A solution of the equation $f'(x)=\lambda^2 f(\lambda x), \lambda > 1$ (Functional Equations and Complex Systems)

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A solution of the equation \( f'(x) = \lambda^2 f(\lambda x), \lambda > 1 \).

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1. INTRODUCTION

The purpose of this paper is to give solutions for the functional-differential equation of advanced type

\[
\begin{cases}
  f'(x) = \lambda^2 f(\lambda x), & x \in \mathbb{R} = (-\infty, +\infty), \\
  f(0) = 0,
\end{cases}
\]

where \( \lambda \) is a constant, \( \lambda > 1 \). Our solutions are infinitely differentiable on \( \mathbb{R} \). Moreover, if \( \lambda \geq 2 \), then the solutions are bounded and have arbitrarily large zeros. Our methods give numerical data readily.

Frederickson [1, 2] (1971) investigated functional-differential equations of advanced type

\[
f'(x) = af(\lambda x) + \lambda f(x),
\]


Frederickson [1] provided a global existence theorem for equations

\[
f'(x) = F(f(2x)), \quad x \in \mathbb{R},
\]

where \( F \) is an odd, continuous function with \( F(s) > 0 \) for \( s > 0 \), by application of the Schauder fixed point theorem. He showed that the absolute value of the solution \( |f(x)| \) is periodic for \( x \geq 0 \). Frederickson [2] also provided a constructive method for solutions for equations

\[
f'(z) = af(\lambda z) + bf(z),
\]
where \(a, b \in \mathbb{C}\) and \(\lambda > 1\). He further gave solutions in the form of a Dirichlet series
\[
\varphi(z, \beta) = \sum_{n \in \mathbb{Z}} c_ne^{\beta \lambda^n z}, \quad \Re(\beta z) \leq 0,
\]
where \(\beta\) is allowed to vary as a parameter. In the case of \(b = 0\) and \(\beta = i\), the solution is analytic in the upper half plane \(\Im z > 0\), continuous on \(\Im z \geq 0\), and the line \(\Im z = 0\) is a natural boundary. From his result it follows that our solutions of (1.1) cannot be real analytic.

Ivanov, Kitamura, Kusano and Shevelo [3] (1982) investigated the higher order functional-differential equations of the form
\[
f^{(n)}(x) = p(x)F(f(g(x))),
\]
where \(p, F\) and \(g\) satisfy appropriate conditions. Kusano [6] (1984) also investigated the functional differential equation
\[
f^{(n)}(x) = p(x)f(g(x))
\]
where \(n\) is even, \(p : [0, \infty) \to \mathbb{R}\) and \(g : [0, \infty) \to \mathbb{R}\) are continuous, \(p(t) > 0\), \(g(t)\) is nondecreasing and \(\lim_{t \to \infty} g(t) = \infty\). They [3, 6] gave sufficient conditions that the solutions are oscillatory.

If \(f\) is a solution of (1.1), then \(f\) is also a solution of the equations
\[
f''(x) = \lambda^4 \lambda f(\lambda^2 x), \quad x \in \mathbb{R},
\]
and
\[
f'''(x) = \lambda^6 \lambda^3 f(\lambda^3 x), \quad x \in \mathbb{R}.
\]
However, (1.5) and (1.6) don't satisfy the sufficient conditions in [3, 6].

Recently, the author [9] constructed solutions of (1.1) with \(\lambda = 2\) by using a little different method from this paper.

We state the main theorem (Theorem 2.3) and application in next section. We can easily apply the solution for the case \(\lambda = 2\) to Friedrichs' mollifier theorem and we can rewrite differential operator. For the proof of main theorem, see [10]. In the third section, we give graphs of solutions for the case of \(\lambda = 4, 3, 2, 31/16, 15/8, 7/4, 2/3, 5/4\).
In the last section, we will give Mathematica programs.

2. Main Results

First, we state two lemmas. Let

\[ f(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx, \quad \mathcal{F}^{-1}[f](\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{ix\xi} \, dx, \]

and

\[ \text{sinc}\, \xi = \begin{cases} \frac{\sin(\pi\xi)}{(\pi\xi)}, & \xi \neq 0, \\ 1, & \xi = 0. \end{cases} \]

Lemma 2.1. The product

\[ \prod_{k=1}^{\infty} \text{sinc} \left( \frac{\xi}{2\lambda^k \pi} \right), \quad \xi \in \mathbb{R} \]

is converges pointwise and in \( L^1(\mathbb{R}) \).

Lemma 2.2. Let

(2.1) \quad u = \mathcal{F}^{-1}[U], \quad U(\xi) = \exp\left( -\frac{i\xi}{2(\lambda-1)} \right) \prod_{k=1}^{\infty} \text{sinc} \left( \frac{\xi}{2\lambda^k \pi} \right). \]

Then \( u \) has the following properties:

\[ u \in C^\infty(\mathbb{R}), \]
\[ u(x) > 0 \quad \text{for} \quad x \in \left( 0, \frac{1}{\lambda-1} \right), \quad u(x) = 0 \quad \text{for} \quad x \notin \left( 0, \frac{1}{\lambda-1} \right), \]
\[ u(x) = u(1/(\lambda-1) - x), \]
\[ \int_{\mathbb{R}} u(x) \, dx = 1, \]

and

(2.2) \quad u'(x) = \lambda^2 u(\lambda x) \quad \text{for} \quad x \in \left[ 0, \min\left( \frac{1}{\lambda}, \frac{1}{\lambda(\lambda-1)} \right) \right]. \]

Let we define the operator \( T : L^1 \to L^1 \) as follows.

(2.3) \quad Tf(x) = \lambda \left( \chi_{[0,1]} * f \right)(\lambda x), \quad f \in L^1. \]

Then the function \( u \) in Lemma 2.2 is given by the following equation.

(2.4) \quad u = \lim_{k} T^k \chi_{[0,1]}.
Secondly, we define sequences \( \{n_k\}_{k=1}^{\infty} \) and \( \{y_k\}_{k=1}^{\infty} \) as follows:

\[
\begin{align*}
&n_1 = 0, \quad n_2 = 1, \\
&n_{2k-1} = 1, \quad n_{2k} = 0, \quad \text{if} \quad n_k = 1 \quad (k \geq 2), \\
&n_{2k-1} = 0, \quad n_{2k} = 1, \quad \text{if} \quad n_k = 0 \quad (k \geq 2),
\end{align*}
\]

and

\[
y_k = \sum_{l=1}^{\infty} C_{k,l} \lambda^{l-1}, \quad k = 1, 2, 3, \ldots,
\]

where \( C_{k,l} \in \{0, 1\} \) \( (l = 1, 2, 3, \ldots) \) are coefficients of the binary system such that

\[
k - 1 = \sum_{l=1}^{\infty} C_{k,l} 2^{l-1}, \quad k = 1, 2, 3, \ldots.
\]

Then we have the following relations.

\[
\begin{align*}
&(-1)^{n_{2k-1}} = (-1)^{n_k}, \\
&(-1) \cdot (-1)^{n_{2k}} = (-1)^{n_k},
\end{align*}
\]

and

\[
\begin{align*}
&y_{2k-1}/\lambda = y_k, \\
&y_{2k}/\lambda = y_k + 1/\lambda,
\end{align*}
\]

and

\[
y_k \geq \lambda^j \quad \text{if} \quad k - 1 \geq 2^j, \quad j = 0, 1, 2, \ldots.
\]

Hence \( \lim_{k \to \infty} y_k = \infty \). If \( \lambda \geq 2 \), then \( y_k \) is strictly increasing. For example,

\[
\begin{align*}
&\{n_k\}_{k=1}^{\infty} = \{0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, \ldots\}, \\
&\{y_k\}_{k=1}^{\infty} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots\} \quad \text{for} \quad \lambda = 2, \\
&\{y_k\}_{k=1}^{\infty} = \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, 68, 69, 80, \ldots\} \quad \text{for} \quad \lambda = 4, \\
&\{y_k\}_{k=1}^{\infty} = \{0, 1, 3/2, 4/4, 5/8, 9/16, 13/32, 15/64, 19/128, 27/256, 35/512, 39/1024, 47/2048, 45/4096, \ldots\} \quad \text{for} \quad \lambda = 3/2.
\end{align*}
\]

Our main result is the following:

**Theorem 2.3.** Let \( \lambda > 1 \). Then a solution \( f \) of (1.1) can be found as

\[
f(x) = \sum_{k=1}^{\infty} (-1)^{n_k} u(x - y_k),
\]

where \( u \), \( \{n_k\}_{k=1}^{\infty} \) and \( \{y_k\}_{k=1}^{\infty} \) are as in (2.1), (2.5) and (2.6), respectively. The solution \( f \) is in \( C^\infty(\mathbb{R}) \) and \( f(x) = 0 \) for \( x \leq 0 \). If \( \lambda \geq 2 \), then \( f \) is bounded.
Let we define a function space $L^{1,\nu}(\mathbb{R})$.

$$L^{1,\nu} = \{ f \in L_{loc}^{1}; ||f||_{L^{1,\nu}} < \infty \}$$

$$||f||_{L^{1,\nu}} = \sup_{r>0} \frac{1}{r^\nu} \int_{-r}^{+r} |f(x)| dx$$

**Theorem 2.4.** The solution $f$ of (1.1) is in $C^\infty \cap L^{1,1/\log_{2}\lambda}$.

**Remark 2.1.** The solution of (1.1) is tempered distribution.

**Remark 2.2.** A constant times $f$ is also a solution.

**Theorem 2.5.** Let $f$ be the solution in Theorem 2.3 for $\lambda = 2$ and

$$G_{k,\epsilon}(x) = (2^{k(k-1)/2} \epsilon^{k+1})^{-1} (f \chi_{[0,2^k]})(x/\epsilon).$$

If $v \in C^k(\mathbb{R})$ or $v \in L^p(\mathbb{R})$ ($k \geq 0$, $1 \leq p < \infty$), then

$$\frac{d^k v}{dx^k} = \lim_{\epsilon \to 0} v * G_{k,\epsilon},$$

uniformly on each compact subset in $\mathbb{R}$ or in $L^p(\mathbb{R})$, respectively.

**Remark 2.3.** $G_{k,\epsilon}$ is in $C^\infty(\mathbb{R})$ with compact support. To prove the theorem we use Friedrichs' molifier $\frac{d^k v}{dx^k} * u_\delta = v * \frac{d^k u_\delta}{dx^k}$, where $u_\delta = u(x/\delta)/\delta$, $\delta > 0$, and $u$ is the function in Lemma 2.2 ($\lambda = 2$).

3. Examples

In this section we give graphs for $\lambda = 4, 3, 2, 31/16, 15/8, 7/4, 3/2, 5/4$.

If $\lambda = 2$, then $\{ x > 0 : f(x) = 0 \} = \{ 1, 2, 3, \cdots \}$. If $\lambda > 2$, then $\{ x > 0 : f(x) = 0 \} = \bigcup_{k=1}^{\infty} [y_k + 1/(\lambda - 1), y_{k+1}]$ and its measure is infinity, since $1/(\lambda - 1) < 1 < y_{k+1} - y_{k}$, $k = 1, 2, 3, \cdots$. 
4. Program of $u(x)$

Mathematica program (Part 1): The function $u$ [ by using $u_{n+1} = T(u_n)$ ]
FIGURE 4. $f'(x) = 2^2 f(2x)$

FIGURE 5. $f'(x) = (31/16)^2 f(31x/16)$

FIGURE 6. $f'(x) = (15/8)^2 f(15x/8)$

FIGURE 7. $f'(x) = (7/4)^2 f(7x/4)$

* Setting lambda ($i < \text{lam} < 9$)

In[1]: \[\text{lam} = 1.75;\]

* Calculation of the data; \[u\{0\}, \ldots, u\{50\}\]

In[2]: \[\text{udata}[0] = \text{Table}[\text{If}[0 < i - 10000 \leq 1000, \text{lam} - 1, 0], \{i, 1, 20000\}];\]

In[3]: \[\text{Timing[Do[udata}[k] = \text{Table}[\text{If}[1 \leq j - 10000 \leq 1000, \text{lam} \cdot \text{Sum}[\text{udata}[k = 1]][i + 10000], \{i, \text{Round}[\text{lam} \cdot (j - 10000)] - \text{Round}[\text{(lam} - 1) \cdot 1000] + 1,\]

\]}]}
FIGURE 8. \( f'(x) = (3/2)^2 f(3x/2) \)

FIGURE 9. \( f'(x) = (5/4)^2 f(5x/4) \)

Round[\text{lam} \times (j - 10000)]; \{0.001/(\text{lam} - 1), 0]\,
\{j, 1, 20000\}, \{k, 1, 50\}]

Out[3]: \{121.14 \text{ Second, Null}\}

* Graph of \( u_{50} \)

In[4]: \text{ulist}[k_] := \text{Table}[\{(i - 10000) \times 0.001/(\text{lam} - 1),}
\text{Part[udata}[k], i]}\}, \{i, 10000, 11000\}]

In[5]: \text{ListPlot[ulist[50], PlotJoined -> True,}
\text{PlotRange} \rightarrow \{0, 1.1 \times \text{lam}\}]

* Save the data
In[6]: udata[50] >> c:/mda/u7ov4-50
In[7]: ulist[50] >> c:/mda/ulist7ov4-50
In[8]: Export["c:/mda/u7ov4.eps",
   ListPlot[ulist[50], PlotJoined -> True,
   PlotRange -> {0, 1.1 * lam}]]

5. PROGRAM OF F(X)

Mathematica program (Part 2): The solution f on the interval [0,tau]

* Setting lambda (1<lam<9) and tau
In[1]: lam = 1.75; tau = 30;
In[2]: kk = Round[Log[lam, tau] + 0.5]
Out[2]: 7

* Load the data
In[3]: udata = << c:/mda/u7ov4-50;
In[4]: ud = Table[Part[udata, i], {i, 10000, 11000}];

* Sequences m_{k} and y_{k}
In[5]: m[1] = 0; m[2] = 1;
   Do[m[k] = If[Mod[k, 2]==0, Mod[m[k /2]+ 1, 2], m[(k + 1)/2]],
   {k, 3, 2^kk + 1}]
In[6]: Do[b[k, 1] = k - 1; Do[c[k, l] = Mod[b[k, l], 2];
   b[k, l + 1] = (b[k, l] - c[k, l])/2, {l, 1, kk + 1}],
   {k, 1, 2^kk + 1}]
In[7]: Do[y[k] =
   Sum[c[k, l]* lam^(l - 1), {l, 1, kk + 1}], {k, 1, 2^kk + 1}]

* Calculation of the solution
   as the sum of (-1)^{m_{k}}u(x-y_{k}), k=1, 2, ..., 2^kk.
In[8]: Do[yy[k] = Round[y[k]*1000*(lam - 1)], {k, 1, 2^kk + 1}]

In[9]: zz[1] = Table[0, {i, 1, yy[2^kk]}]; Do[z[k] =
    Table[0, {i, 1, yy[k]}], {k, 2, 2^kk}];
    Do[zz[k] = Table[0, {i, 1, yy[2^kk] - yy[k]}], {k, 2, 2^kk}];

In[10]: udy[1] = Join[ud, zz[1]]; Do[udy[k] =
    Join[z[k], ud* (-1)^m[k], zz[k]], {k, 2, 2^kk}]

In[11]: fd = Sum[udy[k], {k, 1, 2^kk}];

* Save the graph of the solution

In[12]: ii = tau*(lam - 1)*1000;
In[13]: flist = Table[{i*0.001/(lam - 1),
    Part[fd, i]}, {i, 1, ii}];
In[14]: Export["c:/mda/7ov4.eps",
    ListPlot[flist, PlotJoined -> True, AspectRatio -> Automatic]];

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