in\mathbb{N}} \cos_m$ *from* $\mathbb{R}$ *to* $\mathbb{R}$ *can be constructed.*

2. *Define* $\{d_n\}_{n\in\mathbb{N}}$ *as* $d_n = n \cdot (-1)^{n+1}$ *and define* $t(m)$ *as* $t(m) = 1 - 1/m$. *Then for all* $m \in \mathbb{N}$, $\sum_{n=0}^{\infty} |d_n| t(m)^n$ *is convergent. Define* $U_m = \{x \mid |x-1| < t(m)\}$. *On* U_m, *define* $\log_m(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$. *Then by Corollary 3.18,* $\log_m(x)$ *is an analytic function from* U_m *to* $\mathbb{R}$. *Hence by Lemma 3.19, an analytic function* $\log = \bigcup_{m\in\mathbb{N}} \log_m$ *from* $(0,2)$ *to* $\mathbb{R}$ *can be constructed.*

Next, we define Riemann integral and prove the termwise integration theorem. A modulus of uniform continuity plays a key role to integrate a continuous function.

Definition 3.8 (modulus of uniform continuity). *The following definition is made in* RCA_0. *Let* U *be an open or closed subset of* $\mathbb{R}^n$, *and let* $\mathbf{f}$ *be a continuous function from* U *to* $\mathbb{R}^m$. *A modulus of uniform continuity on* U *for* $\mathbf{f}$ *is a function* h *from* $\mathbb{N}$ *to* $\mathbb{N}$ *such that for all* $n \in \mathbb{N}$ *and for all* $\mathbf{x}, \mathbf{y} \in U$, *if* $\|\mathbf{x} - \mathbf{y}\| < 2^{-h(n)}$, *then* $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < 2^{-n}$.

A modulus of uniform continuity for f guarantees rather strong uniform continuity of f than usual sense.

Definition 3.9 (Riemann integral: [8] Lemma IV.2.6). *The following definition is made in* RCA_0. *Let* f *be a continuous function from* $[a, b]$ *to* $\mathbb{R}$. *Then, define the Riemann integral* $\int_a^b f(x)\,dx$ *as*

$$\int_a^b f(x)\,dx = \lim_{|\Delta| \to 0} \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})$$

if this limit exists. Here, Δ *is a partition of* $[a, b]$, *i.e.* $\Delta = \{a = x_0 < x_1 < \cdots < x_n = b\}$, $x_{k-1} \le \xi_k \le x_k$ *and* $|\Delta| = \max\{x_k - x_{k-1} \mid 1 \le k \le n\}$.

Lemma 3.20. *The following is provable in* RCA_0. *Let* f *be a continuous function from* $[a, b]$ *to* $\mathbb{R}$ *which has a modulus of uniform continuity. Then* $\int_a^b f(x)\,dx$ *exists.*

Proof. Obvious. □

Theorem 3.21 (termwise integration). *The following is provable in* RCA_0. *Let* $\sum_{n=0}^{\infty} \alpha_n$ *be nonnegative convergent series, and let* $\{f_n\}_{n \in \mathbb{N}}$ *be a sequence of continuous functions from* $[a, b]$ *to* $\mathbb{R}$ *which satisfies the following:*

$$\forall x \in [a, b]\ \forall n \in \mathbb{N}\ |f_n(x)| \le \alpha_n.$$

Then by Theorem 3.16.??, there exists a continuous function $f = \sum_{n=0}^{\infty} f_n$ *from* $[a, b]$ *to* $\mathbb{R}$.

If each f_n *has a modulus of uniform continuity, then* $f = \sum_{n=0}^{\infty} f_n$ *has a modulus of uniform continuity and* $\{f_n\}_{n \in \mathbb{N}}$ *and* f *satisfy the following:*

$$\int_a^b f(x)\,dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)\,dx. \tag{13}$$

(By Lemma 3.20, $\int_a^b f(x)\,dx$ *and* $\int_a^b f_n(x)\,dx$ *exist.)*

Proof. We reason within RCA_0. Let h_n be a modulus of uniform continuity for f_n. Let $\sum_{n=0}^{\infty} \alpha_n = \alpha$. Define $k(n)$ as the following:

$$k(n) = \min\left\{ k \;\middle|\; \left(\alpha - \sum_{i=0}^{k} \alpha_i\right)_k < 2^{-n-2} - 2^{-k+1} \right\}.$$

(Here, $(\alpha)_k$ is the k-th approximation of α.) Then

$$\sum_{i=k(n)+1}^{\infty} \alpha_i < 2^{-n-2}. \tag{14}$$

Now define h as

$$h(n) = \max\{h_i(n+2+i) \mid i \leq k(n)\}.$$

Then for all $x, y \in [a, b]$, $|x - y| < 2^{-h(n)}$ implies

$$\forall i \leq k(n)\ |f(x) - f(y)| < 2^{-n-2-i}. \tag{15}$$

Hence by (14) and (15), for all $n \in \mathbb{N}$, if $|x - y| < 2^{-h(n)}$, then

$$\begin{aligned}
|f(x) - f(y)| &\leq \sum_{i=k(n)+1}^{\infty} (|f_i(x)| + |f_i(y)|) + \sum_{i=0}^{k(n)} |f(x) - f(y)| \\
&\leq \sum_{i=k(n)+1}^{\infty} 2a_i + \sum_{i=0}^{k(n)} 2^{-n-2-i} \\
&< 2 \cdot 2^{-n-2} + 2^{-n-1} \\
&= 2^{-n}.
\end{aligned}$$

This means h is a modulus of uniform continuity for f.

To prove (13), for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\left| \int_a^b f(x)\,dx - \sum_{i=0}^{k(n)} \int_a^b f_i(x)\,dx \right| \\
= &\left| \int_a^b (f(x) - \sum_{i=0}^{k(n)} f_i(x))\,dx \right| \\
= &\left| \int_a^b \sum_{i=k(n)+1}^{\infty} f_i(x)\,dx \right| \\
\leq &\int_a^b \sum_{i=k(n)+1}^{\infty} \alpha_i(x)\,dx \\
\leq &|b - a| 2^{-n-2}.
\end{aligned}$$

This implies (13), and this completes the proof. □

3.4 Inverse function theorem and implicit function theorem

In this section, we prove the inverse function theorem and the implicit function theorem in RCA_0. Differentiable condition functions again play a key role.

Theorem 3.22 (inverse function theorem and implicit function theorem). *The following assertions are provable in* RCA_0.

1. *Let U be an open subset of $\mathbb{R}^n$, and let $\mathbf{f}$ be a C^r $(r \geq 1)$ or C^∞-function from U to $\mathbb{R}^n$. Let $\mathbf{a}$ be a point of U such that $|\mathbf{f}'(\mathbf{a})| \neq 0$. Then, there exist open subsets of $\mathbb{R}^n$ V, W and a C^r or C^∞-function $\mathbf{g}$ from W to V such that $\mathbf{a} \in V$, $\mathbf{f}(\mathbf{a}) \in W$ and*

$$\forall \mathbf{x} \in V \quad \mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x},$$
$$\forall \mathbf{y} \in W \quad \mathbf{f}(\mathbf{g}(\mathbf{y})) = \mathbf{y}.$$

2. *Let U be an open subset of $\mathbb{R}^n \times \mathbb{R}^m$, and let $\mathbf{F}$ be a C^r $(r \geq 1)$ or C^∞-function from U to $\mathbb{R}^m$. Let $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2)$ be a point of U such that $\mathbf{F}(\mathbf{a}) = \mathbf{0}$ and $|\mathbf{F}_{x_{n+1}\dots x_{n+m}}(\mathbf{a})| \neq 0$. Then there exist open subsets $V \subseteq \mathbb{R}^n$, $W \subseteq \mathbb{R}^m$ and a C^r or C^∞-function $\mathbf{f}$ from W to V such that $\mathbf{a}_1 \in V$, $\mathbf{a}_2 \in W$ and*

$$\mathbf{f}(\mathbf{a}_1) = \mathbf{a}_2,$$
$$\forall \mathbf{v} \in V \ \mathbf{F}(\mathbf{v}, \mathbf{f}(\mathbf{v})) = \mathbf{0}.$$

Here, $|\mathbf{f}'(\mathbf{a})|$ and $|\mathbf{F}_{x_{n+1}\dots x_{n+m}}(\mathbf{a})|$ are the Jacobians, i.e.,

$$\begin{aligned} |\mathbf{f}'(\mathbf{a})| &= \det\left(\frac{\partial f_i}{\partial x_j}\right)_{1 \leq i,j \leq n}, \\ |\mathbf{F}_{x_{n+1}\dots x_{n+m}}(\mathbf{a})| &= \det\left(\frac{\partial F_i}{\partial x_{n+j}}\right)_{1 \leq i,j \leq m}. \end{aligned}$$

Proof. We reason within RCA_0. We first prove 1. By Theorem 3.3 and Corollary 3.15, we may assume the following condition:

$$\mathbf{a} = \mathbf{f}(\mathbf{a}) = \mathbf{0};$$
$$\forall \mathbf{x} \in U \ |\mathbf{f}'(\mathbf{x})| > 0;$$
$$\frac{\partial f_i}{\partial x_j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Define $\mathbf{u}$ from U to $\mathbb{R}^n$ as $\mathbf{u}(\mathbf{x}) = \mathbf{x} - \mathbf{f}(\mathbf{x})$. Then $\mathbf{u}$ is C^1, hence we can construct the differentiable condition function $\mathbf{e}_\mathbf{u}$ for $\mathbf{u}$. Then for all $\mathbf{x}, \mathbf{y} \in U$,

$$\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{u}_{x_i}(\mathbf{x})(y_i - x_i) + \mathbf{e}_\mathbf{u}(\mathbf{x}, \mathbf{y})\|\mathbf{y} - \mathbf{x}\|.$$

Hence

$$\|\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})\| \le \left(\sum_{i=1}^{n}\|\mathbf{u}_{x_i}(\mathbf{x})\| + \|\mathbf{e}_{\mathbf{u}}(\mathbf{x},\mathbf{y})\|\right)\|\mathbf{y}-\mathbf{x}\|.$$

Here, $\sum_{i=1}^{n}\|\mathbf{u}_{x_i}(\mathbf{0})\| = 0$ and $\|\mathbf{e}_{\mathbf{u}}(\mathbf{0},\mathbf{0})\| = 0$. Hence by continuity of $\sum_{i=1}^{n}\|\mathbf{u}_{x_i}\|$ and $\|\mathbf{e}_{\mathbf{u}}\|$, we can get $\varepsilon > 0$ such that

$$\begin{aligned} W_0 := \{\mathbf{x}\in\mathbb{R}^n \mid \|\mathbf{x}-\mathbf{0}\| < \varepsilon\} &\subseteq U, \\ \forall\mathbf{x}\in W_0\ \sum_{i=1}^{n}\|\mathbf{u}_{x_i}(\mathbf{x})\| &< \frac{1}{4}, \\ \forall\mathbf{x},\mathbf{y}\in W_0\ \|\mathbf{e}_{\mathbf{u}x_i}(\mathbf{x},\mathbf{y})\| &< \frac{1}{4}. \end{aligned}$$

Then for all $\mathbf{x},\mathbf{x}\in W_0$,

$$\|\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})\| \le \frac{1}{2}\|\mathbf{y}-\mathbf{x}\|; \tag{16}$$

$$\begin{aligned} \|\mathbf{y}-\mathbf{x}\| &= \|\mathbf{u}(\mathbf{y})+\mathbf{f}(\mathbf{y})-\mathbf{u}(\mathbf{x})-\mathbf{f}(\mathbf{x})\| \\ &\le \|\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})\| + \|\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})\| \\ &\le \|\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})\| + \frac{1}{2}\|\mathbf{y}-\mathbf{x}\|. \end{aligned} \tag{17}$$

Hence

$$\|\mathbf{y}-\mathbf{x}\| \le 2\|\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})\|. \tag{18}$$

Define open sets V and W as

$$\begin{aligned} W &:= \left\{\mathbf{x}\in\mathbb{R}^n \,\middle|\, \|\mathbf{x}-\mathbf{0}\| < \frac{\varepsilon}{2}\right\}, \\ V &:= \mathbf{f}^{-1}(W)\cap W_0. \end{aligned}$$

Claim 3.22.1. *For all $\mathbf{y}\in W$, there exists a unique $\mathbf{x}\in V$ such that $\mathbf{f}(\mathbf{x})=\mathbf{y}$.*

To prove this claim, let $\mathbf{y}$ be a point of W. Define $\mathbf{v_y}$ from W_0 to $\mathbb{R}^n$ as $\mathbf{v_y}(\mathbf{x}) = \mathbf{y}+\mathbf{u}(\mathbf{x})$. Then by (16), for all $\mathbf{x}',\mathbf{x}''\in W_0$,

$$\|\mathbf{v_y}(\mathbf{x}'')-\mathbf{v_y}(\mathbf{x}')\| \le \frac{1}{2}\|\mathbf{x}''-\mathbf{x}'\|. \tag{19}$$

Especially,

$$\|\mathbf{v_y}(\mathbf{x}')-\mathbf{y}\| = \|\mathbf{v_y}(\mathbf{x}')-\mathbf{v_y}(\mathbf{0})\| \le \frac{1}{2}\|\mathbf{x}'\| < \frac{\varepsilon}{2}. \tag{20}$$

On the other hand, $\mathbf{y}\in W$ implies $\|\mathbf{y}\| < \varepsilon/2$. Hence by (20),

$$\forall\mathbf{x}'\in W_0\ \|\mathbf{v_y}(\mathbf{x}')\| < \varepsilon. \tag{21}$$

(19) and (21) mean that $\mathbf{h_y}$ is a contraction map from W_0 to W_0. Hence by contraction mapping theorem (particular version of [8] Theorem IV.8.3), there exists a unique $\mathbf{x} \in W_0$ such that $\mathbf{h_y}(\mathbf{x}) = \mathbf{x}$. This implies $\mathbf{f}(\mathbf{x}) = \mathbf{y}$ and then $\mathbf{x} \in V$. This completes the proof of the claim.

Next, we construct a code for the local inverse function. Let $\mathbf{F}$ be a code for $\mathbf{f}$. Let $\varphi(\mathbf{b}, s, \mathbf{a}, r)$ be a Σ^0_1 formula which expresses that $\|\mathbf{b}\| + s < \varepsilon/2$ and there exists $(m', \mathbf{a}', r', \mathbf{b}', s') \in \mathbf{F}$ such that $\|\mathbf{b} - \mathbf{b}'\| + s < s'$ and $\|\mathbf{a} - \mathbf{a}'\| + 4s' < r$. Write

$$\varphi(\mathbf{b}, s, \mathbf{a}, r) \equiv \exists m \theta(m, \mathbf{b}, s, \mathbf{a}, r)$$

where θ is Σ^0_0. By Δ^0_1 comprehension, define $\mathbf{G}$ as $(m, \mathbf{b}, s, \mathbf{a}, r) \in \mathbf{G} \leftrightarrow \theta(m, \mathbf{b}, s, \mathbf{a}, r)$.

Claim 3.22.2. $\mathbf{G}$ *is a code for a continuous (partial) function (in the sense of remark 3.1).*

We can easily check that the condition 2 and 3 holds. We must check the condition 1. Assume $(\mathbf{b}, s)\mathbf{G}(\mathbf{a}_1, r_1)$ and $(\mathbf{b}, s)\mathbf{G}(\mathbf{a}_2, r_2)$. By the previous claim, we can take a unique $\mathbf{a}_0 \in V$ such that $\mathbf{f}(\mathbf{a}_0) = \mathbf{b}$. By the definition of $\mathbf{G}$, there exist $(\mathbf{a'}_i, r'_i, \mathbf{b'}_i, s'_i)$ $(i = 1, 2)$ such that $(\mathbf{a'}_i, r'_i)\mathbf{F}(\mathbf{b'}_i, s'_i)$, $\|\mathbf{b} - \mathbf{b'}_i\| + s < s'_i$ and $\|\mathbf{a}_i - \mathbf{a'}_i\| + 4s'_i < r_i$ $(i = 1, 2)$. Then

$$\|\mathbf{f}(\mathbf{a}_0) - \mathbf{f}(\mathbf{a'}_i)\| = \|\mathbf{b} - \mathbf{f}(\mathbf{a'}_i)\| \leq \|\mathbf{b} - \mathbf{b'}_i\| + \|\mathbf{b'}_i - \mathbf{f}(\mathbf{a'}_i)\| < 2s'_i.$$

Hence by (18),

$$\|\mathbf{a}_0 - \mathbf{a'}_i\| < 4s'_i.$$

This implies $\|\mathbf{a}_0 - \mathbf{a}_i\| < r_i$ $(i = 1, 2)$ and then $\|\mathbf{a}_1 - \mathbf{a}_2\| < r_1 + r_2$. This completes the proof of the claim.

Claim 3.22.3. *Let* $\mathbf{g}$ *be the continuous function coded by* $\mathbf{G}$. *Then for all* $\mathbf{y} \in W$, $\mathbf{y} \in \mathrm{dom}(\mathbf{g})$.

For all $\mathbf{y} \in W$ and for all $\delta > 0$, we need to show that there exists $(\mathbf{b}, s, \mathbf{a}, r)$ such that $(\mathbf{b}, s)\mathbf{G}(\mathbf{a}, r)$, $\|\mathbf{b} - \mathbf{y}\| < s$ and $r < \delta$. Take $\mathbf{x} \in V$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. Then there exists $(\mathbf{a}', r', \mathbf{b}', s')$ such that $(\mathbf{a}', r')\mathbf{F}(\mathbf{b}', s')$, $\|\mathbf{a}' - \mathbf{x}\| < r'$ and $\|\mathbf{b}' - \mathbf{y}\| < s' < \delta/8$. Then, there exists n such that the following conditions holds:

$$\begin{aligned} \|\mathbf{y}_n - \mathbf{b}'\| + 2^{-n+1} &< s'; \\ \|\mathbf{y}_n\| + 2^{-n+1} &< \frac{\varepsilon}{2}. \end{aligned}$$

Here, $\mathbf{y}_n$ is a n-th approximation of $\mathbf{y}$. These conditions can be expressed by Σ_1^0 formula, hence we can take $n = n_0$ which satisfies them. Define $(\mathbf{b}, s, \mathbf{a}, r)$ as

$$\begin{aligned} \mathbf{b} &:= \mathbf{y}_{n_0}; \\ s &:= 2^{-n_0+1}; \\ \mathbf{a} &:= \mathbf{a}'; \\ r &:= 5s'. \end{aligned}$$

Then $\|\mathbf{a} - \mathbf{a}'\| + 4s' < r$, hence $(\mathbf{b}, s)\mathbf{G}(\mathbf{a}, r)$. Also $\|\mathbf{b} - \mathbf{y}\| < s$ and $r < \delta$ hold. This completes the proof of the claim.

Claim 3.22.4. $\mathbf{g}$ *is the local inverse of* $\mathbf{f}$, *i.e.*,

$$\forall \mathbf{x} \in V \quad \mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}, \tag{22}$$

$$\forall \mathbf{y} \in W \quad \mathbf{f}(\mathbf{g}(\mathbf{y})) = \mathbf{y}. \tag{23}$$

We first show (22). Let $\mathbf{x} \in V$ and $\mathbf{y} = \mathbf{f}(\mathbf{x})$. To prove $\mathbf{x} = \mathbf{g}(\mathbf{y})$, we need to show that $(\mathbf{b}, s)\mathbf{G}(\mathbf{a}, r)$ and $\|\mathbf{y} - \mathbf{b}\| < s$ imply $\|\mathbf{x} - \mathbf{a}\| < r$. Assume $(\mathbf{b}, s)\mathbf{G}(\mathbf{a}, r)$ and $\|\mathbf{y}-\mathbf{b}\| < s$. Then by the definition of $\mathbf{G}$, there exist $(\mathbf{a}', r', \mathbf{b}', s')$ such that $(\mathbf{a}', r')\mathbf{F}(\mathbf{b}', s')$, $\|\mathbf{b} - \mathbf{b}'\| + s < s'$ and $\|\mathbf{a} - \mathbf{a}'\| + 4s' < r$. Then

$$\begin{aligned} \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}')\| &= \|\mathbf{y} - \mathbf{f}(\mathbf{a}')\| \\ &\leq \|\mathbf{y} - \mathbf{b}\| + \|\mathbf{b} - \mathbf{b}'\| + \|\mathbf{b}' - \mathbf{f}(\mathbf{a}')\| \\ &< 2s'. \end{aligned}$$

Hence by (18),

$$\|\mathbf{x} - \mathbf{a}'\| < 4s'.$$

Therefore

$$\|\mathbf{x} - \mathbf{a}\| \leq \|\mathbf{x} - \mathbf{a}'\| + \|\mathbf{a}' - \mathbf{a}\| < r.$$

(23) is immediate from (22) since $\mathbf{f}$ is bijective on V. This completes the proof of the claim.

Now we expand $\mathbf{g}$ into a C^r or C^∞-function. We can easily define the derivatives of $\mathbf{g}$. For example, define the first derivatives as

$$\left(\frac{\partial g_i}{\partial x_j}\right)_{1 \leq i,j \leq n} = \left(\left(\frac{\partial f_i}{\partial x_j}\right)_{1 \leq i,j \leq n}\right)^{-1}.$$

It remains to prove that $\mathbf{g}$ and their derivatives surely satisfy the conditions for C^r or C^∞. Using the differentiable condition function for $\mathbf{f}$, this can be achieved as usual. This completes the proof of 1.

We can imitate the usual proof to show the implication $1 \to 2$. □

Mathematics in RCA_0 is concerned with constructive mathematics. The constructive proof of implicit function theorem is in Bridges, Calude, Pavlov and Ştefănescu [4]. For details of constructive mathematics, see Bishop and Bridges [1].

The inverse function theorem for Banach spaces is provable in WKL_0 plus a certain version of Baire category theorem [2]. See also [3].

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