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Infinite games and set existence axioms

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Abstract

The aim of this paper is to investigate the logical strength of the determinacy of infinite games whose complexity lies in the finite difference hierarchy over Σ_2^0 . This is a continuation of Tanaka's papers [12] and [13], and will be succeeded by our further studies on Δ_3^0 games.

1 Introduction

For $A \subset \omega^\omega$, we associate a two-person *game* G_A (or simply denote A) described as follows: player I and player II alternately choose natural numbers (starting with I) to form an infinite sequence $f \in \omega^\omega$ and I (resp. II) wins iff $f \in A$ (resp. $f \notin A$). We say that A is *determinate* if one of the players has a *winning strategy*, namely, a map $\sigma : \omega^{<\omega} \rightarrow \omega$ such that the player is guaranteed to win each play f in which he makes a move $f(n) = \sigma((f(0), \dots, f(n-1)))$ at each stage n of his turn.

Much work has been done regarding determinacy in set theory. Gale and Stewart [4] proved that open sets are determined. Their result were soon extended by Wolfe [14] to the F_σ -sets, then by Davis [2] to $F_{\sigma\delta}$ -sets. The next major advance was made by Martin [7] who showed that all Borel sets are determinate. It has also shown that one must assume some strong large cardinal axioms to prove the determinacy of games beyond the class of Borel sets. See [6] for more information.

The determinacy of games with various logical complexities can be formally stated within second order arithmetic. Then, we are interested in the following question motivated by Reverse Mathematics (cf. Simpson [10]): what set existence axioms are needed to prove the determinacy of some classes of formulas in second order arithmetic. While Friedman [3] proved that Σ_5^0 -determinacy is not provable

in the full system of second order arithmetic, Steel [11] (cf. Tanaka [12] and Simpson [10]) proved that \mathbf{ATR}_0 is necessary and sufficient for Σ_1^0 -determinacy. Later, Tanaka [13] proved that the axiom of Σ_1^1 -monotone inductive definitions is equivalent to Σ_2^0 -determinacy over \mathbf{ATR}_0 .

Meanwhile, Burgess [1] has obtained a characterization of R -sets by applying the game quantifier to the difference hierarchy over G_δ -sets. Hinman [4] defined an effective counterpart of the finite level of R -hierarchy and showed that this hierarchy can be defined by finite iteration of Kolmogorov's operator \mathcal{R} . These provide us with useful indication to extend Tanaka's characterization of Σ_2^0 -determinacy, though it does not seem to be easy to transform them directly into our setting.

The paper is structured as follows. Section 2 contains some preliminaries. Section 3 represents the main part in this paper, we introduce the inductive definition of a combination of finately many Σ_1^1 -operators, then show that it is equivalent to the determinacy of a Boolean combination of Σ_2^0 -sets.

The proofs here are only sketched. A full treatment will appear elsewhere.

2 Preliminaries

In this section, we introduce some subsystems of second order arithmetic.

Definition 2.1 The *language of second order arithmetic* consists of the following symbols:

- *number variables* $x, y, z \dots$ intended to range over ω ,
- *set variables* X, Y, Z, \dots intended to range over the subsets of ω ,
- *constant symbols* $0, 1$,
- *function symbols* $+, \cdot$,
- *relation symbols* $=, <, \in$.

Number terms are build from number variables and constant symbols $0, 1$ by using $+, \cdot$. *Atomic formulas* are of the form $t_1 = t_2, t_1 < t_2, t_1 \in X$, where t_1, t_2 are number terms. *Formulas* are built up from atomic formulas by means of propositional connectives, number quantifiers $\forall x, \exists x$, and set quantifiers $\forall X, \exists X$. The formulas can be classified as follows:

- φ is *bounded* (Π_0^0) if it is built up from atomic formulas by using propositional connectives and bounded number quantifiers $(\forall x < t), (\exists x < t)$,

- φ is Π_0^1 if it does not contain any set quantifier. Π_0^1 formulas are called *arithmetical* formulas,
- φ is Σ_n^i if $\varphi \equiv \neg\psi$ where ψ is a Π_n^i -formula ($i \in \{0, 1\}, n \in \omega$),
- φ is Π_{n+1}^0 if $\varphi \equiv \forall x_1 \cdots \forall x_k \psi$ where ψ is a Σ_n^0 -formula ($n \in \omega$),
- φ is Π_{n+1}^1 if $\varphi \equiv \forall X_1 \cdots \forall X_k \psi$ where ψ is a Σ_n^1 -formula ($n \in \omega$).

The Π_n^i -comprehension axiom, denoted $(\Pi_n^i\text{-CA})$, is defined to be $\exists X \forall x (x \in X \leftrightarrow \varphi(x))$, where φ is Π_n^i .

Definition 2.2 The system \mathbf{ACA}_0 consists of the discretely ordered semi-ring axioms for $(\mathbb{N}, +, \cdot, 0, 1, <)$ together with $(\Pi_0^1\text{-CA})$ and the following *induction axiom*:

$$\forall X ((0 \in X \wedge \forall x (x \in X \rightarrow (x+1) \in X)) \rightarrow \forall x (x \in X)).$$

\mathbf{ACA}_0 is a conservative extension of Peano-arithmetic. It is also a nice base theory when we deal with classical analysis.

Finally, we define the subsystem \mathbf{ATR}_0 . We start by defining the axiom $(\Pi_n^i\text{-TR})$.

Definition 2.3 The axiom $(\Pi_n^i\text{-TR})$ is defined as follows: for any set $X \subset \omega$ and for any well-ordering $<$, there exists a set $H \subset \omega$ such that

1. if b is the $<$ -least element, then $(H)_b = X$,
2. if b is the immediate successor of a w.r.t. $<$, then $\forall n (n \in (H)_b \leftrightarrow \psi(n, (H)_a))$,
3. if b is a limit, then $\forall a \forall n ((n, a) \in (H)_b \leftrightarrow (a < b \wedge n \in (H)_a))$,

where ψ is a Π_n^i -formula, $(H)_a = \{n : (n, a) \in H\}$ and, $(n, a) = ((n+a)(n+a+1)/2) + n$.

The subsystem \mathbf{ATR}_0 consists of $\mathbf{ACA}_0 + (\Pi_0^1\text{-TR})$. Obviously, \mathbf{ATR}_0 is stronger than \mathbf{ACA}_0 .

3 $(\Sigma_2^0)_k$ -determinacy and $[\Sigma_1^1]^k$ -ID

An operator $\Gamma : P(\omega) \rightarrow P(\omega)$ is said to be *monotone* iff $\Gamma(X) \subset \Gamma(Y)$ whenever $X \subset Y$. Γ is a Σ_1^1 -operator iff its graph $\{(x, X) : x \in \Gamma(X)\}$ is Σ_1^1 . We will use $\text{mon}\Sigma_1^1$ to denote the class of Σ_1^1 monotone operators. A relation R is a *pre-ordering* iff it is reflexive, connected and transitive. R is a *pre-wellordering* iff it is pre-ordering and well-founded. The *field* of R is the set $\{x : \exists y (x, y) \in R \vee (y, x) \in R\}$.

We characterize the determinacy of a Boolean combination of Σ_2^0 -formulas in term of inductive definition of a finite combination of Σ_1^1 -operators. First, let us look how to iterate a pair of operators Γ_1, Γ_2 . We assume Γ_1 has a distinct parameter X , and so it is also denoted by Γ_1^X . We start with the empty set, and iterate applying Γ_1^\emptyset until we get a fixed point F_0 , i.e, $\Gamma_1^\emptyset(F_0) \subset F_0$. Then re-start iterating $\Gamma_1^{\Gamma_2(F_0)}$, get a fixed point F_1 . Again, iterate $\Gamma_1^{\Gamma_2(F_0) \cup \Gamma_2(F_1)}$ and so on. The procedure stops when we get a fixed point F such that if F' is the least fixed point of Γ_1^F then $\Gamma_2(F') \subset F$. This procedure can be viewed as an inductive definition by a single operator $[\Gamma_1^X, \Gamma_2]$.

Generally, the iteration of $\Gamma_1, \dots, \Gamma_{k-1}, \Gamma_k$ can be described as follows. We iterate $[\Gamma_1, \dots, \Gamma_{k-1}]^\emptyset$ until we get a fixed point, say F_0 . Then restart iterating $[\Gamma_1, \dots, \Gamma_{k-1}]^{\Gamma_k(F_0)}$, get another fixed point F_1 , again iterate $[\Gamma_1, \dots, \Gamma_{k-1}]^{\Gamma_k(F_0) \cup \Gamma_k(F_1)}$ and so on. Eventually, we stop when we get a fixed point F of $[\Gamma_1, \dots, \Gamma_{k-1}, \Gamma_k]$.

Now, let S_1, \dots, S_k be collections of operators. By $[S_1, S_2, \dots, S_k]$ -ID, we denote the axiom which guarantees the existence of a fixed point of $[\Gamma_1, \dots, \Gamma_k]$ for any $\Gamma_1 \in S_1, \dots, \Gamma_k \in S_k$. We here give a formal definition for $k = 2$ as follows.

Definition 3.1 $[S_1, S_2]$ -ID asserts that for any $\Gamma_1 \in S_1, \Gamma_2 \in S_2$, there exist pre-wellorderings W, V' on their fields F, F' respectively such that

$$\Gamma_2(F') \subset F \wedge \Gamma_1^F(F') \subset F',$$

and also a pre-wellordering V^m on its field F^m for each $m \in F$ such that

- $\forall y \in F^m, V_y^m = \Gamma_1^{W_{<^m}}(V_{<^m}^m) \cup V_{<^m}^m$
- $W_m = \Gamma_2(F^m) \cup W_{<^m}$
- $\Gamma_1^{W_{<^m}}(F^m) \subset F^m$.

For convenience, we introduce some abbreviations. By $[S_1^{k_1}, S_2^{k_2}, \dots, S_i^{k_i}]$ -ID, we mean

$$\underbrace{[S_1, \dots, S_1]}_{k_1 \text{ times}}, \underbrace{[S_2, \dots, S_2]}_{k_2 \text{ times}}, \dots, \underbrace{[S_i, \dots, S_i]}_{k_i \text{ times}} \text{-ID.}$$

We also write $[S]^k$ -ID for $[S^k]$ -ID.

We here remark that our definition of combinations of inductive operators is different from one studied in Richter and Aczel [8].

Next, we define the class $(\Sigma_2^0)_k$ of formulas as follows. For $k = 1$, $(\Sigma_2^0)_1 = \Sigma_2^0$. For $k > 1$, $\psi \in (\Sigma_2^0)_k$ iff it can be written as $\psi_1 \wedge \psi_2$, where $\neg\psi_1 \in (\Sigma_2^0)_{k-1}$ and $\psi_2 \in \Sigma_2^0$. The goal of this section is to prove the equivalence between $(\Sigma_2^0)_k$ -determinacy and $[\Sigma_1^1]^k$ -ID. We start by the following theorem.

Theorem 3.1 $\text{ATR}_0 \vdash [\text{mon}\Sigma_1^1, (\Sigma_1^0)^{k-1}\text{-ID}] \rightarrow (\Sigma_2^0)_k\text{-Det}$

Proof. We only show the case $k = 2$ since the case $k > 2$ can be treated by simple meta-induction. Let $A(f)$ be a $(\Sigma_2^0)_2$ -formula. There are a Σ_2^0 -formula φ_0 and a Π_2^0 -formula φ_1 such that for all $f \in \omega^\omega$, $A(f) \equiv \varphi_0(f) \wedge \varphi_1(f)$. Since φ_0 is a Σ_2^0 , there is a Π_2^0 relation R such that for all $f \in \omega^\omega$, $\varphi_0(f) \equiv \exists x \forall y R(x, f[y])$, where $f[y]$ is a code for $\langle f(0), f(1), \dots, f(y-1) \rangle$.

By $[\text{mon}\Sigma_1^1, \Sigma_1^0\text{-ID}]$, we define a transfinite sequence $\langle W_\alpha, \alpha \in Y \rangle$ of sure winning positions for player I as follows: for any ordinal $\alpha \in Y$,

$$u \in W_\alpha \leftrightarrow \underbrace{\exists x}_{(1)} \underbrace{(\text{I has a winning strategy for } A_{u,\alpha,x})}_{(2)},$$

where $A_{u,\alpha,x} = \{f \in \omega^\omega : \forall y (R(x, (u * f)[y]) \vee (u * f)[y] \in W_{<\alpha}) \wedge \varphi_1^u(f)\}$. Here, $u * f$ denotes the concatenation of u and f , $\varphi_1^u(f) \leftrightarrow \varphi_1(u * f)$ and $W_{<\alpha} = \bigcup \{W_\beta : \beta < \alpha\}$. We put $W_\infty = \bigcup_{\alpha \in Y} W_\alpha$.

Now, let us look at the above definition of W_α . Clearly, part(1) of the right hand side of the definition is Σ_1^0 -operator. Part(2) can be viewed as the complement of a fixed point of Σ_1^1 -monotone operator, since $A_{u,\alpha,x}$ is a Π_2^0 game (see [4]). Thus, W_α is defined by a combination of Σ_1^0 -operator and Σ_1^1 -monotone operator. We talked about a set of ordinals Y , but there is no harm of this at all, because we can simply identify the set Y with the pre-wellordering constructed by $[\text{mon}\Sigma_1^1, \Sigma_1^0\text{-ID}]$. Moreover, it is not difficult to show that for any $u \in \omega^{<\omega}$, we have

1. $u \in W_\infty \rightarrow$ player I has a winning strategy for A^u ,
2. $u \notin W_\infty \rightarrow$ player II has a winning strategy for A^u .

Here, A^u is defined by $f \in A^u \leftrightarrow u * f \in A$.

From this claim, one of the players has a winning strategy for A^u . Thus $A = A^\emptyset$ is determinate. This completes the proof of the main theorem. \square

Next, we prove the converse.

Theorem 3.2 $\text{ACA}_0 \vdash (\Sigma_2^0)_k\text{-Det} \rightarrow [\Sigma_1^1]^k\text{-ID}$.

Proof. As in the previous theorem, we only sketch the prove for the case $k = 2$. Let Γ_1, Γ_2 be Σ_1^1 -operators.

First, we construct a $(\Sigma_2^0)_2$ -game G as follows:

- Player I starts by playing y^* asking whether y^* is in the field of the structure inductively defined by $[\Gamma_1, \Gamma_2]$, then II has to answer either *yes* (in this case, he will be called **Pro**) or *no* (in this case, I will be called **Pro**).

- Pro is requested to construct a pre-ordering W , and $\langle V^m, m \in \text{field}(W) \rangle$ such that every V^m is pre-wellorderings, with $y \in \text{field}(W)$ or $y \in \text{field}(V^{m,y})$ for some m .
- We give Con (the opponent of Pro) a chance to win if he points out a mistake in the $(\leq y)$ -segment.
- We show that I has no winning strategy and then by the $(\Sigma_2^0)_2$ -determinacy, we deduce that player II has a winning strategy.
- Using II's winning strategy, we build \bar{W} , $\langle \bar{V}^m, m \in \text{field}(\bar{W}) \rangle$ such that for any y accepted by II, $\bar{W}_{<y} = W_{<y}$ and $\bar{V}^m = V^m$ for any $m \in \bar{W}_{<y}$.
- Using the fact that I can never win, we show that \bar{W} , $\langle \bar{V}^m, m \in \text{field}(\bar{W}) \rangle$ satisfy the condition of Definition 3.1. \square

Finally, we have the following corollary:

Corollary 3.1 *For any $k > 0$, we have*

$$\text{ATR}_0 \vdash (\Sigma_2^0)_k\text{-Det} \leftrightarrow [\text{mon}\Sigma_1^1, (\Sigma_1^0)^{k-1}\text{-ID}] \leftrightarrow [\Sigma_1^1]^k\text{-ID} .$$

References

- [1] J.P.Burgss, *Classical hierarchies from a modern standpoint*, Part II: R -sets, *Fundamenta Mathematica* 105 (1983), pp.97-105.
- [2] M.Davis *Infinite game of perfect information*, *Annals of Mathematical Studies* 52(1964) pp.85-101.
- [3] H.M. Friedman, *Higher set theory and mathematical practice*, *Annals of Mathematical Logic* 2(1971), pp. 325-357.
- [4] D. Gale and F.M.Stewart *Infinite game of perfect information*, *Annals of Mathematical Studies* 28(1964), pp.245-266.
- [5] P.G. Hinman, *The finite levels of the hierarchy of effective R -sets*, *Fundamenta Mathematica* 74(1973), pp.1-10.
- [6] A.Kanamori, *The Higher Infinite*, Springer(1997).
- [7] D.A.Martin *Borel determinacy*, *Annals of Mathematics*, 102, 363-371.
- [8] W.Richter and P.Aczel, *Inductive definitions and reflecting properties of admissible ordinals*, J.E Fenstand and P.G. Hinman, eds., *General Recursion Theory* (1974), pp.301-381.

- [9] J.H. Schmerl and S.G. Simpson, *On the role of Ramsey quantifiers in first order arithmetic*, Journal of Symbolic Logic 47(1982), pp. 423-435.
- [10] S.G. Simpson, *Subsystems of Second Order Arithmetic*, Springer(1999).
- [11] J.R. Steel, *Determinateness and subsystems of analysis*, Ph.D. thesis, Berkeley University(1977).
- [12] K. Tanaka, *Weak axioms of determinacy and subsystems of analysis I: Δ_2^0 -games*, Zeitschrift für mathematische Logik und Grundlagen der Mathematik 36(1990), pp. 481-491.
- [13] K. Tanaka, *Weak axioms of determinacy and subsystems of analysis II: Σ_2^0 -games*, Annals of Pure and Applied Logic 52(1991), pp. 181-193.
- [14] P. Wolfe, *The strict determinateness of certain infinite games*, Pacific Journal of Mathematics 5(1955), pp 841-847.