Stability Properties for Linear Volterra Difference Equations
with Convolution Kernels in a Banach Space (Functional Equations and Complex Systems)

Murakami, Satoru; Nagabuchi, Yutaka

数理解析研究所講究録 (2005), 1445: 143-153

http://hdl.handle.net/2433/47629

Departmental Bulletin Paper
Stability Properties for Linear Volterra Difference Equations with Convolution Kernels in a Banach Space

Satoru Murakami

Department of Applied Mathematics,
Okayama University of Science

Yutaka Nagabuchi

Department of Applied Science,
Okayama University of Science

1. INTRODUCTION

We consider the Volterra difference equations of convolution type

\[ x(n + 1) = \sum_{r=0}^{n} B(n - r)x(r), \quad n \in \mathbb{Z}^+, \quad (E_0) \]

and

\[ y(n + 1) = \sum_{r=-\infty}^{n} B(n - r)y(r), \quad n \in \mathbb{Z}, \quad (E_\infty) \]

where \( B(n) \) \((n \in \mathbb{Z}^+, \) the nonnegative integers) are bounded linear operators on a Banach space \( X \) over the field \( \mathbb{C} \). The study of Volterra difference equations has actively been done. Indeed, in the case where \( X \) is of finite dimension, the equations have extensively been treated in the book [1] and some results on stability properties and so on were obtained; for more details we refer the reader to [1, 2, 3] and the references therein.

Also, in [4, 5], Volterra difference equations with infinite dimensional \( X \) were discussed in connection with some partial differential equations with piecewise continuous delays, and uniform asymptotic stability for \( (E_\infty) \) was investigated in connection with the invertibility of the characteristic operator together with the summability of the fundamental solution, under additional conditions such as the mutual commutativity of the operators \( B(n) \), \( n \in \mathbb{Z}^+ \) or the exponential decay of the norm \( ||B(n)||\).

In this paper, we give a nice result on the stability properties of the zero solution of \( (E_0) \) or \( (E_\infty) \) in the context above. Indeed, without the additional conditions imposed in [4, 5], we will establish an equivalence relation among the uniform asymptotic stability of the zero solution of \( (E_0) \) or \( (E_\infty) \), the summability of the fundamental solution and the invertibility of the characteristic operator outside the unit circle in the complex plane.

*Partly supported by the Grant-in-Aid for Scientific Research (C), No.16540177, of the Japanese Ministry of Education, Culture, Sports, Science and Technology.
2. NOTATIONS

Let $X$ be a (complex) Banach space with the norm $|\cdot|$. We denote by $\mathcal{L}(X)$ the space of all bounded linear operators on $X$. Clearly, $\mathcal{L}(X)$ is a Banach space equipped with the operator norm $||\cdot||$, which is defined by

$$||T|| = \sup\{|Tx| : x \in X, |x| = 1\}$$

for any $T \in \mathcal{L}(X)$.

For any interval $J \subset \mathbb{R}$ we use the same notation $J$ meaning the discrete one $J \cap \mathbb{Z}$, e.g. $[0, \sigma) = \{0, 1, \ldots, \sigma\}$ for $\sigma \in \mathbb{Z}^+$. Also, for an $X$-valued function $\xi$ on a discrete interval $J$, its norm is denoted by $||\xi||_J := \sup\{|\xi(j)| : j \in J\}$. Let $\sigma \in \mathbb{Z}^+$ and a function $\phi : [0, \sigma] \rightarrow X$ be given. We denote by $x(n;\sigma, \phi)$ the solution $x(n)$ of $(E_0)$ satisfying $x(n) = \phi(n)$ on $[0, \sigma]$. Similarly, for $\tau \in \mathbb{Z}$ and a function $\psi : (\infty, \tau] \rightarrow X$, we denote by $y(n;\tau, \psi)$ the solution $y(n)$ of $(E_\infty)$ satisfying $y(n) = \psi(n)$ on $(-\infty, \tau]$.

**Definition 1.** The zero solution of $(E_0)$ is said to be

(i) uniformly stable if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $\sigma \in \mathbb{Z}^+$ and $\phi$ is an initial function on $[0, \sigma]$ with $||\phi||_{[0, \sigma]} < \delta$ then $|x(n;\sigma, \phi)| < \varepsilon$ for all $n \geq \sigma$.

(ii) uniformly asymptotically stable if it is uniformly stable, and if there exists a $\mu > 0$ such that, for any $\varepsilon > 0$ there exists an $N = N(\varepsilon) \in \mathbb{Z}^+$ with the property that, if $\sigma \in \mathbb{Z}^+$ and $\phi$ is an initial function on $[0, \sigma]$ with $||\phi||_{[0, \sigma]} < \mu$ then $|x(n;\sigma, \phi)| < \varepsilon$ for all $n \geq \sigma + N$.

**Definition 2.** The zero solution of $(E_\infty)$ is said to be

(i) uniformly stable if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $\tau \in \mathbb{Z}$ and $\psi$ is an initial function on $(-\infty, \tau]$ with $||\psi||_{(-\infty, \tau]} < \delta$ then $|y(n;\tau, \psi)| < \varepsilon$ for all $n \geq \tau$.

(ii) uniformly asymptotically stable if it is uniformly stable, and if there exists a $\mu > 0$ such that, for any $\varepsilon > 0$ there exists an $N = N(\varepsilon) \in \mathbb{Z}^+$ with the property that, if $\tau \in \mathbb{Z}$ and $\psi$ is an initial function on $(-\infty, \tau]$ with $||\psi||_{(-\infty, \tau]} < \mu$ then $|y(n;\tau, \psi)| < \varepsilon$ for all $n \geq \tau + N$.

The fundamental solution of $(E_0)$ is a family in $\mathcal{L}(X)$ satisfying the relation

$$R(n + 1) = \sum_{j=0}^{n} B(n - j)R(j), \quad n \in \mathbb{Z}^+$$

and $R(0) = I$. Then, for instance, the solution $y(n;\tau, \psi)$ of $(E_\infty)$ is given by the variation of constant formula as follows:

$$y(n;\tau, \psi) = R(n - \tau)\psi(\tau) + \sum_{r=\tau}^{n-1} R(n - r - 1) \left( \sum_{s=-\infty}^{r-1} B(r - s)\psi(s) \right). \quad (1)$$
3. **Main Results**

In what follows, we assume that \( B := \{B(n)\} \subset \mathcal{L}(X) \) is summable, that is, the condition \( \sum_{n=0}^{\infty} \|B(n)\| < \infty \) holds, and study stability properties of the zero solution of Eq. \((E_{\infty})\), together with those of the zero solution of Eq. \((E_{0})\). Here and subsequently, \( \hat{B}(z) \) denotes the \(Z\)-transform of \( B \); that is, \( \hat{B}(z) := \sum_{n=0}^{\infty} B(n) z^{-n} \) for \( |z| \geq 1 \).

In [5, Theorem 2] and [4, Theorem 2], the equivalence among the uniform asymptotic stability of the zero solution of Eq. \((E_{\infty})\), the summability of the fundamental solution \( R = \{R(n)\} \) of Eq. \((E_{0})\), and the invertibility of the characteristic operator \( zI - \hat{B}(z) \) associated with Eq. \((E_{0})\) has been established under some restrictions such as the mutual commutativity of the operators \( B(n) \), \( n \in \mathbb{Z}^{+} \) or the exponential decay of the norm \( \|B(n)\| \). We will show in the following theorem that [5, Theorem 2] and [4, Theorem 2] hold true without such restrictions.

**Theorem 1.** Let \( B = \{B(n)\}_{n=0}^{\infty} \in l^{1}(\mathbb{Z}^{+}) := l^{1}(\mathbb{Z}^{+}; \mathcal{L}(X)) \), and assume that \( B(n) \), \( n \in \mathbb{Z}^{+} \), are all compact. Then the following statements are equivalent.

(i) The zero solution of Eq. \((E_{0})\) is uniformly asymptotically stable.

(ii) The zero solution of Eq. \((E_{\infty})\) is uniformly asymptotically stable.

(iii) \( R = \{R(n)\}_{n=0}^{\infty} \in l^{1}(\mathbb{Z}^{+}) \).

(iv) For any \( z \) such that \( |z| \geq 1 \), the operator \( zI - \hat{B}(z) \) is invertible in \( \mathcal{L}(X) \).

In order to prove the theorem, we need the following preparatory results.

**Proposition 1.** Let \( K = \{K(n)\}_{n=-\infty}^{\infty} \in l^{1}(\mathbb{Z}) := l^{1}(\mathbb{Z}; \mathcal{L}(X)) \), and assume that \( I - \tilde{K}(\rho) \) is invertible for each \( \rho \in \mathbb{R} \), where \( \tilde{K}(\rho) := \sum_{n=-\infty}^{\infty} K(n) e^{-i\rho n} \). Then there is an \( R \in l^{1}(\mathbb{Z}) \) such that

\[
\tilde{K}(\rho)(I - \tilde{K}(\rho))^{-1} = \tilde{R}(\rho), \quad \forall \rho \in \mathbb{R}.
\]

**Proof.** (1-Step) For each (small) \( \epsilon > 0 \) we define a \(2\pi\)-periodic function \( \tilde{\phi}_{\epsilon} \) by

\[
\tilde{\phi}_{\epsilon}(t) = \begin{cases}
1 & (|t| \leq \epsilon) \\
0 & (2\epsilon \leq |t| \leq \pi) \\
(2\epsilon - t)/\epsilon & (\epsilon < t < 2\epsilon) \\
(2\epsilon + t)/\epsilon & (-2\epsilon < t < -\epsilon).
\end{cases}
\]

One can easily check that Fourier coefficients of \( \tilde{\phi}_{\epsilon} \) are given by

\[
d_{l} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\phi}_{\epsilon}(t)e^{ilt} dt = \begin{cases}
\frac{2}{\pi\epsilon^{2}} \sin \frac{3\epsilon l}{2} \sin \frac{l}{2} & (l = \pm1, \pm2, \ldots) \\
\frac{3\epsilon}{2\pi} & (l = 0).
\end{cases}
\]
Clearly, the sequence $\phi_{\epsilon} := \{d_{l}\}_{l=-\infty}^{\infty}$ is summable, and by the Fourier expansion theorem we get
\[
\sum_{l=-\infty}^{\infty} d_{l}e^{-itl} = \phi_{\epsilon}(t), \quad \forall t \in \mathbb{R}.
\]
Hence it follows that for any $t_{0} \in \mathbb{R}$,
\[
\tilde{\phi}_{\epsilon}(t - t_{0}) \equiv \sum_{l=-\infty}^{\infty} c_{l}e^{-ilt},
\]
where $c := \{c_{l}\}_{l \in \mathbb{Z}}$ is a sequence defined by $c_{l} = d_{l}e^{it_{0}l} = \phi_{\epsilon}(l)e^{it_{0}l}$ for $l \in \mathbb{Z}$. Notice that the sequence $c$ is summable.

(2-Step) Consider the function $f(x)$ defined by
\[
f(x) = \begin{cases} 
\frac{1}{\pi x} \sin 3x \sin x & (x \neq 0) \\
\frac{3}{\pi} & (x = 0).
\end{cases}
\]
One can easily see that $f$ is continuously differentiable. In fact, $f'(x)$ is given by
\[
f'(x) = \begin{cases} 
-\frac{2}{\pi x^{3}} \sin 3x \sin x + \frac{1}{\pi x^{2}}(3 \cos 3x \sin x + \sin 3x \cos x) & (x \neq 0) \\
0 & (x = 0).
\end{cases}
\]
Since $\lim_{|x| \to \infty}(|f(x)| + |f'(x)|) = 0$, there exists a constant $H > 0$ such that $\sup_{-\infty < x < \infty}(|f(x)| + |f'(x)|) = H$. Moreover,
\[
\int_{0}^{\infty} |f'(x)| \, dx = \int_{0}^{1} |f'(x)| \, dx + \int_{1}^{\infty} |f'(x)| \, dx \\
\leq H + \int_{1}^{\infty} \frac{6}{\pi x^{2}} \, dx \\
\leq H + C,
\]
where $C = 6/\pi$.

(3-Step) Put $M = \sup_{\rho \in \mathbb{R}} \|(I - \tilde{K}(\rho))^{-1}\|$, and take a positive integer $N$ such that
\[
\frac{23M}{\pi} \sum_{|\tau| \geq N+1} \|K(\tau)\| \leq \frac{1}{4}.
\]
Moreover, take a positive integer $k_{0}$, $k_{0} \geq 3$, such that
\[
2NM(H + C)\pi \sum_{l \in \mathbb{Z}} \|K(l)\| < \frac{3k_{0}}{4},
\]
and set $\epsilon = \pi/(3k_{0})$ and $\rho_{n} = 3n\epsilon$, $n = 0, 1, \ldots$. Then $\rho_{2k_{0}} = 2\pi$, and the following relation holds:
\[
\sum_{n=0}^{2k_{0}-1} \phi_{\epsilon}(\rho - \rho_{n}) \equiv 1, \quad \forall \rho \in \mathbb{R}.
\]
where \( \tilde{\phi}_\epsilon \) is the one introduced in 1-Step. Set \( F(\rho) = \tilde{K}(\rho)(I - \tilde{K}(\rho))^{-1} \) and \( F_n(\rho) = \tilde{\phi}_\epsilon(\rho - \rho_n)F(\rho), \ \rho \in \mathbb{R} \). Then

\[
F(\rho) \equiv \sum_{n=0}^{2k_0-1} F_n(\rho).
\]

Therefore, in order to establish the proposition it suffices only to certify that for each \( n \) there exists an \( R_n \in l^1(\mathbb{Z}) \) such that \( F_n(\rho) \equiv \tilde{R}_n(\rho) \). We now set

\[
K_n(l) = \left[ \left( (\phi_{2\epsilon}e^{i\rho_n}) * K \right)(l) - \phi_{2\epsilon}(l)e^{i\rho_n} \tilde{K}(\rho_n) \right] (I - \tilde{K}(\rho_n))^{-1}, \quad l \in \mathbb{Z},
\]

where \( * \) denotes the convolution in \( l^1(\mathbb{Z}) \). Then \( K_n \in l^1(\mathbb{Z}) \), and moreover

\[
\tilde{K}_n(\rho) = \left[ (\phi_{2\epsilon}e^{i\rho_n})(\rho) \tilde{K}(\rho) - (\phi_{2\epsilon}e^{i\rho_n})(\rho) \tilde{K}(\rho_n) \right] (I - \tilde{K}(\rho_n))^{-1}
\]

\[
= \phi_{2\epsilon}(\rho - \rho_n)(\tilde{K}(\rho) - \tilde{K}(\rho_n))(I - \tilde{K}(\rho_n))^{-1}.
\]

Observe that \( \tilde{\phi}_\epsilon(\rho - \rho_n) \neq 0 \) implies \( \phi_{2\epsilon}(\rho - \rho_n) = 1 \), and hence

\[
\tilde{K}_n(\rho) = (\tilde{K}(\rho) - \tilde{K}(\rho_n))(I - \tilde{K}(\rho_n))^{-1}
\]

\[
= (\tilde{K}(\rho) - I)(I - \tilde{K}(\rho_n))^{-1} + I,
\]

or

\[
I - \tilde{K}_n(\rho) = (I - \tilde{K}(\rho))(I - \tilde{K}(\rho_n))^{-1},
\]

which implies

\[
(I - \tilde{K}(\rho))^{-1} = (I - \tilde{K}(\rho_n))^{-1}(I - \tilde{K}(\rho_n))^{-1}.
\]

This observation leads to

\[
F_n(\rho) \equiv \tilde{\phi}_\epsilon(\rho - \rho_n)\tilde{K}(\rho)(I - \tilde{K}(\rho))^{-1}
\]

\[
\equiv \tilde{\phi}_\epsilon(\rho - \rho_n)\tilde{K}(\rho)(I - \tilde{K}(\rho_n))^{-1}(I - \tilde{K}_n(\rho))^{-1}.
\]

We claim that

\[
|K_n|_1 := \sum_{l=-\infty}^{\infty} \|K_n(l)\| < \frac{1}{2}, \quad (2)
\]

If the claim is true, then the series \( \sum_{r=0}^{\infty} K_n^{*r} := e + K_n + K_n * K_n + K_n * K_n * K_n + \cdots \) (here \( e \) is the unit element in \( l^1(\mathbb{Z}) \)), converges in \( l^1(\mathbb{Z}) \) with \( (I - \tilde{K}_n(\rho))^{-1} \equiv (\sum_{r=0}^{\infty} K_n^{*r})(\rho) \), and hence we may set \( R_n = (\phi_{2\epsilon}e^{i\rho_n}) * K * \{(I - \tilde{K}(\rho_n))^{-1}\sum_{r=0}^{\infty} K_n^{*r}\} \) to get the equality \( F_n = \tilde{R}_n \) with \( R_n \in l^1(\mathbb{Z}) \).
In what follows we will evaluate $|K_n|_1$ to establish (2). It follows that

$$|K_n|_1 \leq M \sum_{l=-\infty}^{\infty} \| \sum_{\tau=-\infty}^{\infty} K(\tau)(\phi_{2\epsilon}(l-\tau) e^{i\rho_n(l-\tau)} - \phi_{2\epsilon}(l) e^{i\rho_n l}) \|$$

$$\leq M \sum_{l=1}^{N} \| K(\tau) \| \sum_{l=-\infty}^{\infty} |\phi_{2\epsilon}(l-\tau) - \phi_{2\epsilon}(l)|$$

$$+ M \sum_{|\tau| \geq N+1} \| K(\tau) \| \sum_{l=-\infty}^{\infty} |\phi_{2\epsilon}(l-\tau) - \phi_{2\epsilon}(l)|$$

$$=: I_1 + I_2.$$ 

Noting $0 < \epsilon < 1/2$, we get

$$I_2 \leq 2M \sum_{|\tau| \geq N+1} \| K(\tau) \| \times \sum_{l=-\infty}^{\infty} |\phi_{2\epsilon}(l)|$$

$$\leq 2M \sum_{|\tau| \geq N+1} \| K(\tau) \| \times \left( \frac{3\epsilon}{\pi} + \sum_{k=1}^{\infty} \frac{2}{\pi k^2 \epsilon} |\sin 3\epsilon k \sin \epsilon k| \right)$$

$$\leq 2M \sum_{|\tau| \geq N+1} \| K(\tau) \| \times \left( \frac{3\epsilon}{\pi} + \frac{2}{\pi \epsilon} \sum_{k=1}^{[1/\epsilon]} \frac{1}{k^2} |\sin 3\epsilon k \sin \epsilon k| + \frac{2}{\pi} \sum_{k=1/\epsilon}^{\infty} \frac{1}{k^2} \right)$$

$$\leq 2M \sum_{|\tau| \geq N+1} \| K(\tau) \| \times \left( \frac{3\epsilon}{\pi} + \frac{6\epsilon}{\pi} \left[ \frac{1}{\epsilon} \right] + \frac{2}{\pi \epsilon} \left[ \frac{1}{\epsilon} \right] \right)$$

$$\leq \frac{23M}{\pi} \sum_{|\tau| \geq N+1} \| K(\tau) \|$$

$$\leq \frac{1}{4},$$

where $\left[ \frac{1}{\epsilon} \right]$ denotes the largest integer which does not exceed $1/\epsilon$. Also, using the function $f$ introduced in 2-Step we get

$$I_1 \leq M|K|_1 \sup_{1 \leq |\tau| \leq N} \left( \sum_{l=-\infty}^{\infty} |f((l-\tau)\epsilon) - f(l\epsilon)| \epsilon \right)$$

$$\leq M\epsilon|K|_1 \sup_{1 \leq |\tau| \leq N} \left( \sum_{m=0}^{[\tau]-1} \sum_{s=-\infty}^{\infty} |f((s+1)\epsilon |\tau| + m\epsilon) - f((s+1)\epsilon |\tau| + m\epsilon)| \right)$$

$$\leq M\epsilon|K|_1 \sup_{1 \leq |\tau| \leq N} \left( \sum_{m=0}^{[\tau]-1} \sum_{s=-\infty}^{[\tau]-1} \int_{(s+1)\epsilon |\tau| + m\epsilon}^{(s+1)\epsilon |\tau| + m\epsilon} |f'(x)| dx \right)$$
\[ M \leq |K|_1 \sup_{1 \leq |r| \leq N} \left( \sum_{m=0}^{\lfloor r \rfloor - 1} \int_{-\infty}^{\infty} |f'(x)| \, dx \right) \]
\[ \leq M \epsilon |K|_1 N \int_{-\infty}^{\infty} |f'(x)| \, dx \]
\[ < \frac{2(H + C)MN \pi |K|_1}{3k_0} \]
\[ < \frac{1}{4}. \]

Thus \(|K_n|_1 \leq I_1 + I_2 < 1/4 + 1/4 = 1/2\), as required. \(\square\)

**Proposition 2.** Let \( B = \{B(n)\}^{\infty}_{n=0} \in l^1(\mathbb{Z}^+) \), and assume that \( I - B(0) \) is invertible and that \( I - \hat{B}(z) \) is invertible for each \( z \in \mathbb{C} \) with \(|z| \geq 1\), where \( \hat{B}(z) := \sum_{n=0}^{\infty} B(n) z^{-n} \).

Then there is an \( R \in l^1(\mathbb{Z}^+) \) such that

\[ \hat{B}(z)(I - \hat{B}(z))^{-1} = \hat{R}(z), \quad \forall |z| \geq 1. \]

**Proof.** Consider the sequence \( B' \in l^1(\mathbb{Z}) \) defined by \( B'(n) = B(n) \) if \( n \geq 0 \), and \( B'(n) = 0 \) if \( n < 0 \). Then

\[ I - \tilde{B}'(\rho) = I - \sum_{n=0}^{\infty} B'(n) e^{-i\rho n} = I - \hat{B}(e^{i\rho}), \quad \forall \rho \in \mathbb{R}. \]

Hence \( I - \hat{B}'(\rho) \) is invertible for each \( \rho \in \mathbb{R} \), and consequently there exists a \( Q \in l^1(\mathbb{Z}) \) such that \( \hat{B}'(\rho)(I - \hat{B}'(\rho))^{-1} = \tilde{Q}(\rho), \forall \rho \in \mathbb{R} \), by Proposition 1. Define an element \( Q_+ \) in \( l^1(\mathbb{Z}^+) \) by \( Q_+(n) = Q(n) \) for any \( n \in \mathbb{Z}^+ \). The function \( \tilde{Q}_+(z) \) is bounded and continuous on the domain \(|z| \geq 1\), and it is analytic on \(|z| > 1\). Similarly, the function \( \sum_{n=1}^{\infty} Q(-n) z^n \) is bounded and continuous on the domain \(|z| \leq 1\), and it is analytic on \(|z| < 1\). Moreover, if \(|z| = 1\) with \( z = e^{i\rho} \), then

\[ \hat{B}(z)(I - \hat{B}(z))^{-1} - \tilde{Q}_+(z) = \hat{B}'(\rho)(I - \hat{B}'(\rho))^{-1} - \sum_{n=0}^{\infty} Q(n) e^{-i\rho n} \]
\[ = \tilde{Q}(\rho) - \sum_{n=0}^{\infty} Q(n) e^{-i\rho n} \]
\[ = \sum_{n=-\infty}^{1} Q(n) e^{-i\rho n} \]
\[ = \sum_{n=1}^{\infty} Q(-n) e^{i\rho n} \]
\[ = \sum_{n=1}^{\infty} Q(-n) z^n. \]
Therefore, the function $G(z)$ defined by

$$G(z) = \begin{cases} \hat{B}(z)(I - \hat{B}(z))^{-1} - \hat{Q}_+(z) & (|z| \geq 1) \\ \sum_{n=1}^{\infty} Q(-n)z^n & (|z| < 1) \end{cases}$$

is analytic on the entire domain by Morera's theorem. Observe that $I - \hat{B}(z) \rightarrow I - B(0)$ in $\mathcal{L}(X)$ as $|z| \rightarrow \infty$. Since $I - B(0)$ is invertible by the assumption, it follows that $\lim_{|z|\rightarrow\infty} \|(I - \hat{B}(z))^{-1}\| = \|(I - B(0))^{-1}\|$, and consequently $\sup_{|z|\geq 1} \|(I - \hat{B}(z))^{-1}\| < \infty$. Therefore, $G(z)$ is bounded on the entire domain, and hence $G(z)$ is a constant function by Liouville's theorem. Then $G(z) \equiv G(0) = 0$, and hence it follows that $\hat{B}(z)(I - \hat{B}(z))^{-1} = \hat{Q}_+(z)$ for any $z$ with $|z| \geq 1$. Thus we may set $Q_1 = R$ to establish the proposition.

We are now in a position to prove the theorem.

Clearly, the implication $[(ii) \implies (i)]$ holds true. Also, the implications $[(iii) \implies (ii)]$ and $[(ii) \implies (iv)]$ have already been proved in [5, Theorem 2] and [4, Theorem 2]. In what follows, we will prove the implications $[(iv) \implies (iii)]$ and $[(i) \implies (ii)]$.

Proof of $[(iv) \implies (iii)]$. Let us consider the sequence $D \in l^1(\mathbb{Z}^+)$ defined by $D(n) = B(n - 1)$ if $n \geq 1$, and $D(n) = 0$ if $n = 0$. Clearly, $I - D(0)$ is invertible. For any $z \in \mathbb{C}$ with $|z| \geq 1$, we get

$$\hat{D}(z) = \sum_{n=0}^{\infty} D(n)z^{-n} = z^{-1}\hat{B}(z),$$

and hence

$$I - \hat{D}(z) = \frac{1}{z}(zI - \hat{B}(z)).$$

Thus $I - \hat{D}(z)$ is invertible for each $z \in \mathbb{C}$ with $|z| \geq 1$, and it satisfies the relation

$$(I - \hat{D}(z))^{-1} = z(zI - \hat{B}(z))^{-1}, \quad |z| \geq 1.$$ 

By virtue of Proposition 2, there exists a $Q \in l^1(\mathbb{Z}^+)$ such that $\hat{D}(z)(I - \hat{D}(z))^{-1} = \hat{Q}(z)$, $|z| \geq 1$, and hence we get

$$I + \hat{Q}(z) = I + \hat{D}(z)(I - \hat{D}(z))^{-1}$$

$$= (I - \hat{D}(z))^{-1}$$

$$= z(zI - \hat{B}(z))^{-1}$$

for all $|z| \geq 1$. Consider the sequence $S = \{S(n)\}_{n=0}^{\infty}$ defined by

$$S(n) = \begin{cases} I + Q(0) & (n = 0) \\ Q(n) & (n \geq 1). \end{cases}$$
Then $S \in l^1(\mathbb{Z}^+)$, and \( \hat{S}(z) = I + \hat{Q}(z) = z(zI - \hat{B}(z))^{-1} \) for all $|z| \geq 1$. Notice that the fundamental solution $R$ is bounded exponentially, that is, $\sup_{n \geq 0} e^{-n\omega} \|R(n)\| < \infty$ for some constant $\omega \geq 0$. Hence the $Z$-transform $\sum_{n=0}^{\infty} R(n)z^{-n}$ of $R$ converges for $|z| > e^\omega$. Let us consider the $Z$-transform of both sides in the equation $R(n+1) = \sum_{k=0}^{\infty} B(n-k)R(k)$ with $R(0) = I$ to get the relation $z(\hat{R}(z) - I) = \hat{B}(z)\hat{R}(z)$, or $(zI - \hat{B}(z))\hat{R}(z) = zI$ for $|z| > e^\omega$. Thus it follows that $\hat{R}(z) = z(zI - \hat{B}(z))^{-1} = \hat{S}(z)$ for all $|z| > e^\omega$. By the uniqueness of the $Z$-transform, we get $R(n) \equiv S(n)$, $n \in \mathbb{Z}^+$, which shows the summability of $R$, as required.

**Proof of [(i)$\Rightarrow$ (ii)].** Let $\tau \in \mathbb{Z}$, and $\phi, \psi : (-\infty, \tau] \to X$ be given in such a way that

$$||\phi||_{(-\infty,\tau]} < \delta(\epsilon/2) \quad \text{and} \quad ||\psi||_{(-\infty,\tau]} < \min\{\delta(1/2), \mu\},$$

where $\delta(\cdot)$ and $\mu$ are those in Definition 1. Let us take a sequence $\{n_j\} \subset \mathbb{Z}^+$ such that $n_j \to \infty$ ($j \to \infty$). We may assume that $\tau + n_j > 0$ for $j = 1, 2, \ldots$. Define $\phi^j : [0, \tau + n_j] \to X$ by

$$\phi^j(n) := \phi(n-n_j), \quad n \in [0, \tau + n_j],$$

and $x^j(n)$ by

$$x^j(n) := \begin{cases} x(n + n_j; \tau + n_j, \phi^j) & (n \geq -n_j) \\ \phi(n) & (n < -n_j) \end{cases}$$

for $j = 1, 2, \ldots$. Since $x^j(n) = \phi^j(n+n_j) = \phi(n)$ for $n \in [-n_j, \tau]$, the uniform asymptotic stability of the zero solution of $(E_0)$ yields

$$|x^j(n)| < \frac{\epsilon}{2} \quad \text{for} \quad n \geq \tau. \quad (3)$$

Let any $n \in \mathbb{Z}$ be given. We now assert that the sequence $\{x^j(n)\}_j$ contains a convergent subsequence. Indeed, in case of $n \leq \tau$, we get $x^j(n) = \phi(n)$, and hence the assertion clearly holds. Let us consider the case $\tau < n$. It follows that

$$x^j(n) = \sum_{k=0}^{n_j+n-1} B(n_j + n - 1 - k)x(k; \tau + n_j, \phi^j)$$

$$= \sum_{s=-n_j}^{n-1} B(n-1-s)x^j(s)$$

$$= \sum_{s=-\infty}^{n-1} B(n-1-s)x^j(s) + \sum_{s=-n_j}^{-n_j-1} B(n-1-s)x^j(s).$$

By virtue of the summability of $B = \{B(n)\}_{n=0}^{\infty}$, it is easy to certify that the term $\sum_{s=-\infty}^{-n_j} B(n-1-s)x^j(s)$ tends to 0 as $j \to \infty$. Moreover, since the operator $B(n-1-s)$ is compact, we see that the sequence $\{\sum_{j=-\infty}^{n-1} B(n-1-s)x^j(s)\}_j$ contains a convergent
subsequence. This observation leads to that the sequence \( \{x_j(n)\}_j \) contains a convergent subsequence, which completes the proof of the assertion.

Now one can select a subsequence \( \{x_j(n)\}_j \), denoted by the same notation \( x_j(n) \), which converges to some \( \tilde{y}(n) \) on \( \mathbb{Z} \) as \( j \to \infty \). Obviously \( \tilde{y}(n) = \phi(n) \) for \( n \in (-\infty, \tau] \). Moreover, it follows that \( \lim_{j \to \infty} \sum_{s=-n}^{n} B(n-s)x^{j}(s) = \sum_{s=-\infty}^{n} B(n-s)\tilde{y}(s) \). Thus we obtain that

\[
\tilde{y}(n + 1) = \lim_{j \to \infty} x^{j}(n + 1) = \lim_{j \to \infty} x(n + 1 + n_j; \tau + n_j, \phi^j) = \lim_{j \to \infty} \sum_{r=0}^{n+n_j} B(n + n_j - r)x(r; \tau + n_j, \phi^j) = \lim_{j \to \infty} \sum_{s=-n_j}^{n} B(n-s)x^{j}(s) = \sum_{s=-\infty}^{n} B(n-s)\tilde{y}(s),
\]

which implies that \( \tilde{y}(n) = y(n; \tau, \phi) \) on \( \mathbb{Z} \). Letting \( j \to \infty \) in (3) we get

\[
|y(n; \tau, \phi)| \leq \frac{\varepsilon}{2} < \varepsilon \quad \text{for} \quad n \geq \tau. \tag{4}
\]

Furthermore, by the same argument we see that

\[
|y(n; \tau, \psi)| < \frac{\varepsilon}{2} \quad \text{for} \quad n \geq \tau + N(\varepsilon/2), \tag{5}
\]

where \( N(\cdot) \) is the one in Definition 1. The inequality (4), together with (5), shows that the zero solution of \( (E_\infty) \) is uniformly asymptotically stable. \( \square \)

**Remark 1.** One can see from the proof that in Theorem 1, the implications (iv) \( \implies \) (iii) \( \implies \) (ii) \( \implies \) (i) hold true without the assumption that \( B(n), n \in \mathbb{Z}^+ \), are compact. It is an interesting problem to ask whether or not the implication (ii) \( \implies \) (iii) (or (ii) \( \implies \) (iv)) holds good without the compactness condition on \( B(n) \). But the problem is still open for the authors.

**Remark 2.** We can apply Theorem 1 to establish the existence of bounded (resp. asymptotically almost periodic) solutions for forced equations of \( (E_\infty) \) with a bounded (resp. asymptotically almost periodic) forcing term, provided that the zero solution of \( (E_0) \) is uniformly asymptotically stable. Details will be discussed in a forthcoming paper [6].

**REFERENCES**


