A solution of the equation $f'(x)=\lambda^2f(\lambda x), \lambda>1$ (Functional Equations and Complex Systems)

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1. INTRODUCTION

The purpose of this paper is to give solutions for the functional-differential equation of advanced type

$$\begin{cases} f'(x) = \lambda^2 f(\lambda x), \quad x \in \mathbb{R} = (-\infty, +\infty), \\ f(0) = 0, \end{cases}$$

where $\lambda$ is a constant, $\lambda > 1$. Our solutions are infinitely differentiable on $\mathbb{R}$. Moreover, if $\lambda \geq 2$, then the solutions are bounded and have arbitrarily large zeros. Our methods give numerical data readily.

Frederickson [1, 2] (1971) investigated functional-differential equations of advanced type

$$f'(x) = af(\lambda x) + \lambda f(x),$$


Frederickson [1] provided a global existence theorem for equations

$$f'(x) = F(f(2x)), \quad x \in \mathbb{R},$$

where $F$ is an odd, continuous function with $F(s) > 0$ for $s > 0$, by application of the Schauder fixed point theorem. He showed that the absolute value of the solution $|f(x)|$ is periodic for $x \geq 0$. Frederickson [2] also provided a constructive method for solutions for equations

$$f'(z) = af(\lambda z) + bf(z),$$
where $a$, $b \in \mathbb{C}$ and $\lambda > 1$. He further gave solutions in the form of a Dirichlet series

$$\varphi(z, \beta) = \sum_{n \in \mathbb{Z}} c_n e^{\beta \lambda^n z}, \quad \Re(\beta z) \leq 0,$$

where $\beta$ is allowed to vary as a parameter. In the case of $b = 0$ and $\beta = i$, the solution is analytic in the upper half plane $\Im z > 0$, continuous on $\Im z \geq 0$, and the line $\Im z = 0$ is a natural boundary. From his result it follows that our solutions of (1.1) cannot be real analytic.

Ivanov, Kitamura, Kusano and Shevelo [3] (1982) investigated the higher order functional-differential equations of the form

$$(1.3) \quad f^{(n)}(x) = p(x) F(f(g(x))),$$

where $p$, $F$ and $g$ satisfy appropriate conditions. Kusano [6] (1984) also investigated the functional differential equation

$$(1.4) \quad f^{(n)}(x) = p(x) f(g(x))$$

where $n$ is even, $p : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ are continuous, $p(t) > 0$, $g(t)$ is nondecreasing and $\lim_{t \to \infty} g(t) = \infty$. They [3, 6] gave sufficient conditions that the solutions are oscillatory.

If $f$ is a solution of (1.1), then $f$ is also a solution of the equations

$$(1.5) \quad f''(x) = \lambda^4 \lambda f(\lambda^2 x), \quad x \in \mathbb{R},$$

and

$$(1.6) \quad f'''(x) = \lambda^6 \lambda^3 f(\lambda^3 x), \quad x \in \mathbb{R}.$$
In the last section, we will give Mathematica programs.

2. Main Results

First, we state two lemmas. Let

\[ \hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx, \quad \mathcal{F}^{-1}[f](\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{ix\xi} \, dx, \]

and

\[ \text{sinc} \xi = \begin{cases} \sin(\pi \xi) / (\pi \xi), & \xi \neq 0, \\ 1, & \xi = 0. \end{cases} \]

Lemma 2.1. The product

\[ \prod_{k=1}^{\infty} \text{sinc} \left( \frac{\xi}{2\lambda^{k}\pi} \right), \quad \xi \in \mathbb{R} \]

is converges pointwise and in \( L^1(\mathbb{R}) \).

Lemma 2.2. Let

\[ u = \mathcal{F}^{-1}[U], \quad U(\xi) = \exp\left( -\frac{\mathrm{i} \xi}{2(\lambda - 1)} \right) \prod_{k=1}^{\infty} \text{sinc} \left( \frac{\xi}{2\lambda^{k}\pi} \right). \]

Then \( u \) has the following properties:

\[ u \in C^\infty(\mathbb{R}), \]

\[ u(x) > 0 \text{ for } x \in \left( 0, \frac{1}{\lambda - 1} \right), \quad u(x) = 0 \text{ for } x \notin \left( 0, \frac{1}{\lambda - 1} \right), \]

\[ u(x) = u(1/(\lambda - 1) - x), \]

\[ \int_{\mathbb{R}} u(x) \, dx = 1, \]

and

\[ u'(x) = \lambda^2 u(\lambda x) \text{ for } x \in \left[ 0, \min \left( \frac{1}{\lambda}, \frac{1}{\lambda(\lambda - 1)} \right) \right]. \]

Let we define the operator \( T : L^1 \rightarrow L^1 \) as follows.

\[ Tf(x) = \lambda \left( \chi_{[0,1]} * f \right)(\lambda x), \quad f \in L^1. \]

Then the function \( u \) in Lemma 2.2 is given by the following equation.

\[ u = \lim_{k} T^k \chi_{[0,1]}. \]
Secondly, we define sequences $\{n_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ as follows:

(2.5) $\begin{cases}
    n_1 = 0, & n_2 = 1, \\
    n_{2k-1} = 1, & n_{2k} = 0, \text{ if } n_k = 1 \ (k \geq 2), \\
    n_{2k-1} = 0, & n_{2k} = 1, \text{ if } n_k = 0 \ (k \geq 2),
\end{cases}$

and

(2.6) $y_k = \sum_{l=1}^{\infty} C_{k,l} \lambda^{l-1}, \quad k = 1, 2, 3, \cdots,$

where $C_{k,l} \in \{0, 1\}$ $(l = 1, 2, 3, \cdots)$ are coefficients of the binary system such that

$k - 1 = \sum_{l=1}^{\infty} C_{k,l} 2^{l-1}, \quad k = 1, 2, 3, \cdots.$

Then we have the following relations.

(2.7) $\begin{cases}
    (-1)^{n_{2k-1}} = (-1)^{n_k}, \\
    (-1)^{n_{2k}} = (-1)^{n_k},
\end{cases} \quad k = 1, 2, 3, \cdots,$

(2.8) $\begin{cases}
    y_{2k-1}/\lambda = y_k, \\
    y_{2k}/\lambda = y_k + 1/\lambda,
\end{cases} \quad k = 1, 2, 3, \cdots,$

and

(2.9) $y_k \geq \lambda^j \quad \text{if} \quad k - 1 \geq 2^j, \quad j = 0, 1, 2, \cdots.$

Hence $\lim_{k \to \infty} y_k = \infty$. If $\lambda \geq 2$, then $y_k$ is strictly increasing. For example,

$\{n_k\}_{k=1}^{\infty} = \{0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \cdots\},$

$\{y_k\}_{k=1}^{\infty} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \cdots\}$ for $\lambda = 2$,

$\{y_k\}_{k=1}^{\infty} = \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, 68, 69, 80, \cdots\}$ for $\lambda = 4$,

$\{y_k\}_{k=1}^{\infty} = \left\{0, 1, \frac{3}{2}, \frac{5}{2}, \frac{9}{4}, \frac{13}{4}, \frac{15}{4}, \frac{19}{8}, \frac{27}{8}, \frac{35}{8}, \frac{39}{8}, \frac{47}{8}, \frac{45}{8}, \cdots\right\}$ for $\lambda = 3/2$.

Our main result is the following:

**Theorem 2.3.** Let $\lambda > 1$. Then a solution $f$ of (1.1) can be found as

$f(x) = \sum_{k=1}^{\infty} (-1)^{n_k} u(x - y_k),$

where $u$, $\{n_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ are as in (2.1), (2.5) and (2.6), respectively. The solution $f$ is in $C^\infty(\mathbb{R})$ and $f(x) = 0$ for $x \leq 0$. If $\lambda \geq 2$, then $f$ is bounded.
Let we define a function space $L^{1,\nu}(\mathbb{R})$.

$$L^{1,\nu} = \{ f \in L_{ac}^{1}; ||f||_{L^{1,\nu}} < \infty \}$$

$$||f||_{L^{1,\nu}} = \sup_{r>0} \frac{1}{r^\nu} \int_{-r}^{+r} |f(x)| \, dx$$

**Theorem 2.4.** The solution $f$ of (1.1) is in $C^\infty \cap L^{1,1/\log_2 \lambda}$.

**Remark 2.1.** The solution of (1.1) is tempered distribution.

**Remark 2.2.** A constant times $f$ is also a solution.

**Theorem 2.5.** Let $f$ be the solution in Theorem 2.3 for $\lambda = 2$ and

$$G_{k,\epsilon}(x) = \left(2^{k(k-1)/2} \epsilon^{k+1}\right)^{-1} (f\chi_{[0,2^k]})(x/\epsilon).$$

If $v \in C^k(\mathbb{R})$ or $v \in L^p_k(\mathbb{R})$ ($k \geq 0$, $1 \leq p < \infty$), then

$$\frac{d^kv}{dx^k} = \lim_{\epsilon \to 0} v * G_{k,\epsilon},$$

uniformly on each compact subset in $\mathbb{R}$ or in $L^p(\mathbb{R})$, respectively.

**Remark 2.3.** $G_{k,\epsilon}$ is in $C^\infty(\mathbb{R})$ with compact support. To prove the theorem we use Friedrichs' molifier $\frac{d^kv}{dx^k} * u_\delta = v * \frac{d^k u_\delta}{dx^k}$, where $u_\delta = u(x/\delta)/\delta$, $\delta > 0$, and $u$ is the function in Lemma 2.2 ($\lambda = 2$).

3. **Examples**

In this section we give graphs for $\lambda = 4, 3, 2, 31/16, 15/8, 7/4, 3/2, 5/4$.

If $\lambda = 2$, then $\{x > 0 : f(x) = 0\} = \{1, 2, 3, \cdots\}$. If $\lambda > 2$, then $\{x > 0 : f(x) = 0\} = \cup_{k=1}^\infty[y_k + 1/(\lambda - 1), y_{k+1}]$ and its measure is infinity, since $1/(\lambda - 1) < 1 < y_{k+1} - y_k$, $k = 1, 2, 3, \cdots$. 
4. Program of $u(x)$

Mathematica program (Part 1): The function $u$ [by using $u_{n+1} = T(u_n)$]
4. $f'(x) = 2^2 f(2x)$

5. $f'(x) = (31/16)^2 f(31x/16)$

6. $f'(x) = (15/8)^2 f(15x/8)$

7. $f'(x) = (7/4)^2 f(7x/4)$

* Setting lambda ($1 < \lambda < 9$)

In[1]: lam = 1.75;

* Calculation of the data; $u_{-0}, \ldots, u_{-50}$

In[2]: udata[0] = Table[If[0 < i - 10000 <= 1000, lam - 1, 0], {i, 1, 20000}];

In[3]: Timing[ Do[udata[k] = Table[If[1 <= j - 10000 <= 1000, 
  lam * Sum[udata[k = 1] [[i + 10000]], 
  {i, Round[lam * (j - 10000)] - Round[(lam - 1)* 1000] + 1, 
  {j - 10000}, {i, lam - 1}, {k, 1, 50}]}}, 
  {k, 1, 50}];

* Calculation of the data; $u_{-0}, \ldots, u_{-50}$
Figure 8. $f'(x) = (3/2)^2 f(3x/2)$

Figure 9. $f'(x) = (5/4)^2 f(5x/4)$

Round$[\text{lam} \times (j - 10000)] \times 0.001/(\text{lam} - 1), 0],$

$j, 1, 20000], \{k, 1, 50\}]$

Out[3]: {121.14 Second, Null}

* Graph of u_{50}

In[4]: u\{k\}_ = Table[(i - 10000) \times 0.001/(\text{lam} - 1),

Part[udata\{k\}, i], \{i, 10000, 11000\}]

In[5]: ListPlot[ulist\{50\}, PlotJoined -> True,

PlotRange -> {0, 1.1 \times \text{lam}]}]

* Save the data
In[6]: udata[50] >> c:/mdata/udata7ov4-50
In[7]: ulist[50] >> c:/mdata/ulist7ov4-50
In[8]: Export["c:/mda/t/u7ov4.eps",
   ListPlot[ulist[50], PlotJoined -> True,
   PlotRange -> {0, 1.1 * lam}]]

5. PROGRAM OF F(X)

Mathematica program (Part 2): The solution f on the interval [0,tau]

* Setting lambda (1<lam<9) and tau
In[1]: lam = 1.75; tau = 30;
In[2]: kk = Round[Log[lam, tau] + 0.5]
Out[2]: 7

* Load the data
In[3]: udata = << c:/mdata/udata7ov4-50;
In[4]: ud = Table[Part[udata, i], {i, 10000, 11000}];

* Sequences m_{k} and y_{k}
In[5]: m[1] = 0; m[2] = 1;
   Do[m[k] = If[Mod[k, 2] == 0, Mod[m[k/2] + 1, 2], m[(k + 1)/2]],
   {k, 3, 2^kk + 1}]
In[6]: Do[b[k, 1] = k - 1; Do[c[k, l] = Mod[b[k, l], 2];
   b[k, l + 1] = (b[k, l] - c[k, l])/2, {l, 1, kk + 1}],
   {k, 1, 2^kk + 1}]
In[7]: Do[y[k] =
   Sum[c[k, l]* lam^(l - 1), {l, 1, kk + 1}], {k, 1, 2^kk + 1}]

* Calculation of the solution
   as the sum of (-1)^{m(k)} u(x-y_{k}), k=1, 2, ... ,2^kk.
In[8]:  Do[yy[k] = Round[y[k] * 1000 * (lam - 1)], {k, 1, 2^kk + 1}]
In[9]:  zz[1] = Table[0, {i, 1, yy[2^kk]}]; Do[z[k] =
    Table[0, {i, 1, yy[k]}], {k, 2, 2^kk}];
    Do[zz[k] = Table[0, {i, 1, yy[2^kk] - yy[k]}], {k, 2, 2^kk}];
In[10]:  udy[1] = Join[ud, zz[1]]; Do[udy[k] =
    Join[z[k], ud* (-1)^m[k], zz[k]], {k, 2, 2^kk}]
In[11]:  fd = Sum[udy[k], {k, 1, 2^kk}];

* Save the graph of the solution

In[12]:  ii = tau*(lam-1)*1000;
In[13]:  flist = Table[{i * 0.001/(lam - 1),
    Part[fd, i]}, {i, 1, ii}];
In[14]:  Export["c:/mdata/f7ov4.eps",
    ListPlot[flist, PlotJoined -> True, AspectRatio -> Automatic]];


[7] Zhi-cheng Wang, Ionannis P. Stavroulakis, Xiang-zheng Qian


[8] Hiroshi Onose


[9] T. Yoneda, *On the functional-differential equation of advanced type* \( f'(x) = af(2x) \) with \( f(0) = 0 \), preprint.

[10] T. Yoneda, *On the functional-differential equation of advanced type* \( f'(x) = af(\lambda x) \), \( \lambda > 1 \) with \( f(0) = 0 \), preprint.


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