A solution of the equation $f'(x) = \lambda^2 f(\lambda x), \lambda > 1$ (Functional Equations and Complex Systems)

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A solution of the equation $f'(x) = \lambda^2 f(\lambda x), \lambda > 1$.

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1. Introduction

The purpose of this paper is to give solutions for the functional-differential equation of advanced type

\begin{equation}
\begin{cases}
f'(x) = \lambda^2 f(\lambda x), & x \in \mathbb{R} = (-\infty, +\infty), \\
f(0) = 0,
\end{cases}
\end{equation}

where $\lambda$ is a constant, $\lambda > 1$. Our solutions are infinitely differentiable on $\mathbb{R}$. Moreover, if $\lambda \geq 2$, then the solutions are bounded and have arbitrarily large zeros. Our methods give numerical data readily.

Frederickson [1, 2] (1971) investigated functional-differential equations of advanced type

\begin{equation}
f'(x) = af(\lambda x) + \lambda f(x),
\end{equation}


Frederickson [1] provided a global existence theorem for equations

\begin{equation}
f'(x) = F(f(2x)), \quad x \in \mathbb{R},
\end{equation}

where $F$ is an odd, continuous function with $F(s) > 0$ for $s > 0$, by application of the Schauder fixed point theorem. He showed that the absolute value of the solution $|f(x)|$ is periodic for $x \geq 0$. Frederickson [2] also provided a constructive method for solutions for equations

\begin{equation}
f'(z) = af(\lambda z) + bf(z),
\end{equation}
where \( a, b \in \mathbb{C} \) and \( \lambda > 1 \). He further gave solutions in the form of a Diriclet series
\[
\varphi(z, \beta) = \sum_{n \in \mathbb{Z}} c_n e^{\beta \lambda^n x}, \quad \Re(\beta z) \leq 0,
\]
where \( \beta \) is allowed to vary as a parameter. In the case of \( b = 0 \) and \( \beta = i \), the solution is analytic in the upper half plane \( \Im z > 0 \), continuous on \( \Im z \geq 0 \), and the line \( \Im z = 0 \) is a natural boundary. From his result it follows that our solutions of (1.1) cannot be real analytic.

Ivanov, Kitamura, Kusano and Shevelo [3] (1982) investigated the higher order functional-differential equations of the form
\[
f^{(n)}(x) = p(x)F(f(g(x))),
\]
where \( p, F \) and \( g \) satisfy appropriate conditions. Kusano [6] (1984) also investigated the functional differential equation
\[
f^{(n)}(x) = p(x)f(g(x))
\]
where \( n \) is even, \( p : [0, \infty) \to \mathbb{R} \) and \( g : [0, \infty) \to \mathbb{R} \) are continuous, \( p(t) > 0 \), \( g(t) \) is nondecreasing and \( \lim_{t \to \infty} g(t) = \infty \). They [3, 6] gave sufficient conditions that the solutions are oscillatory.

If \( f \) is a solution of (1.1), then \( f \) is also a solution of the equations
\[
f''(x) = \lambda^4 \lambda f(\lambda^2 x), \quad x \in \mathbb{R},
\]
and
\[
f'''(x) = \lambda^6 \lambda^3 f(\lambda^3 x), \quad x \in \mathbb{R}.
\]
However, (1.5) and (1.6) don’t satisfy the sufficient conditions in [3, 6].

Recently, the author [9] constructed solutions of (1.1) with \( \lambda = 2 \) by using a little different method from this paper.

We state the main theorem (Theorem 2.3) and application in next section. We can easily apply the solution for the case \( \lambda = 2 \) to Friedrichs’ mollifier theorem and we can rewrite differential operator. For the proof of main theorem, see [10]. In the third section, we give graphs of solutions for the case of \( \lambda = 4, 3, 2, 31/16, 15/8, 7/4, 2/3, 5/4 \).
In the last section, we will give Mathematica programs.

2. MAIN RESULTS

First, we state two lemmas. Let

\[ f(x) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx, \quad \mathcal{F}^{-1}[f](\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{ix\xi} \, dx, \]

and

\[ \text{sinc} \, \xi = \begin{cases} \frac{\sin(\pi \xi)}{\pi \xi}, & \xi \neq 0, \\
1, & \xi = 0. \end{cases} \]

Lemma 2.1. The product

\[ \prod_{k=1}^{\infty} \text{sinc} \left( \frac{\xi}{2\lambda^{k}\pi} \right), \quad \xi \in \mathbb{R} \]

is converges pointwise and in \( L^{1}(\mathbb{R}) \).

Lemma 2.2. Let

(2.1) \quad u = \mathcal{F}^{-1}[U], \quad U(\xi) = \exp(-\frac{\mathrm{i}\xi}{2(\lambda-1)}) \prod_{k=1}^{\infty} \text{sinc} \left( \frac{\xi}{2\lambda^{k}\pi} \right).

Then \( u \) has the following properties:

\[ u \in C^\infty(\mathbb{R}), \]

\[ u(x) > 0 \text{ for } x \in \left(0, \frac{1}{\lambda - 1}\right), \quad u(x) = 0 \text{ for } x \notin \left(0, \frac{1}{\lambda - 1}\right), \]

\[ u(x) = u(1/(\lambda - 1) - x), \]

\[ \int_{\mathbb{R}} u(x) \, dx = 1, \]

and

(2.2) \quad u'(x) = \lambda^{2} u(\lambda x) \quad \text{for} \quad x \in \left[0, \min \left(\frac{1}{\lambda}, \frac{1}{\lambda(\lambda - 1)}\right)\right].

Let we define the operator \( T : L^{1} \to L^{1} \) as follows.

(2.3) \quad Tf(x) = \lambda \left(\chi_{[0,1]} * f\right)(\lambda x), \quad f \in L^{1}.

Then the function \( u \) in Lemma 2.2 is given by the following equation.

(2.4) \quad u = \lim_{k} T^{k} \chi_{[0,1]}.
Secondly, we define sequences $\{n_k\}_{k=1}^\infty$ and $\{y_k\}_{k=1}^\infty$ as follows:

$$\begin{cases}
n_1 = 0, & \quad \text{n}_2 = 1, \\
n_{2k-1} = 1, & \quad \text{n}_{2k} = 0, \quad \text{if} \quad n_k = 1 \quad (k \geq 2), \\
n_{2k-1} = 0, & \quad \text{n}_{2k} = 1, \quad \text{if} \quad n_k = 0 \quad (k \geq 2),
\end{cases} \tag{2.5}$$

and

$$y_k = \sum_{l=1}^\infty C_{k,l} \lambda^{l-1}, \quad k = 1, 2, 3, \ldots, \tag{2.6}$$

where $C_{k,l} \in \{0, 1\}$ ($l = 1, 2, 3, \ldots$) are coefficients of the binary system such that

$$k - 1 = \sum_{l=1}^\infty C_{k,l} 2^{l-1}, \quad k = 1, 2, 3, \ldots.$$ 

Then we have the following relations.

$$\begin{aligned}
(-1)^{n_{2k-1}} &= (-1)^{n_k}, \\
(-1) \cdot (-1)^{n_{2k}} &= (-1)^{n_k}, \\
k - 1 &= \sum_{l=1}^\infty C_{k,l} 2^{l-1}, \quad k = 1, 2, 3, \ldots.
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
y_{2k-1}/\lambda &= y_k, \\
y_{2k}/\lambda &= y_k + 1/\lambda,
\end{aligned} \tag{2.8}$$

and

$$y_k \geq \lambda^j \quad \text{if} \quad k - 1 \geq 2^j, \quad j = 0, 1, 2, \ldots. \tag{2.9}$$

Hence $\lim_{k \to \infty} y_k = \infty$. If $\lambda \geq 2$, then $y_k$ is strictly increasing. For example,

$$\begin{aligned}
\{n_k\}_{k=1}^\infty &= \{0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, \ldots\}, \\
\{y_k\}_{k=1}^\infty &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots\} \quad \text{for} \quad \lambda = 2, \\
\{y_k\}_{k=1}^\infty &= \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, 68, 69, 80, \ldots\} \quad \text{for} \quad \lambda = 4, \\
\{y_k\}_{k=1}^\infty &= \{0, 1, 0.3, 0.5, 0.9, 1.5, 19, 27, 35, 39, 47, 45, \ldots\} \quad \text{for} \quad \lambda = 3/2.
\end{aligned}$$

Our main result is the following:

**Theorem 2.3.** Let $\lambda > 1$. Then a solution $f$ of (1.1) can be found as

$$f(x) = \sum_{k=1}^\infty (-1)^{n_k} u(x - y_k),$$

where $u$, $\{n_k\}_{k=1}^\infty$ and $\{y_k\}_{k=1}^\infty$ are as in (2.1), (2.5) and (2.6), respectively. The solution $f$ is in $C^\infty(\mathbb{R})$ and $f(x) = 0$ for $x \leq 0$. If $\lambda \geq 2$, then $f$ is bounded.
Let we define a function space $L^{1,\nu}(\mathbb{R})$.

$$L^{1,\nu} = \{ f \in L_{loc}^{1}; ||f||_{L^{1,\nu}} < \infty \}$$

$$||f||_{L^{1,\nu}} = \sup_{r>0} \frac{1}{r^\nu} \int_{-r}^{+r} |f(x)| dx$$

**Theorem 2.4.** The solution $f$ of (1.1) is in $C^\infty \cap L^{1,1/\log_2 \lambda}$.

**Remark 2.1.** The solution of (1.1) is tempered distribution.

**Remark 2.2.** A constant times $f$ is also a solution.

**Theorem 2.5.** Let $f$ be the solution in Theorem 2.3 for $\lambda = 2$ and

$$G_{k,\epsilon}(x) = \left(2^{k(k-1)/2} \epsilon^{k+1}\right)^{-1} f(x/\epsilon).$$

If $v \in C^k(\mathbb{R})$ or $v \in L^p(\mathbb{R})$ ($k \geq 0$, $1 \leq p < \infty$), then

$$\frac{d^k v}{dx^k} = \lim_{\epsilon \to 0} v * G_{k,\epsilon},$$

uniformly on each compact subset in $\mathbb{R}$ or in $L^p(\mathbb{R})$, respectively.

**Remark 2.3.** $G_{k,\epsilon}$ is in $C^\infty(\mathbb{R})$ with compact support. To prove the theorem we use Friedrichs' molifier $\frac{d^k v}{dx^k} * u_\delta = v * \frac{d^k u_\delta}{dx^k}$, where $u_\delta = u(x/\delta)/\delta$, $\delta > 0$, and $u$ is the function in Lemma 2.2 ($\lambda = 2$).

3. **EXAMPLES**

In this section we give graphs for $\lambda = 4, 3, 2, 31/16, 15/8, 7/4, 3/2, 5/4$.

If $\lambda = 2$, then $\{ x > 0 : f(x) = 0 \} = \{ 1, 2, 3, \cdots \}$. If $\lambda > 2$, then $\{ x > 0 : f(x) = 0 \} = \bigcup_{k=1}^\infty [y_k + 1/(\lambda - 1), y_{k+1}]$ and its measure is infinity, since $1/(\lambda - 1) < 1 < y_{k+1} - y_k$, $k = 1, 2, 3, \cdots$. 
Figure 1. $u(\lambda = 4, 2, 3/2)$

Figure 2. $f'(x) = 4^2 f(4x)$

Figure 3. $f'(x) = 3^2 f(3x)$

4. Program of $u(x)$

Mathematica program (Part 1): The function $u$ [by using $u_{n+1} = T(u_n)$]
* Setting lambda (1<\text{lambda}<9)

In[1]: lam = 1.75;

* Calculation of the data ; u_{0}, \ldots , u_{50}

In[2]: udata[0] =

Table[If[0 < i - 10000 <= 1000, lam - 1, 0], {i, 1, 20000}];

In[3]: Timing[ Do[udata[k] = Table[If[1 <= j - 10000 <= 1000,
                lam * Sum[udata[k = 1] [[i + 10000]],
                {i,Round[lam * (j - 10000)] - Round[(lam - 1)* 1000} + 1,

FIGURE 4. $f'(x) = 2^2 f(2x)\\

FIGURE 5. $f'(x) = (31/16)^2 f(31x/16)\\

FIGURE 6. $f'(x) = (15/8)^2 f(15x/8)\\

FIGURE 7. $f'(x) = (7/4)^2 f(7x/4)\\

In[3]: Timing[ Do[udata[k] = Table[If[1 <= j - 10000 <= 1000,
                lam * Sum[udata[k = 1] [[i + 10000]],
                {i,Round[lam * (j - 10000)] - Round[(lam - 1)* 1000} + 1,
Figure 8. $f'(x) = (3/2)^2 f(3x/2)$

Figure 9. $f'(x) = (5/4)^2 f(5x/4)$

Round[$\lambda * (j - 10000) * 0.001 / (\lambda - 1)$, 0],
{j, 1, 20000}, {k, 1, 50}]

Out[3]: {121.14 Second, Null}

* Graph of $u_{\{50\}}$

In[4]: ulist[k_] := Table[((i - 10000) * 0.001/(\lambda - 1)),
            Part[udata[k], i], {i, 10000, 11000}]

In[5]: ListPlot[ulist[50], PlotJoined -> True,
           PlotRange -> {0, 1.1 * \lambda}]

* Save the data
In[6]: udata[50] >> c:/mdata/udata7ov4-50
In[7]: ulist[50] >> c:/mdata/ulist7ov4-50
In[8]: Export["c:/mdata/u7ov4.eps",
ListPlot[ulist[50], PlotJoined -> True,
PlotRange -> {0, 1.1 * lam}]]

5. PROGRAM OF F(X)

Mathematica program (Part 2): The solution f on the interval [0,tau]

* Setting lambda (1<lam<9) and tau
In[1]: lam = 1.75; tau = 30;
In[2]: kk = Round[Log[lam, tau]+ 0.5]
Out[2]: 7

* Load the data
In[3]: udata = << c:/mdata/udata7ov4-50;
In[4]: ud = Table[Part[udata, i], {i, 10000, 11000}];

* Sequences m_{k} and y_{k}
In[5]: m[1]= 0; m[2] = 1;
    Do[m[k]= If[Mod[k, 2]==0, Mod[m[k/2]+ 1, 2], m[(k + 1)/2]],
       {k, 3, 2^kk + 1}]
In[6]: Do[b[k, 1]= k - 1; Do[c[k, 1]= Mod[b[k, 1], 2];
       b[k, 1 + 1]= (b[k, 1] - c[k, 1])/2, {1, 1, kk + 1}],
       {k, 1, 2^kk + 1}]
In[7]: Do[y[k]=
       Sum[c[k, 1]* lam^(1 - 1), {1, 1, kk + 1}], {k, 1, 2^kk + 1}]

* Calculation of the solution
as the sum of (-1)^m_{k}u(x-y_{k}), k=1, 2, ... ,2^kk.
In[8]: Do[yy[k] = Round[y[k]* 1000 * (lam - 1)], {k, 1, 2^kk + 1}]
In[9]: zz[1] = Table[0, {i, 1, yy[2^kk]}]; Do[z[k] =
         Table[0, {i, 1, yy[k]}], {k, 2, 2^kk}];
Do[zz[k] = Table[0, {i, 1, yy[2^kk]- yy[k]}], {k, 2, 2^kk}];
In[10]: udy[1] = Join[ud, zz[1]]; Do[udy[k] =
         Join[z[k], ud* (-1)~m[k], zz[k]], {k, 2, 2^kk}]
In[11]: fd = Sum[udy[k], {k, 1, 2^kk}];

* Save the graph of the solution

In[12]: ii=tau*(lam-1)*1000;
In[13]: flist = Table[{i * 0.001/(lam - 1),
         Part[fd, i]}, {i, 1, ii}];
In[14]: Export["c:/mdata/f7ov4.eps",
         ListPlot[flist, PlotJoined -> True, AspectRatio -> Automatic]];


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