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Infinite games and set existence axioms

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Abstract

The aim of this paper is to investigate the logical strength of the determinacy of infinite games whose complexity lies in the finite difference hierarchy over $\Sigma^0_2$. This is a continuation of Tanaka's papers [12] and [13], and will be succeeded by our further studies on $\Delta^0_3$ games.

1 Introduction

For $A \subseteq \omega^\omega$, we associate a two-person game $G_A$ (or simply denote $A$) described as follows: player I and player II alternately choose natural numbers (starting with I) to form an infinite sequence $f \in \omega^\omega$ and I (resp. II) wins iff $f \in A$ (resp. $f \not\in A$). We say that $A$ is determinate if one of the players has a winning strategy, namely, a map $\sigma : \omega^{<\omega} \rightarrow \omega$ such that the player is guaranteed to win each play $f$ in which he makes a move $f(n) = \sigma((f(0), \ldots, f(n-1)))$ at each stage $n$ of his turn.

Much work has been done regarding determinacy in set theory. Gale and Stewart [4] proved that open sets are determined. Their result were soon extended by Wolfe [14] to the $F_\omega$-sets, then by Davis [2] to $F_{\sigma}\delta$-sets. The next major advance was made by Martin [7] who showed that all Borel sets are determinate. It has also shown that one must assume some strong large cardinal axioms to prove the determinacy of games beyond the class of Borel sets. See [6] for more information.

The determinacy of games with various logical complexities can be formally stated within second order arithmetic. Then, we are interested in the following question motivated by Reverse Mathematics (cf. Simpson [10]): what set existence axioms are needed to prove the determinacy of some classes of formulas in second order arithmetic. While Friedman [3] proved that $\Sigma^0_2$-determinacy is not provable.
in the full system of second order arithmetic, Steel [11] (cf. Tanaka [12] and Simpson [10]) proved that $\mathbf{ATR}_0$ is necessary and sufficient for $\Sigma^0_1$-determinacy. Later, Tanaka [13] proved that the axiom of $\Sigma^1_1$-monotone inductive definitions is equivalent to $\Sigma^0_2$-determinacy over $\mathbf{ATR}_0$.

Meanwhile, Burgess [1] has obtained a characterization of $R$-sets by applying the game quantifier to the difference hierarchy over $G_5$-sets. Hinman [4] defined an effective counterpart of the finite level of $R$-hierarchy and showed that this hierarchy can be defined by finite iteration of Kolmogorov's operator $\mathcal{R}$. These provide us with useful indication to extend Tanaka's characterization of $\Sigma^0_2$-determinacy, though it does not seem to be easy to transform them directly into our setting.

The paper is structured as follows. Section 2 contains some preliminaries. Section 3 represents the main part in this paper, we introduce the inductive definition of a combination of finitely many $\Sigma^1_1$-operators, then show that it is equivalent to the determinacy of a Boolean combination of $\Sigma^0_2$-sets.

The proofs here are only sketched. A full treatment will appear elsewhere.

\section{Preliminaries}

In this section, we introduce some subsystems of second order arithmetic.

\textbf{Definition 2.1} The language of second order arithmetic consists of the following symbols:

- \textit{number variables} $x, y, z \ldots$ intended to range over $\omega$,
- \textit{set variables} $X, Y, Z \ldots$ intended to range over the subsets of $\omega$,
- \textit{constant symbols} $0, 1$,
- \textit{function symbols} $+, \cdot$,
- \textit{relation symbols} $=, <, \in$.

Number terms are build from number variables and constant symbols $0, 1$ by using $+, \cdot$. Atomic formulas are of the form $t_1 = t_2, t_1 < t_2, t_1 \in X$, where $t_1, t_2$ are number terms. Formulas are built up from atomic formulas by means of propositional connectives, number quantifiers $\forall x, \exists x$, and set quantifiers $\forall X, \exists X$. The formulas can be classified as follows:

- $\varphi$ is bounded ($\Pi^0_1$) if it is built up from atomic formulas by using propositional connectives and bounded number quantifiers ($\forall x < t, (\exists x < t)$,
• $\varphi$ is $\Pi^1_0$ if it does not contain any set quantifier. $\Pi^1_0$ formulas are called arithmetic formulas,

• $\varphi$ is $\Sigma^1_n$ if $\varphi \equiv \neg \psi$ where $\psi$ is a $\Pi^1_n$-formula ($i \in \{0, 1\}, n \in \omega$),

• $\varphi$ is $\Pi^0_{n+1}$ if $\varphi \equiv \forall x_1 \cdots \forall x_k \psi$ where $\psi$ is a $\Sigma^0_n$-formula ($n \in \omega$),

• $\varphi$ is $\Pi^1_{n+1}$ if $\varphi \equiv \forall X_1 \cdots \forall X_k \psi$ where $\psi$ is a $\Sigma^1_n$-formula ($n \in \omega$).

The $\Pi^1_n$-comprehension axiom, denoted ($\Pi^1_n$-CA), is defined to be $\exists X \forall x (x \in X \leftrightarrow \varphi(x))$, where $\varphi$ is $\Pi^1_n$.

**Definition 2.2** The system $ACA_0$ consists of the discretely ordered semi-ring axioms for $(\mathbb{N}, +, \cdot, 0, 1, <)$ together with ($\Pi^1_0$-CA) and the following induction axiom:

$$\forall X ((0 \in X \land \forall x (x \in X \rightarrow (x + 1) \in X)) \rightarrow \forall x (x \in X)).$$

$ACA_0$ is a conservative extension of Peano-arithmetic. It is also a nice base theory when we deal with classical analysis.

Finally, we define the subsystem $ATR_0$. We start by defining the axiom ($\Pi^1_n$ - TR).

**Definition 2.3** The axiom ($\Pi^1_n$ - TR) is defined as follows: for any set $X \subset \omega$ and for any well-ordering $<$, there exists a set $H \subset \omega$ such that

1. if $b$ is the $<$-least element, then $(H)_b = X,$

2. if $b$ is the immediate successor of $a$ w.r.t. $<$, then $\forall n (n \in (H)_b \leftrightarrow \psi(n, (H)_a)),$

3. if $b$ is a limit, then $\forall a \forall n ((n, a) \in (H)_b \leftrightarrow (a < b \land n \in (H)_a)),$

where $\psi$ is a $\Pi^1_n$-formula, $(H)_a = \{ n : (n, a) \in H \}$ and, $(n, a) = ((n + a)(n + a + 1)/2) + n.$

The subsystem $ATR_0$ consists of $ACA_0 + (\Pi^1_0$ - TR). Obviously, $ATR_0$ is stronger than $ACA_0$.

### 3 (\Sigma^0_2)_k-determinacy and [\Sigma^1_1]^k-ID

An operator $\Gamma : P(\omega) \rightarrow P(\omega)$ is said to be monotone iff $\Gamma(X) \subset \Gamma(Y)$ whenever $X \subset Y$. $\Gamma$ is a $\Sigma^1_1$-operator iff its graph $\{(x, X) : x \in \Gamma(X)\}$ is $\Sigma^1_1$. We will use $\text{mon}\Sigma^1_1$ to denote the class of $\Sigma^1_1$ monotone operators. A relation $R$ is a pre-ordering iff it is reflexive, connected and transitive. $R$ is a pre-wellordering iff it is pre-ordering and well-founded. The field of $R$ is the set $\{ x : \exists y(x, y) \in R \lor (y, x) \in R \}.$
We characterize the determinacy of a Boolean combination of $\Sigma^0_2$-formulas in terms of an inductive definition of a finite combination of $\Sigma^1_1$-operators. First, let us look how to iterate a pair of operators $\Gamma_1, \Gamma_2$. We assume $\Gamma_1$ has a distinct parameter $X$, and so it is also denoted by $\Gamma^X_1$. We start with the empty set, and iterate applying $\Gamma_1^X$ until we get a fixed point $F_0$, i.e., $\Gamma_1^X(F_0) \subseteq F_0$. Then re-start iterating $\Gamma_1^X(F_0)$, get a fixed point $F_1$. Again, iterate $\Gamma_1^X(F_0) \cup \Gamma_1(F_1)$ and so on. The procedure stops when we get a fixed point $F'$ such that if $F'$ is the least fixed point of $\Gamma_1^X$, then $\Gamma_2(F') \subseteq F$. This procedure can be viewed as an inductive definition by a single operator $[\Gamma_1^X, \Gamma_2]$.

Generally, the iteration of $\Gamma_1, \cdots, \Gamma_{k-1}, \Gamma_k$ can be described as follows. We iterate $[\Gamma_1, \cdots, \Gamma_{k-1}]^\emptyset$ until we get a fixed point, say $F_0$. Then restart iterating $[\Gamma_1, \cdots, \Gamma_{k-1}]^{\Gamma_1(F_0)}$, get another fixed point $F_1$, again iterate $[\Gamma_1, \cdots, \Gamma_{k-1}]^{\Gamma_1(F_0) \cup \Gamma_1(F_1)}$ and so on. Eventually, we stop when we get a fixed point $F$ of $[\Gamma_1, \cdots, \Gamma_{k-1}, \Gamma_k]$.

Now, let $S_1, \cdots, S_k$ be collections of operators. By $[S_1, S_2, \cdots, S_k]$-ID, we denote the axiom which guarantees the existence of a fixed point of $[\Gamma_1, \cdots, \Gamma_k]$ for any $\Gamma_1 \in S_1, \cdots, \Gamma_k \in S_k$. We here give a formal definition for $k = 2$ as follows.

**Definition 3.1** $[S_1, S_2]$-ID asserts that for any $\Gamma_1 \in S_1, \Gamma_2 \in S_2$, there exist pre-wellorderings $W, V'$ on their fields $F, F'$ respectively such that

$$\Gamma_2(F') \subseteq F \land \Gamma_1^W(F') \subseteq F',$$

and also a pre-wellordering $V^m$ on its field $F^m$ for each $m \in F$ such that

- $\forall y \in F^m, \ V^m_y = \Gamma_1^{W < m}(V^m_{< y}) \cup V^m_{< 2y}$
- $W_m = \Gamma_2(F^m) \cup W_{< m}$
- $\Gamma_1^{W < m}(F^m) \subseteq F^m$.

For convenience, we introduce some abbreviations. By $[S_1^{k_1}, S_2^{k_2}, \cdots, S_i^{k_i}]$-ID, we mean

$$k_1 \text{ times } k_2 \text{ times } k_i \text{ times }$$

$$[S_1, \cdots, S_i, S_1, \cdots, S_i, \cdots, S_1, \cdots, S_i]$$

We also write $[S]^k$-ID for $[S]^k$-ID.

We here remark that our definition of combinations of inductive operators is different from one studied in Richter and Aczel [8].

Next, we define the class $(\Sigma^0_2)_k$ of formulas as follows. For $k = 1$, $(\Sigma^0_2)_1 = \Sigma^0_2$. For $k > 1, \psi \in (\Sigma^0_2)_k$ iff it can be written as $\psi_1 \land \psi_2$, where $\neg \psi_1 \in (\Sigma^0_2)_{k-1}$ and $\psi_2 \in \Sigma^0_2$.

The goal of this section is to prove the equivalence between $(\Sigma^0_2)_k$-determinacy and $[\Sigma^1_1]^k$-ID. We start by the following theorem.
Theorem 3.1 $\text{ATR}_0 \vdash [\text{mon} \Sigma^1_1, (\Sigma^0_2)^{k-1}]\text{-ID} \rightarrow (\Sigma^0_2)^k\text{-Det}$

Proof. We only show the case $k = 2$ since the case $k > 2$ can be treated by simple meta-induction. Let $A(f)$ be a $(\Sigma^0_2)^2$-formula. There are a $\Sigma^0_2$-formula $\varphi_0$ and a $\Pi^0_2$-formula $\varphi_1$ such that for all $f \in \omega^\omega$, $A(f) \equiv \varphi_0(f) \land \varphi_1(f)$. Since $\varphi_0$ is a $\Sigma^0_2$, there is a $\Pi^0_2$ relation $R$ such that for all $f \in \omega^\omega$, $\varphi_0(f) \equiv \exists x \forall y R(x, f[y])$, where $f[y]$ is a code for $(f(0), f(1), \ldots, f(y-1))$.

By $[\text{mon} \Sigma^1_1, \Sigma^0_2]$-ID, we define a transfinite sequence $(W_\alpha, \alpha \in Y)$ of sure winning positions for player I as follows: for any ordinal $\alpha \in Y$,

$$u \in W_\alpha \leftrightarrow \exists x \left( \text{I has a winning strategy for } A^u_{u,\alpha,x} \right),$$

where $A^u_{u,\alpha,x} = \{ f \in \omega^\omega : \forall y (R(x, (u*f)[y]) \lor (u*f)[y] \in W_{<\alpha}) \land \varphi_1^u(f) \}$. Here, $u*f$ denotes the concatenation of $u$ and $f$, $\varphi_1^u(f) \leftrightarrow \varphi_1(u*f)$ and $W_{<\alpha} = \bigcup\{W_\beta : \beta < \alpha\}$. We put $W_\infty = \bigcup_{\alpha \in Y} W_\alpha$.

Now, let us look at the above definition of $W_\alpha$. Clearly, part(1) of the right hand side of the definition is $\Sigma^0_2$-operator. Part(2) can be viewed as the complement of a fixed point of a $\Sigma^0_2$-monotone operator, since $A^u_{u,\alpha,x}$ is a $\Pi^0_2$ game (see [4]). Thus, $W_\alpha$ is defined by a combination of $\Sigma^0_2$-operator and $\Sigma^1_2$-monotone operator. We talked about a set of ordinals $Y$, but there is no harm of this at all, because we can simply identify the set $Y$ with the pre-wellordering constructed by $[\text{mon} \Sigma^1_1, \Sigma^0_2]$-ID. Moreover, it is not difficult to show that for any $u \in \omega^{<\omega}$, we have

1. $u \in W_\infty \rightarrow \text{player I has a winning strategy for } A^u$,
2. $u \notin W_\infty \rightarrow \text{player II has a winning strategy for } A^u$.

Here, $A^u$ is defined by $f \in A^u \leftrightarrow u*f \in A$.

From this claim, one of the players has a winning strategy for $A^u$. Thus $A = A^\#$ is determinate. This completes the proof of the main theorem. □

Next, we prove the converse.

Theorem 3.2 $\text{ACA}_0 \vdash (\Sigma^0_2)^k\text{-Det} \rightarrow (\Sigma^1_1)^k\text{-ID}$.

Proof. As in the previous theorem, we only sketch the prove for the case $k = 2$. Let $\Gamma_1, \Gamma_2$ be $\Sigma^1_2$-operators.

First, we construct a $(\Sigma^0_2)^2$-game $G$ as follows:

- Player I starts by playing $y^*$ asking whether $y^*$ is in the field of the structure inductively defined by $[\Gamma_1, \Gamma_2, y^*]$, then II has to answer either yes (in this case, $y^*$ will be called Pro) or no (in this case, I will be called Pro).
Pro is requested to construct a pre-ordering $W$, and $(V^m, m \in \text{field}(W))$ such that every $V^m$ is pre-wellorderings, with $y \in \text{field}(W)$ or $y \in \text{field}(V^{m,y})$ for some $m$.

We give Con (the opponent of Pro) a chance to win if he points out a mistake in the $(\leq y)$-segment.

We show that I has no winning strategy and then by the $(\Sigma^0_2)_2$-determinacy, we deduce that player II has a winning strategy.

Using II's winning strategy, we build $\tilde{W}, (\tilde{V}^m, m \in \text{field}(\tilde{W}))$ such that for any $y$ accepted by II, $\tilde{W}_{<y} = W_{<y}$ and $\tilde{V}^m = V^m$ for any $m \in \tilde{W}_{<y}$.

Using the fact that I can never win, we show that $\tilde{W}, (\tilde{V}^m, m \in \text{field}(\tilde{W}))$ satisfy the condition of Definition 3.1. □

Finally, we have the following corollary:

**Corollary 3.1** For any $k > 0$, we have

$$\text{ATR}_0 \vdash (\Sigma^0_2)_k \text{-Det} \leftrightarrow [\text{mon}\Sigma^1_1, (\Sigma^0_1)^{k-1}]\text{-ID} \leftrightarrow [\Sigma^1_1]^k\text{-ID}.$$  

**References**


