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# Mean-variance hedging for discontinuous processes — a survey and examples

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**Abstract**

In this paper, we survey the results of the mean-variance hedging problem for discontinuous asset price processes, based on Arai (2005a, 2005b, 2005c). Furthermore, we introduce some examples satisfying assumptions which are imposed in these papers on the variance-optimal martingale measure.

**Keywords:** Incomplete market, mean-variance hedging, variance-optimal martingale measure
**JEL classification:** G10

## 1 Introduction

In this paper, we summarize the results of Arai (2005a, 2005b, 2005c), and introduce some examples related to these papers. Arai (2005a) has given a representation of mean-variance hedging for the discontinuous asset price process case under some additional assumptions related to the variance-optimal martingale measure (VOM, for short) as an extension of Gouriéroux, Laurent and Pham (1998) (GLP, for short) and Rheinländer and Schweizer (1997) (RS, for short). Moreover, Arai (2005b) gave another proof of Arai (2005a) by using the duality method of Hou and Karatzas (2004) (HK, for short). Finally, Arai (2005c) improved the additional assumptions in Arai (2005a).

We consider an incomplete financial market being composed of one riskless asset and $d$ risky assets. Let $T > 0$ be its maturity. Suppose that the price of the riskless asset is 1 at all times, and the fluctuation of $d$ risky assets is described by an $\mathbf{R}^d$-valued semimartingale $X$. Furthermore, in this paper, we regard an $\mathbf{R}^d$-valued predictable $X$-integrable process $\vartheta$ such that the stochastic integral $G(\vartheta) := \int_0^{\cdot} \vartheta_s dX_s$ is a semimartingale of the space $\mathcal{S}^2$ as a self-financing strategy, and denote the set of all such self-financing strategies by $\Theta$. The process $G(\vartheta)$ means the trading gains induced by a self-financing strategy $\vartheta$. Note that $\vartheta$ means the number of units invested in the risky assets. Let $H$ be an $\mathcal{F}_T$-measurable square integrable random variable. Throughout this paper, we regard $H$ as a contingent claim, that is, payoff at the maturity $T$. Then, we consider a hedger with an initial capital $c \in \mathbf{R}$. She or he intends to hedge against the contingent claim $H$ by means of a self-financing strategy. However, since the market is incomplete, it is impossible to replicate the contingent claim by an appropriate self-financing strategy. Thus, she or he is under the necessity of optimizing her or his strategy in some way. Then, we assume that she or he makes an attempt to minimize the $\mathcal{L}^2(P)$-norm of discrepancy between the underlying contingent claim and the value at the maturity $T$ of a portfolio associated with a self-financing strategy and the initial capital $c$. In other words, she or he considers, for fixed $c \in \mathbf{R}$, the following minimization problem:

**Problem 1**

$$\text{Minimize} \quad E\left[(H - c - G_T(\vartheta))^2\right] \quad \text{over all } \vartheta \in \Theta.$$

The solution of Problem 1 is said to be mean-variance hedging.

GLP and RS have obtained conclusive results for the case where $X$ is given by a continuous semimartingale. By using a change of numéraire and a change of measure, GLP reduced the problem to a martingale framework. On the other hand, RS used weighted norm inequalities and the Galtchouk-Kunita-Watanabe decomposition (GKW decomposition, for short). They obtained a feedback form expression of mean-variance hedging. Moreover, RS discussed the GLP approach as an alternative approach. In addition, HK has introduced a new duality approach to least-square approximation problem of random variables by stochastic integrals. In Theorem 5.1 of HK, they obtained, through their duality approach, mean-variance hedging under the assumption that the asset price process is given by a semimartingale having continuous paths. This result is the same one as Theorem 5.1 of GLP and Theorems 5, 6 of RS.

On the other hand, for the case where $X$ is a discontinuous semimartingale, Arai (2005a) has obtained a representation of mean-variance hedging as an extension of the GLP approach under additional assumptions related to the VOM as follows:

(A1) the VOM is a probability measure,
(A2) its density process satisfies the reverse Hölder inequality,
(A3) it also satisfies a certain condition related to jumps.

He tried to construct a new decomposition of $H$ on $X$ by the same sort of argument as the alternative approach in Section 4 of RS. Unfortunately, the new decomposition, in general, is not an orthogonal one, that is, not a GKW one. However, thanks to the technical condition (A3), he obtained nice properties of each term in the new decomposition. Thereby, he established a similar representation as in the continuous case along the lines of Section 4 of RS.

In the meantime, Arai (2005b) calculated mean-variance hedging strategy for the discontinuous semimartingale case by means of Hou-Karatzas' duality approach. That is, he gave another proof of the results of Arai (2005a). By using the new decomposition in Arai (2005a), he extended Theorem 5.1 of HK to the discontinuous case.

Moreover, Arai (2005c) investigated some properties of the VOM for discontinuous semimartingales. In addition, he discussed relationship with mean-variance hedging for discontinuous asset price process models. Since we cannot check easily whether or not the above assumptions (A1)-(A3) are satisfied, he drove for giving a sufficient condition for these assumptions, which is described by the asset price process. However, it is very difficult for us to give an answer to this question entirely. Therefore, he tried to give a partial solution to our desire.

If the asset price process has continuous paths a.s., then the VOM is given by a probability measure under weak conditions. On the other hand, for the case where the asset price process is discontinuous, the positivity of the density is not ensured. However, since the positivity makes a change of numéraire method available, it is indispensable to calculate mean-variance hedging. Thus, the following question is natural: When does the VOM become a probability measure? Moreover, in the continuous case, if the existence of mean-variance hedging and of the VOM are ensured, (A2) is satisfied. On the other hand, in the discontinuous case, we do not know when (A2) is satisfied. In order to give a partial answer to the above questions, Arai (2005c) proved that, if (A3) and another additional condition are satisfied, so are (A1) and (A2). In other words, he concluded that (A3) is essential. However, we do not know a sufficient condition for (A3). This problem has been postponed to future research.

The paper is structured as follows: Section 2 prepares for some definitions and notations. Section 3 introduces the main results of Arai (2005a), which is a representation of mean-variance hedging for the discontinuous asset price process case under assumptions related to the VOM. In addition, we introduce, in Section 4, the results of Arai (2005b), which are an extension of HK's duality approach to the discontinuous case. Section 5 deals with some examples satisfying assumptions related to the VOM. In Section 6, we concentrate on improvement of these assumptions. The contents of Section 6 are based on Arai (2005c).

## 2 Preliminaries

Let $(\Omega, \mathcal{F}, P; \mathbf{F} = \{\mathcal{F}_t\}_{0\leq t\leq T})$ be a completed filtered probability space with a right-continuous filtration $\mathbf{F}$ such that $\mathcal{F}_0$ is trivial and contains all null sets of $\mathcal{F}$, and $\mathcal{F}_T = \mathcal{F}$. Let $X$ be an $\mathbf{R}^d$-valued $\mathbf{F}$-adapted RCLL special semimartingale being in the space $\mathcal{S}^2$. Also, suppose the locally boundedness of $X$. There is a unique canonical decomposition of $X$ into an $\mathbf{R}^d$-valued square integrable $P$-martingale $M$ starting at 0, namely, $M \in \mathcal{M}_0^2(P)$, and an $\mathbf{R}^d$-valued natural process $A$ of square integrable variation starting at 0. That is, the canonical decomposition of $X$ is represented as $X_t^i = X_0^i + M_t^i + A_t^i$, for $i = 1, \ldots, d$.

Let $Y$ be a stochastic process. A property $\mathcal{P}$ is said to hold locally if there exists a sequence of stopping times $(\tau_n)_{n\geq 1}$ increasing to $T$ a.s. such that $Y^{\tau_n}\mathbf{1}_{\{\tau_n>0\}}$ has property $\mathcal{P}$, each $n \geq 1$. In particular, $Y$ is a local martingale if $Y$ is locally a uniformly integrable martingale. $C$ denotes a constant in $(0, \infty)$ which may vary from line to line. For all unexplained notations, we refer to Dellacherie and Meyer (1982).

We need some preparations. Firstly, for any RCLL process $U$, we define the process $U^*$ by

$$U_t^* := \sup_{0\leq s\leq t} |U_s|.$$

Let $\mathcal{R}^2(P)$ be the set of all adapted RCLL processes $U$ such that

$$\|U\|_{\mathcal{R}^2(P)} := \|U_T^*\|_{\mathcal{L}^2(P)} < \infty.$$

Next, $\Theta$ is defined as the space of all $\mathbf{R}^d$-valued predictable $X$-integrable processes $\vartheta$ such that the stochastic integral

$$G(\vartheta) := \int_0^\cdot \vartheta_s dX_s$$

is in the space $\mathcal{S}^2$ of semimartingales. In financial point of view, $\Theta$ represents the set of all self-financing strategies and the stochastic integral $G(\vartheta)$ is the gain process induced by a self-financing strategy $\vartheta \in \Theta$. Next, we define signed $\Theta$-martingale measures and the variance-optimal martingale measure (VOM).

**Definition 2.1** (1) A signed measure $Q$ on $(\Omega, \mathcal{F})$ is called a signed $\Theta$-martingale measure, if $Q(\Omega) = 1$, $Q \ll P$ with $\frac{dQ}{dP} \in \mathcal{L}^2(P)$, and

$$E\left[\frac{dQ}{dP}G_T(\vartheta)\right] = 0,$$

for all $\vartheta \in \Theta$. We denote by $\mathbf{P}_s(\Theta)$ the set of all such signed $\Theta$-martingale measures. Moreover, we define

$$\mathbf{P}_e(\Theta) := \{Q \in \mathbf{P}_s(\Theta) | Q \sim P \text{ and } Q \text{ is a probability measure}\},$$

and introduce the closed, convex set

$$\mathcal{D} := \left\{D \in \mathcal{L}^2(P) \middle| D = \frac{dQ}{dP}, \text{ some } Q \in \mathbf{P}_s(\Theta)\right\}.$$

(2) A signed martingale measure $\widetilde{P} \in \mathbf{P}_s(\Theta)$ is called the variance-optimal martingale measure (VOM) if $\widetilde{D} = \arg\min_{D\in\mathcal{D}} E[D^2]$, where $\widetilde{D} = \frac{d\widetilde{P}}{dP}$.

The space $G_T(\Theta) := \{G_T(\vartheta)|\vartheta \in \Theta\}$ is a linear subspace of $\mathcal{L}^2(P)$. Then, we denote by $G_T(\Theta)^\perp$ its orthogonal complement, that is, $G_T(\Theta)^\perp := \{D \in \mathcal{L}^2(P)|E[DG_T(\vartheta)] = 0$, for any $\vartheta \in \Theta\}$. Furthermore, $G_T(\Theta)^{\perp\perp}$ denotes the orthogonal complement of $G_T(\Theta)^\perp$, which is the $\mathcal{L}^2(P)$-closure of $G_T(\Theta)$. Throughout this paper, we impose the following standing assumption on the VOM:

**Assumption 2.2** $1 \notin G_T(\Theta)^{\perp\perp}$, equivalently $\mathbf{P}_s(\Theta) \neq \emptyset$.

Let $f$ be the projection of 1 on $G_T(\Theta)^{\perp\perp}$. Then, since $E[f(1-f)] = 0$, we have $E[f] = E[f^2]$. Thus, $0 < E[f] < 1$ by $f \neq 1$. Moreover, the density $\widetilde{D}$ of the VOM $\widetilde{P}$ is expressed by

$$\widetilde{D} = \frac{1-f}{1-E[f]}.$$

Besides, $1 - f \in G_T(\Theta)^\perp$. Now, we define two processes as follows:

$$Z_t := E[\widetilde{D}|\mathcal{F}_t], \qquad \widetilde{Z}_t := \widetilde{E}[\widetilde{D}|\mathcal{F}_t],$$

where $\widetilde{E}$ means the expectation under $\widetilde{P}$. Remark that $\widetilde{D} = Z_T = \widetilde{Z}_T$. We call $Z$ the density process of the VOM.

# 3 A representation of mean-variance hedging

In this section, we survey the results of Arai (2005a), which are giving a feedback form description of mean-variance hedging for the setting in Section 2 along the lines of Section 4 of RS. We consider the case where $c = 0$ in Problem 1, which is the following minimization problem:

**Problem 2**

$$\text{Minimize} \quad E\left[(H - G_T(\vartheta))^2\right] \quad \text{over all } \vartheta \in \Theta.$$

Now, we enumerate the standing assumptions of this paper:

**Assumption 3.1** (1) The VOM $\widetilde{P}$ exists as a probability measure.
(2) The density process $Z$ satisfies the reverse Hölder inequality, that is, there is a constant $C > 0$ such that, for every stopping time $\sigma \leq T$, we have

$$E\left[Z_T^2 \middle| \mathcal{F}_\sigma\right] \leq C Z_\sigma^2.$$

(3) There exists a constant $C$ such that $Z_- \leq CZ$.

Remark that the standing assumptions of RS are
(RS1) the space $G_T(\Theta)$ is closed in $\mathcal{L}^2(P)$,
(RS2) $\mathcal{D}^s \cap \mathcal{L}^2(P) \neq \emptyset$.

Let us discuss Assumption 3.1, especially its relationship with RS's assumptions and its meanings.

First, we consider Condition (1). Note that we can regard this condition as one of absence of arbitrage. In the continuous case, $\widetilde{P}$ is in $\mathbf{M}^e(P)$ if it exists. However, in the discontinuous case, $\widetilde{P}$ is not always a probability measure, so that we assume Condition (1) which is stronger than (RS2). Thanks to Condition (1), $Z$ is square integrable and strictly positive. This fact will simplify our argument in the sequel. Moreover, Condition (1) ensures that $\widetilde{Z}$ is a numéraire. Hence, we can apply the change of numéraire as in GLP to our setting. Remark that, under Condition (1), for some $N^Z \in \mathcal{M}_{0,loc}$, we can represent $Z = \mathcal{E}(N^Z)$. Then, according to Theorem 1 of Schweizer (1995), Condition (1) ensures the local square integrability of $N^Z$ and the following structure condition (SC): there exists an $\mathbf{R}^d$-valued predictable process $\widehat{\lambda}$ satisfying

$$A_t = \int_0^t d\langle M\rangle_s \widehat{\lambda}_s.$$

As for Conditions (2) and (3), we need these for technical reasons. Although, according to Theorem 2 of RS, Conditions (RS1) and (RS2) imply Condition (2) in the continuous case, we cannot extend this relation to the discontinuous case, because Delbaen et al. (1997) have given a counterexample in their Example 3.9. On the other hand, according to Theorem 5.2 of Choulli, Krawczyk and Stricker (1998) (CKS, for short), Condition (2) together with Condition (1) guarantee Condition (RS1). Thus, the solution of Problem 2 always exists under Assumption 3.1. Next, Condition (3) means that there exists a positive constant $\varepsilon$ such that $\Delta N^Z > -1+\varepsilon$. Although Condition (1) yields $\Delta N^Z > -1$, Condition (3) is slightly stronger than this.

Now, we remark that Condition (2) implies that there exists a constant $C$ such that

$$Z \le \widetilde{Z} \le CZ. \tag{3.1}$$

Thus, Condition (3) implies that there exists a constant $C$ such that

$$\widetilde{Z}_- \le C\widetilde{Z}. \tag{3.2}$$

Next, we define a new space of predictable processes.

**Definition 3.2** $\widetilde{\Theta}$ denotes the space of all $\mathbf{R}^d$-valued predictable $X$-integrable processes $\vartheta$ such that $G(\vartheta)$ is a $\widetilde{P}$-martingale satisfying $G_T(\vartheta) \in \mathcal{L}^2(P)$.

Remark that GLP has considered $\widetilde{\Theta}$ as the set of all self-financing strategies. Now, we prepare one lemma.

**Lemma 3.3 (Lemma 4.1 of Arai (2005a))** *For any $\vartheta \in \Theta$, $G(\vartheta)$ is a local $\widetilde{P}$-martingale.*

We can prove the equivalence between $\Theta$ and $\widetilde{\Theta}$ as an extension of Lemma 9 of RS to our setting by using the above lemma:

**Lemma 3.4 (Lemma 9 of RS)** $\widetilde{\Theta} = \Theta$.

Lemma 3.4 implies that we can replace $\Theta$ in Problem 2 with $\widetilde{\Theta}$. Moreover, together with Lemma 1 of Schweizer (1996), $\widetilde{Z}$ is represented as

$$\widetilde{Z}_t = \widetilde{E}\left[\frac{d\widetilde{P}}{dP}\right] + G_t(\widetilde{\zeta}). \tag{3.3}$$

On the other hand, we define an $\mathbf{R}^{d+1}$-valued process $Y$ and a new probability measure $\widetilde{R}$ which is equivalent to $P$ as

$$\begin{aligned} Y^0 &:= \widetilde{Z}^{-1}, \\ Y^i &:= X^i \widetilde{Z}^{-1}, \quad \text{for } i = 1, \ldots, d, \end{aligned}$$

and

$$\frac{d\widetilde{R}}{d\widetilde{P}} := \frac{\widetilde{Z}_T}{\widetilde{Z}_0}.$$

We can now extend Proposition 8 of RS to our setting.

**Proposition 3.5 (Proposition 8 of RS, Proposition 3.2 of GLP)** *We have*

$$\frac{1}{\widetilde{Z}_T} G_T(\widetilde{\Theta}) = \left\{ \int_0^T \psi_s dY_s \middle| \psi \in L^2(Y, \widetilde{R}) \right\},$$

*where $L^2(Y, \widetilde{R})$ is the space of all $\mathbf{R}^{d+1}$-valued predictable $Y$-integrable processes $\psi$ such that $\int \psi dY$ is in $\mathcal{M}_0^2(\widetilde{R})$. Moreover, the relation between $\vartheta \in \widetilde{\Theta}$ and $\psi \in L^2(Y, \widetilde{R})$ is given by*

$$\begin{aligned} \psi^i &:= \vartheta^i \quad \textit{for } i = 1, \ldots, d, \\ \psi^0 &:= G_-(\vartheta) - \vartheta^{\mathrm{tr}} X_-, \end{aligned}$$

*and*

$$\vartheta^i := \psi^i + \widetilde{\zeta}^i \left( \int_0^- \psi dY - \psi^{\mathrm{tr}} Y_- \right) \quad \textit{for } i = 1, \ldots, d. \tag{3.4}$$

Furthermore, for any $\vartheta \in \Theta$, we have

$$\|H - G_T(\vartheta)\|_{\mathcal{L}^2(P)} = \sqrt{\widetilde{Z}_0} \left\| \frac{H}{\widetilde{Z}_T} - \frac{G_T(\vartheta)}{\widetilde{Z}_T} \right\|_{\mathcal{L}^2(\widetilde{R})},$$

by the definition of $\widetilde{R}$. In view of Proposition 3.5, Problem 2 is equivalent to the following problem:

**Problem 3**

$$\text{Minimize} \quad \left\| \frac{H}{\widetilde{Z}_T} - \int_0^T \psi_s dY_s \right\|_{\mathcal{L}^2(\widetilde{R})} \quad \text{over all } \psi \in L^2(Y, \widetilde{R}).$$

Since $\dfrac{H}{\widetilde{Z}_T} \in \mathcal{L}^2(\widetilde{R})$ and $Y \in \mathcal{M}^2_{loc}(\widetilde{R})$, there exists a GKW decomposition of $\dfrac{H}{\widetilde{Z}_T}$ on $Y$ under $\widetilde{R}$ as follows:

$$\frac{H}{\widetilde{Z}_T} = E_{\widetilde{R}}\left[\frac{H}{\widetilde{Z}_T}\right] + \int_0^T \widetilde{\psi}^H_s dY_s + L_T, \tag{3.5}$$

where $\widetilde{\psi}^H \in L^2(Y, \widetilde{R})$, and $L \in \mathcal{M}^2_0(\widetilde{R})$ is $\widetilde{R}$-orthogonal to $Y$. The solution $\psi^{\text{opt}}$ of Problem 3 is given by the integrand $\widetilde{\psi}^H$. Then, the solution $\vartheta^{\text{opt}}$ of Problem 2 is obtained via (3.4).

Now, we have

$$\widetilde{Z}_T E_{\widetilde{R}}\left[\frac{H}{\widetilde{Z}_T}\right] = \widetilde{E}[H]\left(1 + G_T(\widetilde{Z}_0^{-1}\widetilde{\zeta})\right). \tag{3.6}$$

Next, by Proposition 3.5, we have

$$\widetilde{Z}_T \int_0^T \widetilde{\psi}^H_s dY_s = G_T(\widetilde{\vartheta}^H), \tag{3.7}$$

for some $\widetilde{\vartheta}^H \in \Theta$ given from $\widetilde{\psi}^H$ via (3.4). Remark that $\widetilde{\vartheta}^H$ is the solution of Problem 2. By (3.5) (3.7), we can represent $H$ as follows:

$$H = \widetilde{E}[H] + G_T\left(\widetilde{E}[H]\widetilde{Z}_0^{-1}\widetilde{\zeta} + \widetilde{\vartheta}^H + L_-\widetilde{\zeta}\right) + \int_0^T \widetilde{Z}_{s-}dL_s + [\widetilde{Z}, L]_T. \tag{3.8}$$

We denote

$$\overline{L}_t := \widetilde{E}[H]\widetilde{Z}_0^{-1} + L_t, \qquad \widetilde{\eta}^H_t := \widetilde{\vartheta}^H_t + \overline{L}_{t-}\widetilde{\zeta}_t,$$

and

$$N_t := \int_0^t \widetilde{Z}_{s-}dL_s + [\widetilde{Z}, L]_t = \int_0^t \widetilde{Z}_{s-}d\overline{L}_s + [\widetilde{Z}, \overline{L}]_t. \tag{3.9}$$

Thus, we rewrite (3.8) as follows:

$$H = \widetilde{E}[H] + G_T(\widetilde{\eta}^H) + N_T. \tag{3.10}$$

This equation is a new type decomposition of $H$, which is not orthogonal one. However, we can treat this new decomposition (3.10) as orthogonal one by the good properties of $N$ and $\widetilde{\eta}^H$. The following two lemmas will play important roles in the proof of the main theorem of this section.

**Lemma 3.6 (Lemma 4.4 of Arai (2005a))** *$N$ is a $\widetilde{P}$-martingale with $N_0 = 0$ and in $\mathcal{R}^2(P)$.*

**Lemma 3.7 (Lemma 4.5 of Arai (2005a))** *$\widetilde{\eta}^H \in \Theta$.*

After the above preparations, we state the main theorem of this section.

**Theorem 3.8 (Theorem 4.1 of Arai (2005a))** *Under Assumption 3.1, the solution $\vartheta^{\text{opt}}$ of Problem 2 is given by*

$$\vartheta_t^{\text{opt}} = \widetilde{\eta}_t^H - \frac{\widetilde{\zeta}_t}{\widetilde{Z}_{t-}} \left( \widetilde{V}_{t-}^H - G_{t-}(\vartheta^{\text{opt}}) \right),$$

*where* $\widetilde{V}_t^H := \widetilde{E}[H|\mathcal{F}_t]$.

*Proof.* By Lemmas 3.6 and 3.7, we have

$$\widetilde{V}_t^H = \widetilde{E}[H] + G_t(\widetilde{\eta}^H) + N_t.$$

Moreover, by integration by parts,

$$\widetilde{Z}_t \overline{L}_t = \widetilde{E}[H] + G_t(\overline{L}_- \widetilde{\zeta}) + N_t = \widetilde{V}_{t-}^H - G_{t-}(\vartheta^{\text{opt}}).$$

Then, we conclude that

$$\vartheta_t^{\text{opt}} = \widetilde{\eta}_t^H - \frac{\widetilde{\zeta}_t}{\widetilde{Z}_{t-}} \left( \widetilde{V}_{t-}^H - G_{t-}(\vartheta^{\text{opt}}) \right),$$

by $\vartheta^{\text{opt}} = \widetilde{\vartheta}^H = \widetilde{\eta}^H - \overline{L}_- \widetilde{\zeta}$. This completes the proof of Theorem 3.8.

# 4 Another approach

We explain, in this section, the contents of Hou-Karatzas' duality approach, but we will not give any proofs, and extend them to our setting. Remark that this section is a survey of Arai (2005b).

We consider the following:

$$\begin{aligned} \min_{x \in \mathbf{R}} [(H-x)^2 + yx] &= (H - (H - y/2))^2 + y(H - y/2) \\ &= yH - y^2/4, \quad \text{for all } y \in \mathbf{R}. \end{aligned}$$

For given $c \in \mathbf{R}$, arbitrary $\vartheta \in \Theta$, $D \in \mathcal{D}$ and $k \in \mathbf{R}$, substitute $c + G_T(\vartheta)$ and $2kD$ for $x$ and $y$, respectively. Then we obtain

$$(H - c - G_T(\vartheta))^2 + 2kD(c + G_T(\vartheta)) \geq 2kDH - k^2 D^2.$$

Taking expectation, for every $\vartheta \in \Theta$, $D \in \mathcal{D}$ and $k \in \mathbf{R}$,

$$E[(H - c - G_T(\vartheta))^2] \geq -k^2 E[D^2] + 2k(E[DH] - c).$$

Thus, we obtain

$$\begin{aligned} V(c) &:= E[(H - c - G_T(\vartheta^{(c)}))^2] = \inf_{\vartheta \in \Theta} E[(H - c - G_T(\vartheta))^2] \\ &\geq \sup_{D \in \mathcal{D}} \sup_{k \in \mathbf{R}} \left\{ -k^2 E[D^2] + 2k(E[DH] - c) \right\} \\ &= \sup_{D \in \mathcal{D}} \frac{(E[DH] - c)^2}{E[D^2]} =: \widetilde{V}(c), \end{aligned} \tag{4.1}$$

where $\vartheta^{(c)}$ is the solution of Problem 1.

For given $c \in \mathbf{R}$, the following problem would be a duality problem of Problem 1:

**Problem 4**

$$\text{Maximize } \frac{(E[DH]-c)^2}{E[D^2]} \text{ over all } D \in \mathcal{D}.$$

In HK, they proved that there is no duality gap between Problems 1 and 4, namely, $V(c) = \widetilde{V}(c)$ for any $c \in \mathbf{R}$. Now, let us define the projection operator $\pi : \mathcal{L}^2(P) \to (G_T(\Theta))^{\perp}$ with property, for any $H \in \mathcal{L}^2(P)$ and $D \in (G_T(\Theta))^{\perp}$,

$$E[(H - \pi(H))D] = 0.$$

We denote by $\widetilde{D}_c \in \mathcal{D}$ the optimizer for Problem 4, for any $c \neq \dfrac{E[\pi(H)]}{E[\pi(1)]}$. Hence, we have

$$\widetilde{V}(c) = \frac{(E[\widetilde{D}_c H]-c)^2}{E[\widetilde{D}_c^2]} = -k_c^2 E[\widetilde{D}_c^2] + 2k_c(E[\widetilde{D}_c H]-c),$$

where $k_c := \dfrac{E[\widetilde{D}_c H]-c}{E[\widetilde{D}_c^2]}$. In virtue of (4.1), the following equality holds:

$$\begin{aligned} E[(H-c-G_T(\vartheta^{(c)}))^2] &= -k_c^2 E[\widetilde{D}_c^2] + 2k_c(E[\widetilde{D}_c H]-c) \\ &= -k_c^2 E[\widetilde{D}_c^2] + 2k_c(E[\widetilde{D}_c H - \widetilde{D}_c G_T(\vartheta^{(c)})]-c), \end{aligned}$$

namely, we have $E[(H-c-G_T(\vartheta^{(c)}) - k_c\widetilde{D}_c)^2] = 0$. Consequently, we obtain $H - c - G_T(\vartheta^{(c)}) = k_c\widetilde{D}_c$.

On the other hand, from the Cauchy-Schwarz inequality, we have

$$(E[DH]-c)^2 = (E[D\pi(H-c)])^2 \leq E[D^2]E[\{\pi(H-c)\}^2],$$

for any $D \in \mathcal{D}$. Thus, $\widetilde{D}_c = C\pi(H-c)$ holds, where $C > 0$ has to be chosen such that $E[\widetilde{D}_c] = 1$. Therefore, we can conclude that

$$H - c - G_T(\vartheta^{(c)}) = k_c\widetilde{D}_c = \frac{E[\widetilde{D}_c H]-c}{E[\widetilde{D}_c^2]} \frac{\pi(H-c)}{E[\pi(H-c)]} = \pi(H-c).$$

Also, for the case where $c = \dfrac{E[\pi(H)]}{E[\pi(1)]}$, HK proved that $H - c - G_T(\vartheta^{(c)}) = \pi(H-c)$.

As a result, HK obtained, for any $c \in \mathbf{R}$,

$$G_T(\vartheta^{(c)}) = H - c - \pi(H-c), \tag{4.2}$$

where $\vartheta^{(c)}$ is the solution of Problem 1.

Henceforth, we focus on to obtain the same result as Theorem 3.8 by means of Hou-Karatzas' duality approach.

For the process $N$ in the new decomposition (3.10), since the random variable $N_T$ is square $P$-integrable, we can represent $N_T$ as, for some $\vartheta^N \in \Theta$,

$$N_T = G_T(\vartheta^N) + \pi(N_T).$$

Moreover, remark that we can decompose the constant 1 into

$$1 = G_T(\xi^1) + \pi(1) \quad \text{for some } \xi^1 \in \Theta.$$

Thus, we have

$$\begin{aligned}(H-c) - \pi(H-c) &= \widetilde{E}[H] + G_T(\widetilde{\eta}^H) + N_T - c - \widetilde{E}[H]\pi(1) - \pi(N_T) + c\pi(1) \\ &= (\widetilde{E}[H] - c)(1 - \pi(1)) + G_T(\widetilde{\eta}^H) + N_T - \pi(N_T) \\ &= G_T((\widetilde{E}[H] - c)\xi^1 + \widetilde{\eta}^H + \vartheta^N).\end{aligned}$$

Since the mapping $\vartheta \mapsto G_T(\vartheta)$ is injective, we can conclude that the solution $\vartheta^{(c)}$ is given by

$$\vartheta^{(c)} = (\widetilde{E}[H] - c)\xi^1 + \widetilde{\eta}^H + \vartheta^N, \tag{4.3}$$

by the above observation together with (4.2). In addition, we have the following lemma:

**Lemma 4.1 (Lemma 3.1 of Arai (2005b))** *$L$ in (3.5) is a $\widetilde{P}$-martingale being $\widetilde{P}$-orthogonal to $G(\vartheta)$, for any $\vartheta \in \Theta$.*

Therefore, for any $\vartheta \in \Theta$, we have

$$\begin{aligned}E\left[G_T(\vartheta^N + L_-\widetilde{\zeta})G_T(\vartheta)\right] &= E\left[\left(G_T(\vartheta^N) + \widetilde{Z}_T L_T - N_T\right) G_T(\vartheta)\right] \\ &= E\left[\left(G_T(\vartheta^N) - G_T(\vartheta^N) - \pi(N_T)\right) G(\vartheta)\right] \\ &= E\left[-\pi(N_T) G_T(\vartheta)\right] \\ &= 0,\end{aligned}$$

by $N_T = \widetilde{Z}_T L_T - G_T(L_-\widetilde{\zeta})$ and Lemma 4.1. Therefore, we obtain

$$\vartheta^N = -L_-\widetilde{\zeta}.$$

On the other hand, (3.22) of HK implies $\widetilde{\zeta} = -E[\widetilde{Z}_T^2]\xi^1 = -\widetilde{Z}_0\xi^1$. Consequently, the representation of $\vartheta^{(c)}$ is given by

$$\begin{aligned}\vartheta^{(c)} &= (\widetilde{E}[H] - c)\xi^1 + \widetilde{\eta}^H - L_-\widetilde{\zeta} \\ &= \widetilde{\eta}^H + \left[\widetilde{E}[H] - c + E[\widetilde{Z}_T^2]L_-\right]\xi^1.\end{aligned}$$

Remark that, since the predictable processes $\xi^1$, $\widetilde{\eta}^H$ and $\vartheta^N$ are in the space $\Theta$, so is $\vartheta^{(c)}$. As a result, we can assert the following theorem:

**Theorem 4.2 (Theorem 3.1 of Arai (2005b))** *Under Assumption 3.1, the solution $\vartheta^{(c)}$ of Problem 1 is given by*

$$\vartheta^{(c)} = \widetilde{\eta}^H + \left[\widetilde{E}[H] - c + E[\widetilde{Z}_T^2]L_-\right]\xi^1.$$

# 5 Examples

In this section, we give some examples satisfying Assumption 3.1. In particular, we are interested in only models such that the minimal martingale measure (MMM, for short) does not coincide with the VOM. Thus, we introduce some such examples

**Example 5.1** We consider the case where $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = 2^\Omega$, $d = 1$ and $T > 1$. We define the asset price process $X$ as follows: for $t < 1$, $X_t = 0$, and, for $1 \leq t \leq T$, $X_t(\omega_1) = 1$, $X_t(\omega_2) = 0$ and $X_t(\omega_3) = -2$. Let $\mathcal{F}_t = \{\emptyset, \Omega\}$ for $t < 1$ and $\mathcal{F}_t = 2^\Omega$ for $1 \leq t \leq T$. Moreover, we assume that $P(\{\omega_1\}) = P(\{\omega_3\}) = 1/4$. Then, by simple calculations, the VOM $\widetilde{P}$ is given by $\{\widetilde{P}(\{\omega_1\}), \widetilde{P}(\{\omega_2\}), \widetilde{P}(\{\omega_3\})\} = \{48/77, 5/77, 24/77\}$. It is very easy to make sure that Assumption 3.1 is satisfied and $\widetilde{P}$ is not the MMM.

**Example 5.2 (Section 3 of Arai (2005a))** We consider two completed filtered probability spaces $\mathcal{P}^i := (\Omega^i, \mathcal{F}^i, P^i; \mathbf{F}^i = \{\mathcal{F}_t^i\}_{0\leq t\leq T})$ for $i = 1, 2$. The whole probability space $\mathcal{P} := (\Omega, \mathcal{F}, P; \mathbf{F} = \{\mathcal{F}_t\}_{0\leq t\leq T})$ is defined as the product space of $\mathcal{P}^1$ and $\mathcal{P}^2$. Let $N^i$ for $i = 1, 2$, be two independent Poisson processes with intensity 1, defined on the probability space $\mathcal{P}^i$, respectively. In particular, suppose that, for $i = 1, 2$, the filtration $\mathbf{F}^i$ is the $P^i$-augmentation of the filtration generated by $N^i$, and $\mathcal{F}^i = \mathcal{F}_T^i$. We define $\widetilde{N}_t^i := N_t^i - t$, which is the compensated process of $N^i$. We deal with the case where $d = 1$ and the asset price process is given by:

$$X = x + \int \sigma d\widetilde{N}^1 + \int \widehat{\lambda}\sigma^2 dt,$$

where $x$ is a constant, and, $\widehat{\lambda}$ and $\sigma$ are predictable processes defined on $\mathcal{P}$. Assume that there exists a positive constant $\varepsilon$ such that $|\sigma| > \varepsilon$ and $|\widehat{\lambda}\sigma| < 1-\varepsilon$.

We consider only the case where $\widehat{\lambda}$ and $\sigma$ depend only on $\mathcal{P}^2$, which is a kind of stochastic volatility model. In this case, the density $Z_T$ of the VOM is described by

$$Z_T = \frac{\mathcal{E}_T\left(-\int \widehat{\lambda} dX\right)}{E\left[\mathcal{E}_T\left(-\int \widehat{\lambda} dX\right)\right]}, \tag{5.1}$$

that is, the MMM does not coincide with the VOM. Moreover, we can prove that $Z$ satisfies Assumption 3.1.

**Example 5.3** We try to extend Example 5.2 to more general cases. Set $\eta = \widehat{\lambda}\sigma$. Then, we can rewrite $X$ as

$$X = x + \int \sigma d\widetilde{N}^1 + \int \eta\sigma dt.$$

For the case where the predictable process $\eta$ depends only on $\mathcal{P}^2$, $Z_T$ is represented as (5.1) even if $\sigma$ depends on $\mathcal{P}^1$. Moreover, we can generalize the probability space $\mathcal{P}^2$. Let $W = (W^1, W^2, \cdots, W^n)$ be an $n$-dimensional Brownian motion, and $J = (J^1, J^2, \cdots, J^m)$ an $m$-dimensional Poisson process. Assume that $W$ and $J$ are independent of $N^1$. Then, we set that the filtration $\mathbf{F}^2$ is the $P^2$-augmentation of the filtration generated by $W$ and $J$. In light of the martingale representation theorem, we have the same sort of argument as Example 5.2.

# 6 Reduction of Assumption 3.1

In general, it is difficult for us to obtain the density of the VOM explicitly. Thus, to check whether or not Assumption 3.1 holds is impossible. Hence, we wish Assumption 3.1 were relaxed to checkable condition. In this section, we introduce the results of Arai (2005c), which succeeded in giving checkable condition partially. Throughout this section, we assume Assumption 2.2.

**Proposition 6.1 (Proposition 3.2 of Arai (2005c))** *If (3) of Assumption 3.1 holds and the density process $Z$ of the VOM has a stochastic exponential form, then the VOM $\widetilde{P}$ is in $\mathbf{P}_e(\Theta)$.*

**Remark 1** The process $Z$ does not necessarily have a stochastic exponential form. To be accurate, by Theorem 2 of Schweizer (1995), if the structure condition (SC) is satisfied, that is, there exists an $\mathbf{R}^d$-valued predictable process $\widehat{\lambda}$ satisfying

$$A = \int d\langle M\rangle\widehat{\lambda},$$

then $Z$ is given by a solution of the following stochastic differential equation:

$$Z = 1 - \int Z_{-}\widehat{\lambda}dM + R,$$

where $R \in \mathcal{M}_0^2(P)$ is $P$-orthogonal to $M$.

It is not easy to check whether or not conditions of Proposition 6.1 hold. On the other hand, the combination of the following Assumption 6.2 and (3) of Assumption 3.1 is a sufficient condition for the positivity of $Z_T$. Although we shall observe in the sequel, there is a checkable sufficient condition for only Assumption 6.2.

**Assumption 6.2** There exists a probability measure $Q \in \mathbf{P}_e(\Theta)$ satisfying the reverse Hölder inequality.

Now, we assert main theorems of this section as follows:

**Theorem 6.3 (Theorem 3.4 of Arai (2005c))** *Under Assumption 6.2, $Z$ satisfies the reverse Hölder inequality.*

**Theorem 6.4 (Theorem 3.5 of Arai (2005c))** *Under Assumption 6.2 and (3) of Assumption 3.1, the VOM $\widetilde{P}$ is in $\mathbf{P}_e(\Theta)$.*

**Remark 2** In view of Theorems 6.3 and 6.4, the following two conditions are equivalent under (3) of Assumption 3.1:

(1) Assumption 6.2;

(2) the VOM exists as a probability measure, of which the density process satisfies the reverse Hölder inequality.

We can regard this equivalence as an extension of Theorem 2.18 of Delbaen et al. (1997) to the discontinuous case.

**Example 6.5 (Example 3.6 of Arai (2005c))** The converses of Theorems 6.3 and 6.4 do not hold, because there is the following counterexample: Let $d = 1$ and $X$ be given by

$$X_t = x + \int_0^t p_s dB_s + \int_0^t q_s d\widetilde{J}_s + t,$$

where $x \in \mathbf{R}$, $B$ is a one-dimensional Brownian motion, $J$ is a Poisson process with intensity 1, $\widetilde{J}$ is its compensated Poisson process, namely, $\widetilde{J}_t = J_t - t$ and, $p$ and $q$ are predictable processes such that $p^2+q^2 \equiv 1$ and $-1 \leq q < 1$. Remark that the martingale part $M$ of $X$ is given by $M = \int pdB + \int qd\widetilde{J}$, so that we have $\langle M\rangle_t = t$. Hence, we can rewrite $X$ as $X = x + M + \langle M\rangle$, which is the canonical decomposition. In this case, the process $Z$ is given by $Z = \mathcal{E}(-M)$. Thus, $\Delta Z = -Z_-\Delta M = -Z_- q\Delta J$. Consequently, since $\langle M\rangle_T \in \mathcal{L}^\infty(P)$, $Z$ satisfies the reverse Hölder inequality by Proposition 3.7 of CKS, and, since $\Delta Z/Z_- > -1$, $Z > 0$ holds. On the other hand, (3) of Assumption 3.1 is not satisfied.

By Theorems 6.3 and 6.4, we can rewrite (1) and (2) of Assumption 3.1 as Assumption 6.2. In other words, we have the following theorem:

**Theorem 6.6 (Theorem 5.1 of Arai (2005c))** *Under Assumption 6.2 and (3) of Assumption 3.1, Theorem 3.8 holds.*

We need to discuss the question when Assumption 6.2 and (3) of Assumption 3.1 are satisfied. We can give an answer to this question for only Assumption 6.2. We assume (SC), that is, A is given by $\int d\langle M\rangle\widehat{\lambda}$. If $Y := \int \widehat{\lambda}dM \in \mathcal{M}_0^2$ and

$\langle Y \rangle_T \in \mathcal{L}^\infty$, then $\widehat{Z} := \mathcal{E}(-Y)$ satisfies the reverse Hölder inequality. Moreover, we define a signed martingale measure $\widehat{P}$ as $\frac{d\widehat{P}}{dP} := \widehat{Z}_T$. Note that the signed martingale measure $\widehat{P}$ is said to be the MMM. Then, if we suppose that

$$\Delta Y = \sum_{i=1}^{d} \widehat{\lambda}^i \Delta M^i < 1,$$

then $\widehat{P}$ is a probability measure. In this case, Assumption 6.2 is satisfied. That is, together (SC), $Y \in \mathcal{M}_0^2$, $\langle Y \rangle_T \in \mathcal{L}^\infty$ and $\Delta Y < 1$ is a sufficient condition for Assumption 6.2. We can check this sufficient condition by only using the asset price process. On the other hand, a sufficient condition for (3) of Assumption 3.1 has been still open.

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