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Kneser's property in $C^1$-norm for ordinary differential equations

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Let $D$ be an open subset of $\mathbb{R} \times \mathbb{R}^n$. We consider an initial value problem

$$x' = f(t, x), \quad x(0) = \xi,$$

where the prime denotes the differentiation with respect to $t$, $(0, \xi) \in D$ and $f : D \to \mathbb{R}^n$ is continuous. H. Kneser proved the following theorem (see Theorem 4.1, p.15 in [1]).

**Theorem (Kneser).** For every $(0, \xi) \in D$, a set

$$\{x(\tau) ; x \text{ is a solution of (1)}\}$$

is compact and connected in $\mathbb{R}^n$ when $\tau > 0$ is sufficiently small.

For simplicity, we assume that $D = [0, 1] \times \mathbb{R}^n$ and that $f$ is bounded and continuous. Namely, we suppose that there exists a positive constant $M$ satisfying

$$|f(t, x)| \leq M \quad \text{for } (t, x) \in [0, 1] \times \mathbb{R}^n,$$

where $| \cdot |$ denotes any norm in $\mathbb{R}^n$. In this case, the above theorem is reduced to the following theorem.

**Theorem 1.** For every $\xi \in \mathbb{R}^n$, a set

$$\{x(1) ; x \text{ is a solution of (1)}\}$$

is compact and connected in $\mathbb{R}^n$.

For any $a, b \in \mathbb{R}$ with $a < b$, let $C[a, b]$ denote the Banach space of all $\mathbb{R}^n$-valued continuous mappings on $[a, b]$ with the norm $\| \cdot \|$ defined by $\|x\| = \sup_{a \leq t \leq b} |x(t)|$. Similarly, we denote by $C^1[a, b]$ the Banach space of all $\mathbb{R}^n$-valued continuously differentiable mappings on $[a, b]$ with the norm $\| \cdot \|_1$ defined by $\|x\|_1 = \max \{\|x\|, \|x'\|\}$.

It is well known that Theorem 1 is extended to the following theorem.

**Theorem 2.** A set

$$K := \{x ; x \text{ is a solution of (1)}\}$$

is compact and connected in $C[0, 1]$ for every $\xi \in \mathbb{R}^n$. 
Since the set $K$ given in (3) is included in $C^1[0,1]$, it might be natural to discuss the property of the set $K$ in the topology of $C^1[0,1]$. In this article, we shall introduce the following theorem.

**Theorem 3.** The set $K$ given in (3) is compact and connected in $C^1[0,1]$ for every $\xi \in \mathbb{R}^n$.

**Proof.** First we shall show that $K$ is compact in $C^1[0,1]$. Let $\{x_k\}$ be any sequence in $K$. It follows from (2) that $|x_k'(t)| \leq M$ for $0 \leq t \leq 1$, and hence $\{x_k\}$ is equicontinuous and uniformly bounded on $[0,1]$ because $x_k(0) = \xi$. Then we may assume, by Ascoli-Arzelà's theorem, that $\{x_k\}$ converges to some $x$ in $C[0,1]$ by taking a subsequence if necessary. Since $x_k$ satisfies an equality
\[ x_k(t) = \xi + \int_0^t f(s, x_k(s)) \, ds, \]
$x$ satisfies that $x(t) = \xi + \int_0^1 f(s, x(s)) \, ds$, which implies that $x \in K$. Let $L$ be a compact subset of $\mathbb{R}^n$ defined by
\[ L = \{ x \in \mathbb{R}^n ; |x| \leq |\xi| + M \}. \]
Then $x_k(t) \in L$ for $0 \leq t \leq 1$. Since $f$ is uniformly continuous on a compact set $[0,1] \times L$, it follows that
\[ x_k'(t) = f(t, x_k(t)) \to f(t, x(t)) = x'(t) \quad \text{as} \quad k \to \infty \]
uniformly for $t \in [0,1]$. Therefore, $\{x_k\}$ converges to $x$ in $C^1[0,1]$, which shows that $K$ is compact in $C^1[0,1]$.

Now we shall show that $K$ is connected. Suppose that $K$ is not connected. Then there exist two nonempty compact sets $K_1$ and $K_2$ such that $K_1 \cup K_2 = K$ and that $K_1 \cap K_2 = \emptyset$. It is easy to find an open set $G$ in $C^1[0,1]$ satisfying $K_1 \subset G$ and $\bar{G} \cap K_2 = \emptyset$, where $\bar{G}$ denotes the closure of $G$. Therefore, we obtain that
\[ \partial G \cap K = \emptyset, \]
where $\partial G$ denotes the boundary of $G$. Let $x$ and $y$ be any fixed elements in $K_1$ and $K_2$, respectively.

For any fixed small number $\epsilon > 0$ and a number $T$ satisfying $0 \leq T \leq 1$, define a mapping $\varphi : [0,1] \to \mathbb{R}^n$ by
\[ \varphi(t) = \begin{cases} 
  x(t) & \text{for } 0 \leq t \leq T, \\
  x(T) + \int_T^t f(s, x(T)) \, ds & \text{for } T \leq t \leq T + \epsilon \\
  \varphi(T + \epsilon) + \int_{T+\epsilon}^t f(s, \varphi(s - \epsilon)) \, ds & \text{for } T + \epsilon \leq t \leq 1.
\end{cases} \]
It is not difficult to observe that $\varphi$ belongs to $C^1[0,1]$. We denote the mapping $\varphi$
by $\varphi_T$. Clearly, $\varphi_T$ coincides with $x$ when $T = 1$, while $\varphi_T$ does not depend on $x$.

We shall show that the correspondence $T \mapsto \varphi_T$ is a continuous mapping from $[0, 1]$ into $C^1[0, 1]$. Let $T \in [0, 1]$ be fixed, and let $\{T_k\}$ be any sequence in $[0, 1]$ converging to $T$. For simplicity, we denote $\varphi_{T_k}$ and $\varphi_T$, respectively, by $\varphi_k$ and $\varphi$. It will be verified that $\{\varphi_k\}$ converges to $\varphi$ in $C^1[0, 1]$ as $k \to \infty$ in the following two cases where $T_k > T$ holds for $k \in \mathbb{N}$ and $T_k < T$ holds for $k \in \mathbb{N}$. Since $\varepsilon > 0$ and $T_k \to T$ as $k \to \infty$, we may assume that

\begin{equation}
|T_k - T| < \varepsilon \quad \text{for every } k \in \mathbb{N}.
\end{equation}

(i) In the case where $T_k > T$ holds for $k \in \mathbb{N}$. It follows from (6) that $\varphi_k$ is expressed as

\begin{equation}
\varphi_k(t) = \begin{cases} 
  x(t) & \text{for } 0 \leq t \leq T_k, \\
  x(T_k) + \int_{T_k}^{t} f(s, x(T_k)) \, ds & \text{for } T_k \leq t \leq T_k + \varepsilon, \\
  \varphi(T_k + \varepsilon) + \int_{T_k + \varepsilon}^{t} f(s, \varphi_k(s - \varepsilon)) \, ds & \text{for } T_k + \varepsilon \leq t \leq 1.
\end{cases}
\end{equation}

Since $T_k > T$, an equality $\varphi_k(t) = \varphi(t) = x(t)$ holds for $t \in [0, T]$.

We shall observe that

\begin{equation}
|\varphi_k(t) - \varphi(t)| \leq 2M(T_k - T) + \int_{T}^{T + \varepsilon} |f(t, x(T_k)) - f(t, x(T))| \, ds \quad \text{for } t \in [T, T + \varepsilon]
\end{equation}

and

\begin{equation}
|\varphi_k'(t) - \varphi'(t)| \leq \sup_{t \in [T, T_k]} |f(t, x(T_k)) - f(t, x(T))| + \sup_{t \in [T_k + \varepsilon, T + \varepsilon]} |f(t, x(T_k)) - f(t, x(T))| \quad \text{for } t \in [T, T + \varepsilon]
\end{equation}

hold, where $M$ is the positive constant satisfying (2). Here, notice that an inequality $T < T_k < T + \varepsilon$ holds by assumption (7). For any $t \in [T, T_k]$, we have

\begin{equation}
\varphi_k(t) - \varphi(t) = x(T) + \int_{T}^{t} f(s, x(s)) \, ds - \left\{ x(T) + \int_{T}^{T_k} f(s, x(T)) \, ds \right\}
\end{equation}

and hence it follows from (2) that

\begin{equation}
|\varphi_k(t) - \varphi(t)| \leq 2M(T_k - T) \quad \text{for } t \in [T, T_k].
\end{equation}

Furthermore, we have, by (6) and (8), that

\begin{equation}
\varphi_k'(t) - \varphi'(t) = f(t, x(t)) - f(t, x(T)) \quad \text{for } t \in [T, T_k].
\end{equation}
On the other hand, for $t \in [T_k, T + \epsilon]$, it follows, respectively, from (6) and (8) that

$$\varphi_k(t) = x(T_k) + \int_{T_k}^{t} f(s, x(T_k)) \, ds$$

and that

$$\varphi(t) = x(T) + \int_{T}^{t} f(s, x(T)) \, ds$$

and hence, we have

$$|\varphi_k(t) - \varphi(t)| \leq 2M(T_k - T) + \int_{T}^{T+\epsilon} |f(s, x(T_k)) - f(s, x(T))| \, ds.$$  \hspace{1cm} (13)

Furthermore, it is clear that the following equality holds.

$$\varphi'_k(t) - \varphi'(t) = f(t, x(T_k)) - f(t, x(T)) \quad \text{for} \quad t \in [T_k, T + \epsilon].$$ \hspace{1cm} (14)

It then follows from (11) and (13) that (9) holds. Inequality (10) is a direct consequence of (12) and (14). Thus, we obtain, by (9) and (10), that

$$\varphi_k \rightharpoonup \varphi \quad \text{in} \quad C^1[0, T + \epsilon] \quad \text{as} \quad k \rightarrow \infty.$$ \hspace{1cm} (15)

Now we shall estimate $|\varphi_k(t) - \varphi(t)|$ and $|\varphi'_k(t) - \varphi'(t)|$ on the interval $[T + \epsilon, 1]$.

For any $t \in [T + \epsilon, T_{k} + \epsilon]$, it will be verified that the following inequality holds.

$$|\varphi_k(t) - \varphi(t)| \leq 4M(T_k - T) + \sup_{s \in \left[T, T+\epsilon\right]} |f(s, x(T_k)) - f(s, x(T))|$$

$$+ |\varphi_k(T + \epsilon) - \varphi(T + \epsilon)|$$

$$+ \int_{T+\epsilon}^{T_{k}+\epsilon} |f(s, \varphi_k(s - \epsilon)) - f(s, \varphi(s - \epsilon))| \, ds.$$ \hspace{1cm} (16)

When $t \in [T + \epsilon, T_{k} + \epsilon]$, it follows from (6) and (8) that

$$|\varphi_k(t) - \varphi(t)| \leq \int_{T}^{T_k} |f(s, x(s)) - f(s, x(T))| \, ds$$

$$+ \int_{T_k}^{T+\epsilon} |f(s, x(T_k)) - f(s, x(T))| \, ds$$

$$+ \int_{T+\epsilon}^{T_{k}+\epsilon} |f(s, x(T_k)) - f(s, \varphi(s - \epsilon))| \, ds$$

$$\leq 2M(T_k - T) + \int_{T}^{T+\epsilon} |f(s, x(T_k)) - f(s, x(T))| \, ds$$

$$+ \int_{T+\epsilon}^{T_{k}+\epsilon} |f(s, x(T_k)) - f(s, \varphi(s - \epsilon))| \, ds.$$
\[
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\leq 4M(T_k - T) + \int_T^{T+\varepsilon} |f(s, x(T_k)) - f(s, x(T))| \, ds
\]
(17)
\[
\leq 4M(T_k - T) + \varepsilon \sup_{s \in [T, T+\varepsilon]} |f(s, x(T_k)) - f(s, x(T))|.
\]

When \( t \in [T_k + \varepsilon, T + 2\varepsilon] \), \( \varphi_k \) and \( \varphi \) are expressed, respectively, as
\[
\begin{align*}
\varphi_k(t) &= \varphi_k(T_k + \varepsilon) - \varphi_k(T + \varepsilon) + \varphi_k(T + \varepsilon) + \int_{T_k+\varepsilon}^{t} f(s, \varphi_k(s - \varepsilon)) \, ds \\
&= \int_{T+\varepsilon}^{T_k+\varepsilon} f(s, x(T_k)) \, ds + \varphi_k(T + \varepsilon) + \int_{T_k+\varepsilon}^{t} f(s, \varphi_k(s - \varepsilon)) \, ds
\end{align*}
\]
and
\[
\varphi(t) = \varphi(T + \varepsilon) + \int_{T+\varepsilon}^{T_k+\varepsilon} f(s, \varphi(s - \varepsilon)) \, ds + \int_{T_k+\varepsilon}^{t} f(s, \varphi(s - \varepsilon)) \, ds.
\]
Therefore, we have, for \( t \in [T_k + \varepsilon, T + 2\varepsilon] \),
\[
|\varphi_k(t) - \varphi(t)| \leq |\varphi_k(T + \varepsilon) - \varphi(T + \varepsilon)|
\]
\[
+ \int_{T+\varepsilon}^{T_k+\varepsilon} |f(s, x(T_k)) - f(s, \varphi(s - \varepsilon))| \, ds
\]
\[
+ \int_{T_k+\varepsilon}^{t} |f(s, \varphi_k(s - \varepsilon)) - f(s, \varphi(s - \varepsilon))| \, ds
\]
\[
\leq |\varphi_k(T + \varepsilon) - \varphi(T + \varepsilon)| + 2M(T_k - T)
\]
\[
+ \int_{T+\varepsilon}^{T+2\varepsilon} |f(s, \varphi_k(s - \varepsilon)) - f(s, \varphi(s - \varepsilon))| \, ds.
\]

It then follows from this inequality and (17) that (16) holds for \( t \in [T + \varepsilon, T + 2\varepsilon] \).

Thus, we have \( |\varphi_k(t) - \varphi(t)| \rightarrow 0 \) as \( k \rightarrow \infty \) uniformly on \([T + \varepsilon, T + 2\varepsilon]\) because of (15) and the uniform continuity of \( f \) on \([0, 1] \times L\).

We have to confirm that \( |\varphi'_k(t) - \varphi'(t)| \rightarrow 0 \) as \( k \rightarrow \infty \) uniformly on \([T + \varepsilon, T + 2\varepsilon]\).

For \( t \in [T + \varepsilon, T_k + \varepsilon] \), it follows that
\[
\varphi'_k(t) - \varphi'(t) = f(t, x(T_k)) - f(t, \varphi(t - \varepsilon)).
\]
Since \( \varphi(t - \varepsilon) = x(T) + \int_{T}^{t-\varepsilon} f(s, x(T)) \, ds \) and \( T \leq t - \varepsilon \leq T_k \) hold, we have
\[
|x(T_k) - \varphi(t - \varepsilon)| \leq |x(T_k) - x(T)| + M(T_k - T) \text{ for } t \in [T + \varepsilon, T_k + \varepsilon].
\]

For \( t \in [T_k + \varepsilon, T + 2\varepsilon] \), we have evidently that
\[
\varphi'_k(t) - \varphi'(t) = f(t, \varphi_k(t - \varepsilon)) - f(t, \varphi(t - \varepsilon)).
\]
It follows from (18), (19) and (20) that \( |\varphi'_k(t) - \varphi'(t)| \rightarrow 0 \) as \( k \rightarrow 0 \) uniformly for \( t \in [T + \varepsilon, T + 2\varepsilon] \). Therefore, we obtain that \( \varphi_k \rightarrow \varphi \) in \( C^1[0, T + 2\varepsilon] \) as \( k \rightarrow \infty \). Repeating this procedure, we get that \( \varphi_k \rightarrow \varphi \) in \( C^1[0, 1] \) as \( k \rightarrow \infty \).
In the case where $T_k < T$ holds for $k \in \mathbb{N}$. When $t \in [0, T_k]$, we have that $\varphi_k(t) = \varphi(t)$ holds. For $t \in [T_k, T]$, it follows from (2), (6) and (8) that

$$|\varphi_k(t) - \varphi(t)| \leq \int_{T_k}^{T} |f(s, x(T_k)) - f(s, x(s))| ds \leq 2M(T - T_k).$$

Therefore, $\{\varphi_k\}$ converges to $\varphi$ uniformly on $[0, T]$. Furthermore, for $t \in [T_k, T]$, we have that

$$\varphi_k'(t) - \varphi'(t) = f(t, x(T_k)) - f(t, x(t))$$

and that $|x(T_k) - x(t)| \leq M(t - T_k) \leq M(T - T_k) \to 0$ as $k \to \infty$. Therefore, it follows that $\{\varphi'_k\}$ converges to $\varphi'$ uniformly on $[0, T]$, and hence we obtain that

$$\varphi_k \to \varphi \text{ in } C^1[0, T] \text{ as } k \to \infty. \quad (21)$$

Now we shall show that, for $t \in [T, T + \varepsilon]$,

$$|\varphi_k(t) - \varphi(t)| \leq 4M(T - T_k) + \varepsilon \sup_{s \in [T, T + \varepsilon]} |f(s, x(T_k)) - f(s, x(T))|. \quad (22)$$

For $t \in [T, T_k + \varepsilon]$, $\varphi_k$ and $\varphi$ are expressed, respectively, as

$$\varphi_k(t) = x(T_k) + \int_{T_k}^{T} f(s, x(T_k)) ds + \int_{T}^{t} f(s, x(T_k)) ds$$

and

$$\varphi(t) = x(T_k) + \int_{T_k}^{T} f(s, x(s)) ds + \int_{T}^{T_k + \varepsilon} f(s, x(T_k)) ds + \int_{T_k + \varepsilon}^{t} f(s, x(T)) ds,$$

which imply that

$$|\varphi_k(t) - \varphi(t)| \leq 2M(T - T_k) + \int_{T}^{t} |f(s, x(T_k)) - f(s, x(s))| ds$$

$$\leq 2M(T - T_k) + \varepsilon \sup_{s \in [T, T_k + \varepsilon]} |f(s, x(T_k)) - f(s, x(T))|. \quad (23)$$

For $t \in [T_k + \varepsilon, T + \varepsilon]$, $\varphi_k$ and $\varphi$ are expressed, respectively, as

$$\varphi_k(t) = x(T_k) + \int_{T_k}^{T} f(s, x(T_k)) ds + \int_{T}^{T_k + \varepsilon} f(s, x(T_k)) ds + \int_{T_k + \varepsilon}^{t} f(s, \varphi_k(s - \varepsilon)) ds$$

and

$$\varphi(t) = x(T_k) + \int_{T_k}^{T} f(s, x(s)) ds + \int_{T}^{T_k + \varepsilon} f(s, x(T_k)) ds + \int_{T_k + \varepsilon}^{t} f(s, x(T)) ds,$$

which imply that

$$|\varphi_k(t) - \varphi(t)| \leq \int_{T_k}^{T} |f(s, x(T_k)) - f(s, x(s))| ds$$
\[
+ \int_{T}^{T_{k}+\epsilon} |f(s, x(T_{k}))-f(s, x(T))| ds \\
+ \int_{T_{k}+\epsilon}^{t} |f(s, \varphi_{k}(s-\epsilon))-f(s, x(T))| ds
\]
\[
\leq 2M(T-T_{k}) + \int_{T}^{T_{k}+\epsilon} |f(s, x(T_{k}))-f(s, x(T))| ds \\
+ \int_{T_{k}+\epsilon}^{T+\epsilon} |f(s, \varphi_{k}(s-\epsilon))-f(s, x(T))| ds
\]
\[
\leq 4M(T-T_{k}) + \int_{T}^{T_{k}+\epsilon} |f(s, x(T_{k}))-f(s, x(T))| ds \\
\leq 4M(T-T_{k}) + \epsilon \sup_{s \in [T; T+\epsilon]} |f(s, x(T_{k}))-f(s, x(T))|.
\]

It follows from this inequality and (23) that (22) holds. Therefore, we have that

(24) \quad \varphi_{k}(t) \to \varphi(t) \quad \text{uniformly for } t \in [T, T+\epsilon] \quad \text{as } k \to \infty.

On the interval \([T, T+\epsilon]\), we have

\[
\varphi_{k}'(t) - \varphi'(t) = \left\{
\begin{array}{ll}
  f(t, x(T_{k})) - f(t, x(T)) & \text{for } t \in [T, T_{k}+\epsilon], \\
  f(t, \varphi_{k}(t-\epsilon)) - f(t, x(T)) & \text{for } t \in [T_{k}+\epsilon, T+\epsilon].
\end{array}
\right.
\]

For \(t \in [T_{k}+\epsilon, T+\epsilon]\), notice that \(\varphi_{k}(t-\epsilon)\) is expressed as

\[
\varphi_{k}(t-\epsilon) = x(T_{k}) + \int_{T_{k}}^{t-\epsilon} f(s, x(T_{k})) ds,
\]

and hence, we have

\[
|\varphi_{k}(t-\epsilon) - x(T)| \leq |x(T_{k}) - x(T)| + \int_{T_{k}}^{T-\epsilon} |f(s, x(T_{k}))| ds \leq 2M(T-T_{k}).
\]

Therefore, we obtain that \(\varphi_{k}'(t) - \varphi'(t) \to 0\) as \(k \to \infty\) uniformly for \(t \in [T, T+\epsilon]\). Which, together with (21) and (23), implies that

(25) \quad \varphi_{k} \to \varphi \quad \text{in } C^{1}[0, T+\epsilon] \quad \text{as } k \to \infty.

For \(t \in [T+\epsilon, T+2\epsilon]\), \(\varphi_{k}\) and \(\varphi\) are, respectively, expressed as

\[
\varphi_{k}(t) = \varphi(T_{k}+\epsilon) + \int_{T_{k}+\epsilon}^{T+\epsilon} f(s, x(T_{k})) ds + \int_{T+\epsilon}^{t} f(s, \varphi_{k}(s-\epsilon)) ds
\]

and

\[
\varphi(t) = \varphi(T_{k}+\epsilon) + \int_{T_{k}+\epsilon}^{T+\epsilon} f(s, x(T)) ds + \int_{T+\epsilon}^{t} f(s, \varphi(s-\epsilon)) ds,
\]

and hence, it follows that

\[
|\varphi_{k}(t) - \varphi(t)| \leq |\varphi_{k}(T_{k}+\epsilon) - \varphi(T_{k}+\epsilon)| \\
+ \int_{T_{k}+\epsilon}^{T+\epsilon} |f(s, \varphi_{k}(s-\epsilon)) - f(s, x(T))| ds
\]
\[
\int_{T+\epsilon}^{t} |f(s, \varphi_{k}(s-\epsilon)) - f(s, \varphi(s-\epsilon))| \, ds
\leq |\varphi_{k}(T_{k}+\epsilon) - \varphi(T_{k}+\epsilon)| + 2M(T-T_{k})
+ \int_{T+\epsilon}^{T+2\epsilon} |f(s, \varphi_{k}(s-\epsilon)) - f(s, \varphi(s-\epsilon))| \, ds
\]

\[
(26)
\leq |\varphi_{k}(T_{k}+\epsilon) - \varphi(T_{k}+\epsilon)| + 2M(T-T_{k})
+ \epsilon \sup_{s \in [T+\epsilon, T+2\epsilon]} |f(s, \varphi_{k}(s-\epsilon)) - f(s, \varphi(s-\epsilon))|.
\]

We note, by (25), that
\[
\varphi_{k}(s-\epsilon) \rightarrow \varphi(s-\epsilon)
\]
uniformly for \( s \in [T+\epsilon, T+2\epsilon] \) as \( k \rightarrow \infty \),
which shows, by (26), that
\[
|\varphi_{k}(t) - \varphi(t)| \rightarrow 0
\]
uniformly for \( t \in [T+\epsilon, T+2\epsilon] \). Moreover, we also obtain that
\[
\varphi'(t) - \varphi'(t) = f(t, \varphi_{k}(t-\epsilon)) - f(t, \varphi(t-\epsilon)) \rightarrow 0
\]
uniformly for \( t \in [T+\epsilon, T+2\epsilon] \). These facts, together with (25), imply that
\[
\varphi_{k} \rightarrow \varphi
\]
in \( C^{1}[0, T+2\epsilon] \) as \( k \rightarrow \infty \).

Repeating this procedure, we get that \( \{\varphi_{k}\} \) converges to \( \varphi \) in \( C^{1}[0,1] \). Thus, we proved the continuity of the mapping \( T \mapsto \varphi_{T} \).

Similarly to \( \varphi_{T} \), we can define a mapping \( \psi = \psi_{T} : [0,1] \rightarrow \mathbb{R}^{n} \) by using \( y \) instead of \( x \) for the above \( \epsilon > 0 \) and \( T \). We note that \( \psi_{T} \) coincides with \( y \) when \( T = 1 \) while \( \psi_{T} \) does not depend on \( y \) when \( T = 0 \). Moreover, the mapping \( [0,1] \ni T \mapsto \psi_{T} \in C^{1}[0,1] \) is continuous. Here, notice that \( \varphi_{T} \) coincides with \( \psi_{T} \) when \( T = 0 \). Since \( x \in G \) while \( y \not\in G \), we can choose a \( T \) with \( 0 \leq T < 1 \) satisfying
\[
\varphi_{T} \in \partial G \quad \text{or} \quad \psi_{T} \in \partial G.
\]

We denote the above \( T \) by \( T(\epsilon) \). For any fixed sequence \( \{\epsilon_{k}\} \) of positive numbers converging to 0, we denote \( T(\epsilon_{k}) \) by \( T_{k} \). Moreover, the mappings \( \varphi_{T_{k}} \) and \( \psi_{T_{k}} \) will be denoted, respectively, by \( \varphi_{k} \) and \( \psi_{k} \). We may assume, without loss of generality, that the relation \( \varphi_{k} \in \partial G \) holds for every \( k \in \mathbb{N} \). It follows from (6) that \( \varphi_{k} \) satisfies the following three equalities;

\[
\varphi_{k}(t) = x(t) \quad \text{for} \quad t \in [0, T_{k}],
\]

\[
(27)
\varphi_{k}(t) = x(T_{k}) + \int_{T_{k}}^{t} f(s, x(T_{k})) \, ds \quad \text{for} \quad t \in [T_{k}, T_{k} + \epsilon_{k}],
\]

\[
(28)
\varphi_{k}(t) = x(T_{k} + \epsilon_{k}) + \int_{T_{k} + \epsilon_{k}}^{t} f(s, \varphi_{k}(s-\epsilon_{k})) \, ds \quad \text{for} \quad t \in [T_{k} + \epsilon_{k}, 1].
\]
Therefore, we have that $|\varphi_k'(t)| \leq M$ for $t \in [0,1]$ and that $\varphi_k(0) = \xi$, and hence, by Ascoli-Arzela’s theorem, we may assume that $\{\varphi_k\}$ converges to some $\varphi$ in $C[0,1]$ by taking a subsequence if necessary. Furthermore, we may assume that $\{T_k\}$ converges to some $T_0$ in $[0,1]$.

It is clear from (27) that $\varphi(t) = x(t)$ holds for $0 \leq t < T_0$. By letting $k \to \infty$ in (28), we have that $\varphi(T_0) = x(T_0)$. For any $t$ with $T_0 < t \leq 1$, an inequality $T_k < T_k + \epsilon_k < t$ holds for large $k$, it then follows from (29) that

$$\varphi(t) = x(T_0) + \int_{T_0}^{t} f(s, \varphi(s)) \, ds \quad \text{for} \quad T_0 < t \leq 1.$$ 

These facts show that $\varphi$ is a solution of (1), namely, $\varphi \in K$.

Now we shall show that $\{\varphi_k\}$ converges to $\varphi$ in $C^1[0,1]$. For every $k \in \mathbb{N}$, let $\overline{\varphi_k}$ be a mapping defined by

$$\varphi_k(t) = \begin{cases} 
\varphi(t) & \text{for } 0 \leq t \leq T_k \\
\overline{\varphi}(t) & \text{for } T_k \leq t \leq T_k + \epsilon_k \\
\varphi(t - \epsilon_k) & \text{for } T_k + \epsilon_k \leq t \leq 1.
\end{cases}$$

(30)

Then, it is clear that $\overline{\varphi_k}(t) \to \varphi(t)$ uniformly for $t \in [0,1]$ as $k \to \infty$. Furthermore, it follows from (27) through (29) that $\varphi_k'$ satisfies the following equality

$$\varphi_k'(t) = \begin{cases} 
f(t, \varphi_k(t)) & \text{for } 0 \leq t \leq T_k \\
f(t, \varphi_k(T_k)) + f(t, \varphi_k(t)) - f(t, \varphi(t)) & \text{for } T_k \leq t \leq T_k + \epsilon_k \\
f(t, \varphi_k(t - \epsilon_k)) & \text{for } T_k + \epsilon_k \leq t \leq 1.
\end{cases}$$

(31)

Since $\varphi$ is a solution of (1), we have an inequality

$$|\varphi_k'(t) - \varphi'(t)| \leq |\varphi_k'(t) - f(t, \varphi_k(t))| + |f(t, \varphi_k(t)) - f(t, \varphi(t))|.$$ 

(32)

It is clear that the second term of the right hand side in the above tends to 0 as $k \to \infty$. By (30) and (31), we have

$$\varphi_k'(t) - f(t, \varphi_k(t)) = \begin{cases} 
f(t, \varphi_k(t)) - f(t, \varphi(t)) & \text{for } 0 \leq t \leq T_k \\
f(t, \varphi_k(T_k)) - f(t, \varphi(T_k)) & \text{for } T_k \leq t \leq T_k + \epsilon_k \\
f(t, \varphi_k(t - \epsilon_k)) - f(t, \varphi(t - \epsilon_k)) & \text{for } T_k + \epsilon_k \leq t \leq 1.
\end{cases}$$

Since $\{\overline{\varphi_k}\}$ converges to $\varphi$ uniformly on $[0,1]$, we can conclude from (32) and the above equality that $\{\varphi_k'\}$ converges to $\varphi'$ uniformly on $[0,1]$, which assures that $\{\varphi_k\}$ converges to $\varphi$ in $C^1[0,1]$. It then follows from the relation $\varphi_k \in \partial G$ and the closedness of $\partial G$ that $\varphi$ belongs to $\partial G$, which contradicts (5) and the fact that $\varphi \in K$. This completes the proof. \qed
Corollary 1. A set
\[ \{(x(1), x'(1)); x \text{ is a solution of (1)}\} \]
is compact and connected in $\mathbb{R}^{2n}$ for every $\xi \in \mathbb{R}^n$.

Corollary 2. If $E$ is a compact and connected subset of $\mathbb{R}^n$, then a set
\[ \{ x; x \text{ is a solution of (1) with } \xi \in E \} \]
is compact and connected in $C^1[0,1]$.

Example. An initial value problem
\begin{equation}
(33) \quad x' = 2\sqrt{|x|}, \quad x(0) = 0
\end{equation}
admits two solutions $x_1(t) \equiv 0$ and $x_2(t) = t^2$. It follows from Corollary 1 that a compact and connected set
\[ \{(x(1), x'(1)); x \text{ is a solution of (33)}\} \]
contains two points $(x_1(1), x'_1(1)) = (0, 0)$ and $(x_2(1), x'_2(1)) = (1, 2)$. Therefore (33) admits a solution $x$ satisfying
\[ x(1) + x'(1) = 2 \]
because the straight line $x + y = 2$ separates two points $(0, 0)$ and $(1, 2)$.

REFERENCES