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Comparison Theorems for Perturbed Half-linear Euler Differential Equations

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1. Introduction

Oscillatory and nonoscillatory behaviour of Euler differential equation $x'' + \gamma t^{-2} x = 0$ have been well analysed. It is known that this equation is nonoscillatory for $\gamma \leq 1/4$ and oscillatory for $\gamma > 1/4$. The asymptotic behaviour of the solutions of the generalized Euler differential equation

$$\Big(\varphi(x')\Big)' + \frac{\alpha\gamma}{t^{\alpha+1}}\varphi(x) = 0, \quad t > 0, \tag{1.1}$$

is investigated by Elbert in [3] , where γ is a constant, $\alpha > 0$ and $\varphi(x) = |x|^{\alpha-1}x$. It is established that the value

$$E(\alpha) = \frac{\alpha^\alpha}{(1+\alpha)^{\alpha+1}},$$

plays a crusical role for the oscillatory properties of the solutions of the equation (1.1). Namely, for $\gamma \leq E(\alpha)$ the solutions of (1.1) are nonoscillatory, while for $\gamma > E(\alpha)$ all solutions of (1.1) are oscillatory. Nevertheless, this is not the single case of similarity between the second order linear differential equation $(p(t)x'(t))' + q(t)x(t) = 0$ and the half-linear differential equations $[p(t)|x'(t)|^{\alpha-1}x'(t)]' + q(t)|x(t)|^{\alpha-1}x(t) = 0$. Generally, there is striking similarity between those two equations. This similarity was observed for the first time by Elbert [2], who extended Sturmian comparison and separation theorems for the linear differential equation to the half-linear differential equation. Thus, the zeroes of two linearly independent solutions of the half-linear equation separate each other and all nontrivial solutions are oscillatory or nonoscillatory. Thereafter, many authors proceed further in this direction, extending many of the oscillation and nonoscillation criteria as well as comparison theorems for the linear differential equation to the half-linear differential equation. Among numerous papers, we choose to refer to the papers [6], [7], [8], [9], [10] and [11].

Elbert and Schneider in [5] considered a perturbed version of the equation (1.1)

$$\Big(\varphi(x')\Big)' + \frac{E(\alpha)}{t^{\alpha+1}}[\alpha + \delta(t)]\varphi(x) = 0, \tag{1.2}$$

where $\delta(t)$ is positive and continuous function on (t_0, ∞), for some $t_0 \geq 0$. They proved the following oscillation criterion for the equation (1.2).

Theorem A. *The equation* (1.2) *is oscillatory if*

$$\liminf_{t\to\infty} t \int_t^\infty \delta(e^\eta)\, d\eta > \frac{\alpha+1}{2}, \tag{1.3}$$

is satisfied, or nonoscillatory if

$$\limsup_{t\to\infty} t \int_t^\infty \delta(e^\eta)\, d\eta < \frac{\alpha+1}{2}. \tag{1.4}$$

Having sufficient conditions for oscillation and nonoscillation of the perturbed half-linear Euler equation, it seems interesting and useful to compare this equation with the corresponding nonlinear second order differential equation of the form

$$\Big(\varphi(x')\Big)' + \frac{E(\alpha)}{t^{\alpha+1}}\Big(\alpha + \delta(|x|)\Big)\varphi(x) = 0. \tag{1.5}$$

Our main purpose in this paper is to establish comparison theorems between equations (1.2) and (1.5) as well as between two nonlinear equations of the form (1.5).

The paper is organized as follows. In Section 2 we prove auxiliary lemmas which will be used in the proofs of our main results. Further, in Section 3, we prove three main comparison theorems, while in Section 4 we give some examples illustrating and connecting the obtained results.

Note, that in the proofs of the main comparison theorems we are going to use the Schauder-Tychonoff fixed point theorem, for whose formulation and proof we refer to the book of Coppel [1] (pp. 9-10).

2. Auxiliary lemmas

In this section we collect auxiliary lemmas which will be used later.

The first Lemma has been proved in [2] and presents a well-known Nonoscillation Principle presenting a close connection between nonoscillation of a half-linear equation and the existence of the corresponding generalized Riccati equation.

Lemma 2.1. *The half-linear differential equation*

$$\Big(\varphi(x')\Big)' + q(t)\varphi(x) = 0$$

is nonoscillatory if and only if the generalized Riccati equation

$$u' + q(t) + \alpha\, |u|^{1+1/\alpha} = 0$$

has a solution defined for all sufficiently large t.

Lemma 2.2. *The function*

$$F_\alpha(\varrho) = |\varrho|^{1+\frac{1}{\alpha}} - \varrho + E(\alpha), \quad \varrho \in \mathbb{R} \tag{2.1}$$

has the following properties:
(i) *it is nonnegative for all* $\varrho \in \mathbb{R}$;
(ii) $F_\alpha(\varrho) = 0$ *if and only if* $\varrho = D(\alpha) = \left(\dfrac{\alpha}{1+\alpha}\right)^\alpha$.

Lemma 2.3. *Let $x(t)$ be a positive function on $[t_0, \infty)$ satisfying*

$$\Big(\varphi(x')\Big)' + \frac{\alpha\, E(\alpha)}{t^{\alpha+1}}\, \varphi(x) \le 0\,. \tag{2.2}$$

Then

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} x'(t) = 0 \tag{2.3}$$

and

$$\lim_{t\to\infty} t\, \frac{x'(t)}{x(t)} = \frac{\alpha}{\alpha+1}\,. \tag{2.4}$$

PROOF: From the inequality (2.2) it is obvious that $x'(t)$ is decreasing on $[t_0, \infty)$. Using the fact that if a function $x(t) \in C^2[t_0, \infty)$ satisfies $x'(t) < 0$ and $x''(t) < 0$ for all large t, then $x(t) \to -\infty$ as $t \to \infty$, we conclude that $x'(t) > 0$ for all $t \ge t_0$. Since, $x'(t)$ is positive and decreasing, it tends to a finite limit $x'(\infty) \ge 0$. If we integrate (2.2) from t to ∞, we get

$$\Big(x'(t)\Big)^\alpha \ge \Big(x'(\infty)\Big)^\alpha + \alpha\, E(\alpha) \int_t^\infty \frac{x^\alpha(s)}{s^{\alpha+1}}\, ds, \quad t \ge t_0\,, \tag{2.5}$$

from which, using the increasing property of $x(t)$, we see that

$$\Big(x'(t)\Big)^\alpha \ge \frac{E(\alpha)\, x^\alpha(t_0)}{t^\alpha}, \quad t \ge t_0\,,$$

or

$$x'(t) \ge \Big(E(\alpha)\Big)^{1/\alpha} \frac{x(t_0)}{t}, \quad t \ge t_0\,.$$

Integrating again the previous inequality over $[t_0, t]$, we get

$$x(t) \ge x(t_0) + \Big(E(\alpha)\Big)^{1/\alpha} x(t_0)\, \log \frac{t}{t_0}\,, \quad t \ge t_0\,.$$

Therefore, $\lim_{t\to\infty} x(t) = \infty$.

Suppose now, that $x'(\infty) > 0$. Then, $\lim_{t\to\infty} x(t)/t = x'(\infty)$, so that there exists a constant $c > 0$ such that $x(t) \ge c\,t$ for $t \ge t_0$. Then, from (2.5), we have

$$\Big(x'(t_0)\Big)^\alpha > \alpha\, E(\alpha) \int_t^\infty \frac{x^\alpha(s)}{s^{\alpha+1}}\, ds \ge \alpha\, E(\alpha)\, c^\alpha \int_{t_0}^\infty \frac{ds}{s} = \infty\,.$$

This is impossible, so that we prove that $\lim_{t\to\infty} x'(t) = 0$.

Now, for $t \ge t_0$, we define

$$f(t) = -\bigg[\Big(\varphi(x')\Big)' + \frac{\alpha\, E(\alpha)}{t^{\alpha+1}}\, \varphi(x)\bigg] \ge 0, \quad \Phi(t) = t^{\alpha+1}\, \frac{f(t)}{x^\alpha(t)} \ge 0\,.$$

Then, (2.2) can be rewritten in the form

$$\Big(\varphi(x')\Big)' + \frac{1}{t^{\alpha+1}}\Big(\alpha\, E(\alpha) + \Phi(t)\Big)\, \varphi(x) = 0\,, \quad t \ge t_0\,. \tag{2.6}$$

The function $u(t)$ defined for $t \ge t_0$ with

$$u(t) = \left(\frac{x'(t)}{x(t)}\right)^\alpha,$$

satisfies the Riccati equation

$$u'(t) + \alpha(u(t))^{\frac{\alpha+1}{\alpha}} + \frac{1}{t^{\alpha+1}}(\alpha E(\alpha) + \Phi(t)) = 0\,, \quad t \geq t_0\,. \tag{2.7}$$

Since, by (2.3), $u(t) \to 0$ as $t \to \infty$, from (2.7) we obtain

$$u(t) = \alpha \int_t^\infty \Big(u(s)\Big)^{\frac{\alpha+1}{\alpha}} ds + \int_t^\infty \frac{\Phi(s)}{s^{\alpha+1}}\, ds + \frac{E(\alpha)}{t^\alpha}\,, \quad t \geq t_0\,. \tag{2.8}$$

Accordingly, $\Big(u(t)\Big)^{\frac{\alpha+1}{\alpha}} \in L^1[t_0, \infty)$.

If we put $v(t) = t^\alpha u(t)$, from (2.7) we have

$$v'(t) + \frac{\alpha}{t} F_\alpha(v(t)) + \frac{\Phi(t)}{t} = 0\,, \quad t \geq t_0 \tag{2.9}$$

where the function $F_\alpha(\varrho)$ is defined by (2.1). Also, from (2.8) we obtain the following Riccati integral equality for the function $v(t)$

$$v(t) = \alpha t^\alpha \int_t^\infty \frac{(v(s))^{\frac{\alpha+1}{\alpha}}}{s^{\alpha+1}}\, ds + t^\alpha \int_t^\infty \frac{\Phi(s)}{s^{\alpha+1}}\, ds + E(\alpha)\,, \quad t \geq t_0\,. \tag{2.10}$$

By Lemma 2.2, we have that $F_\alpha(v(t)) \geq 0$, so that from (2.9) we see that

$$v'(t) + \frac{\Phi(t)}{t} \leq 0, \quad t \geq t_0\,. \tag{2.11}$$

Consequently, $v(t)$ is positive and decreasing function, so that there exists $\lim_{t\to\infty} v(t) = V < \infty$. Integrating (2.11) over $[t_0, \infty)$ we conclude that $\Phi(t)/t \in L^1[t_0, \infty)$, so that

$$\lim_{t\to\infty} t^\alpha \int_t^\infty \frac{\Phi(s)}{s^{\alpha+1}}\, ds = 0.$$

We now let $t \to \infty$ in (2.10) and we get

$$V = V^{\frac{\alpha+1}{\alpha}} + E(\alpha) \quad \text{i.e.} \quad F_\alpha(V) = 0\,.$$

Applying Lemma 2.2 (ii), we have that $V = D(\alpha)$. Accordingly,

$$\lim_{t\to\infty} \left(t\,\frac{x'(t)}{x(t)}\right)^\alpha = \left(\frac{\alpha}{\alpha+1}\right)^\alpha,$$

which proves (2.4). $\triangle$

Lemma 2.4. *If a positive function $x(t)$ satisfy (2.2), then for any $\varepsilon > 0$ we have*

$$\lim_{t\to\infty} t^{\varepsilon - \frac{\alpha}{\alpha+1}} x(t) = \infty \quad \textit{and} \quad \lim_{t\to\infty} t^{-\varepsilon - \frac{\alpha}{\alpha+1}} x(t) = 0\,. \tag{2.12}$$

PROOF: For a positive function $x(t)$ satisfying (2.2), according to Lemma 2.3, we have (2.4). In fact, (2.4) implies that

$$t\,\frac{x'(t)}{x(t)} = \frac{\alpha}{\alpha+1} + \delta(t)\,, \quad \lim_{t\to\infty} \delta(t) = 0\,. \tag{2.13}$$

If we denote $\sigma = \frac{\alpha}{\alpha+1}$ and integrate (2.13) over $[t_0, t]$, we get

$$x(t) = x(t_0)\exp\Bigg(\int\limits_{t_0}^{t} \frac{\sigma+\delta(s)}{s}\,ds\Bigg), \quad t \geq t_0\,. \tag{2.14}$$

For any $\lambda \in \mathbb{R}$, we have $t^\lambda = \exp(\lambda\,\log t) = t_0^\lambda \exp\Big(\lambda \int_{t_0}^t \frac{ds}{s}\Big)$, which combining with (2.14) yields

$$t^\lambda\, x(t) = c_1 \exp\Bigg(\int\limits_{t_0}^{t} \frac{\sigma+\lambda+\delta(s)}{s}\,ds\Bigg), \quad t \geq t_0\,,$$

where $c_1 = t_0^\lambda\, x(t_0)$. If we now take $\lambda = -\sigma+\varepsilon$ or $\lambda = -\sigma-\varepsilon$, we get

$$\begin{aligned} t^{-\sigma+\varepsilon}\, x(t) &= c_1 \exp\Bigg(\int\limits_{t_0}^{t} \frac{\varepsilon+\delta(s)}{s}\,ds\Bigg), \quad t \geq t_0\,, \\ t^{-\sigma-\varepsilon}\, x(t) &= c_1 \exp\Bigg(\int\limits_{t_0}^{t} \frac{\delta(s)-\varepsilon}{s}\,ds\Bigg), \quad t \geq t_0\,. \end{aligned}$$

Letting $t \to \infty$ and noting that $\delta(t) \to 0$ as $t \to \infty$, we get (2.12). $\triangle$

3. Comparison theorems

Now, we can show two comparison theorems between the perturbed half-linear Euler differential equations and the corresponding nonlinear second order differential equations.

Theorem 3.1. *Consider the equations*

$$\Big(\varphi(x')\Big)' + \frac{1}{t^{\alpha+1}}\Big(\alpha\, E(\alpha) + f(t)\Big)\varphi(x) = 0\,, \quad t \geq a\,, \tag{3.1}$$

and

$$\Big(\varphi(x')\Big)' + \frac{1}{t^{\alpha+1}}\Big(\alpha\, E(\alpha) + f(|x|^\delta)\Big)\varphi(x) = 0\,, \quad t \geq a\,. \tag{3.2}$$

where the function $f : [a,\infty) \to (0,\infty)$ is continuous. Let there exists some $L > 0$, such that $f(t)$ is nonincreasing for all $t \geq L$ and

$$t^\alpha f\Big(t^{\frac{\alpha+1}{\alpha}}\Big) \ \textit{is nondecreasing for all} \ \ t \geq L > 0. \tag{3.3}$$

If the equation (3.1) *is nonoscillatory, then there exists a nonoscillatory solution of the equation* (3.2) *for every $\delta > \frac{\alpha+1}{\alpha}$.*

Proof: Let $X(t)$ be a positive solution of the equation (3.1) on $[t_0, \infty)$ and let ε be an arbitrary constant such that $0 < \varepsilon < \frac{\alpha}{\alpha+1}$ arbitrary constant. Then $X'(t)$ is positive and decreasing function and since the function $X(t)$ satisfies (2.2), by Lemma 2.3, we have that

$$\lim_{t\to\infty} X(t) = \infty \ \text{ and } \ \lim_{t\to\infty} X'(t) = 0\,. \tag{3.4}$$

By applying Lemma 2.4, we also have that

$$\lim_{t\to\infty} t^{\varepsilon-\frac{\alpha}{\alpha+1}} X(t) = \infty\,.$$

Accordingly, there exists $T \geq \max\{t_0, L\}$ such that

$$X(t) > t^{\frac{\alpha}{\alpha+1}-\varepsilon} \quad \text{for} \quad t \geq T\,. \tag{3.5}$$

Denote $\mu = \frac{\alpha+1}{\alpha-\varepsilon(\alpha+1)}$ and notice that $\mu > \frac{\alpha+1}{\alpha}$. Then, by the nonincreasing property of $f(t)$, from (3.5), we obtain

$$f(t) \geq f\Big([X(t)]^{\mu}\Big), \quad t \geq T\,. \tag{3.6}$$

Taking into account (3.4), integration of the equation (3.1) twice, from t to ∞ and then from T to t, yields

$$X(t) = X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(\xi)\Big) X^\alpha(\xi)\, d\xi \right\}^{1/\alpha} ds\,, \quad t \geq T\,. \tag{3.7}$$

From (3.6) and (3.7), we now obtain

$$X(t) \geq X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(X^\mu(\xi))\Big) X^\alpha(\xi)\, d\xi \right\}^{\frac{1}{\alpha}} ds\,, \quad t \geq T\,. \tag{3.8}$$

Let $C[T,\infty)$ be the set of all continuous functions $x : [T,\infty) \to \mathbb{R}$ with the topology of uniform convergence on compact subintervals of $[T,\infty)$. Define the set $\Omega \subset C[T,\infty)$ and the operator $\mathcal{F}_1 : \Omega \to C[T,\infty)$ by

$$\Omega = \{x \in C[T,\infty) \,:\, X(T) \leq x(t) \leq X(t),\ t \geq T\} \tag{3.9}$$

$$\mathcal{F}_1 x(t) = X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(x^\mu(\xi))\Big) x^\alpha(\xi)\, d\xi \right\}^{\frac{1}{\alpha}} ds, \quad t \geq T\,.$$

Because of (3.8), using the assumption (3.3), we have that

$$X(T) \leq \mathcal{F}_1\, x(t) \leq X(t) \quad \text{for all } t \geq T, \quad \text{i.e. } \mathcal{F}_1\, x \in \Omega \text{ for } x \in \Omega\,.$$

Using the Lebesgue dominated convergence theorem it can be shown that $\mathcal{F}_1$ is a continuous mapping. By the Ascoli-Arzela Theorem the set $\mathcal{F}_1(\Omega)$ is relatively compact in $C[t_0,\infty)$, if it is uniformly bounded and locally equicontinuous. Let $t^* > T$ be fixed. If $x \in \Omega$, then $X(T) \leq x(t) \leq X(t) \leq X(t^*)$ for all $t \in [T,t^*]$. This shows that $\mathcal{F}_1(\Omega)$ is uniformly bounded on $[T,t^*]$. Also, if $x \in \Omega$, then for all $t \in [T,t^*]$

$$\begin{aligned}
0 \leq (\mathcal{F}_1\, x)'(t) &= \left\{ \int_t^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f\Big([x(\xi)]^\mu\Big)\Big) x^\alpha(\xi)\, d\xi \right\}^{1/\alpha} \\
&\leq \left\{ \int_t^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f\Big([X(\xi)]^\mu\Big)\Big) X^\alpha(\xi)\, d\xi \right\}^{1/\alpha} \\
&\leq \left\{ \int_t^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(\xi)\Big) X^\alpha(\xi)\, d\xi \right\}^{1/\alpha} = X'(t) \leq X'(t^*)\,.
\end{aligned}$$

This shows that $\mathcal{F}_1(\Omega)$ is equicontinuous on $[T, t^*]$. Consequently, $\mathcal{F}_1(\Omega)$ is a relatively compact subset of $C[T,\infty)$.

Therefore, by the Schauder-Tychonoff fixed point theorem, there exists an element $y \in \Omega$ such that $\mathcal{F}_1 y = y$, or equivalently

$$y(t) = X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(y^\mu(\xi))\Big) y^\alpha(\xi)\, d\xi \right\}^{\frac{1}{\alpha}} ds\,, \quad t \ge T\,.$$

Accordingly, the positive function $y(t)$ is a solution of the equation

$$\Big(\varphi(y')\Big)' + \frac{1}{t^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(|y|^\mu)\Big)\varphi(y) = 0\,.$$

This ensures that the equation (3.2) has a nonoscillatory solution for all $\delta > \dfrac{\alpha+1}{\alpha}$. $\triangle$

Theorem 3.2. *Consider the equations*

$$\Big(\varphi(x')\Big)' + \frac{1}{t^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(|x|)\Big)\varphi(x) = 0\,, \quad t \ge a\,, \tag{3.10}$$

and

$$\Big(\varphi(x')\Big)' + \frac{1}{t^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(t^\delta)\Big)\varphi(x) = 0\,, \quad t \ge a\,, \tag{3.11}$$

where the function $f : [a,\infty) \to (0,\infty)$ is continuous. Let there exist some $L > 0$ such that $f(t)$ is nonincreasing for all $t \ge L$. If the equation (3.10) *has a nonoscillatory solution, then the equation* (3.11) *is nonoscillatory for every $\delta > \frac{\alpha}{\alpha+1}$.*

PROOF: Let $X(t)$ be the positive solution of the equation (3.10) on $[t_0,\infty)$ and let $\varepsilon > 0$ be an arbitrary constant. By Lemma 2.4, we have that

$$\lim_{t\to\infty} t^{-\varepsilon-\frac{\alpha}{\alpha+1}} X(t) = 0\,,$$

so that, there exists $T \ge \max\{t_0, L\}$ such that

$$L \le X(t) < t^{\frac{\alpha}{\alpha+1}+\varepsilon} \quad \text{for} \quad t \ge T\,. \tag{3.12}$$

Denote $\mu = \frac{\alpha}{\alpha+1} + \varepsilon > \frac{\alpha}{\alpha+1}$. Integrating (3.10) from t to ∞, we have

$$X'(t) = \left\{ \int_t^\infty \frac{1}{s^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(X(s))\Big) X^\alpha(s)\, ds \right\}^{\frac{1}{\alpha}}, \quad t \ge T\,. \tag{3.13}$$

We then integrate (3.13) on $[T,t]$ and obtain

$$X(t) = X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(X(\xi))\Big) X^\alpha(\xi)\, d\xi \right\}^{\frac{1}{\alpha}} ds\,, \quad t \ge T\,. \tag{3.14}$$

Therefore, from (3.12) and (3.14), using the nonincreasing property of the function $f(t)$, we have the integral inequality of the form

$$X(t) \ge X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(\xi^\mu)\Big) X^\alpha(\xi)\, d\xi \right\}^{1/\alpha} ds\,, \quad t \ge T\,.$$

Then, if the set Ω is given by (3.9), define the operator $\mathcal{F}_2 : \Omega \to C[T,\infty)$ by

$$\mathcal{F}_2 x(t) = X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(\xi^\mu)\Big) x^\alpha(\xi)\, d\xi \right\}^{1/\alpha} ds\,, \quad t \ge T\,.$$

Accordingly, $\mathcal{F}_2$ maps Ω to itself and it can be shown in the similar way as in the proof of Theorem 3.1 that $\mathcal{F}_2$ is a continuous mapping and that the set $\mathcal{F}_2(\Omega)$ is precompact in $C[T,\infty)$. By the application of the Schauder-Tychonoff fixed point theorem, $\mathcal{F}_2$ has fixed point z in Ω, i.e.

$$z(t) = X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(\xi^\mu)\Big) z^\alpha(\xi)\, d\xi \right\}^{1/\alpha} ds\,.$$

The function $z(t)$ is a positive solution of the half-linear equation

$$\Big(\varphi(z')\Big)' + \frac{1}{t^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(t^\mu)\Big) \varphi(z) = 0\,.$$

This ensures that the perturbed half-linear equation (3.11) is nonoscillatory for all $\delta > \dfrac{\alpha}{\alpha+1}$.
$\triangle$

Moreover, we are able to prove the comparison theorem between the two nonlinear second order differential equations of the form (1.5). Let us compare the following differential equations

$$(A) \qquad \Big(\varphi(x')\Big)' + \frac{1}{t^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(|x|)\Big) \varphi(x) = 0\,,$$

$$(B) \qquad \Big(\varphi(x')\Big)' + \frac{1}{t^{\alpha+1}} \Big(\alpha\, E(\alpha) + g(|x|)\Big) \varphi(x) = 0\,,$$

where the functions $f, g \in C(0,\infty)$ are positive.

Theorem 3.3. *Let there exists some $L > 0$ such that*

$$(3.15) \qquad g(\xi) \textit{ is nonincreasing, } \xi^\alpha g(\xi) \textit{ is nondecreasing for all } \xi \ge L\,,$$

$$(3.16) \qquad f(\xi) \ge g(\xi) \quad \textit{for all} \quad \xi \ge L\,.$$

If the equation (A) has a nonoscillatory solution, then the equation (B) also has a nonoscillatory solution.

Proof: Let $X(t)$ be a positive solution of the equation (A) on $[t_0,\infty)$. Then,

$$\Big(\varphi(X'(t))\Big)' + \frac{\alpha\, E(\alpha)}{t^{\alpha+1}} \varphi(X(t)) = -\frac{f(|X(t)|)}{t^{\alpha+1}} \varphi(X(t)) \le 0, \quad t \ge t_0\,.$$

By Lemma 2.3 we have that

$$\lim_{t\to\infty} X(t) = \infty, \quad \lim_{t\to\infty} X'(t) = 0\,,$$

so that there exists some $T > t_0$, such that $X(t) \ge L$ for all $t \ge T$. As in the proof of Theorem 3.2 we obtain (3.14). By the assumption (3.16), we have, for all $t \ge T$, that

$$(3.17) \qquad X(t) \ge X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + g(X(\xi))\Big) X^\alpha(\xi)\, d\xi \right\}^{\frac{1}{\alpha}} ds\,.$$

Let $C[T,\infty)$ be the set of all continuous functions $x:[T,\infty)\to\mathbb{R}$ with the topology of uniform convergence on compact subintervals of $[T,\infty)$. Define the set $\Omega\subset C[T,\infty)$ by (3.9) and the operator $\mathcal{F}_3:\Omega\to C[T,\infty)$ by

$$\mathcal{F}_3\,x(t)=X(T)+\int_T^t\left\{\int_s^\infty\frac{1}{\xi^{\alpha+1}}\Big(\alpha\,E(\alpha)+g(x(\xi))\Big)x^\alpha(\xi)\,d\xi\right\}^{1/\alpha}ds,\quad t\geq T.$$

Because of (3.17), using the assumption (3.15), we have that

$$X(T)\leq\mathcal{F}_3\,x(t)\leq X(t)\ \text{ for all } t\geq T,\ \text{ i.e. }\ \mathcal{F}_3\,x\in\Omega \text{ for } x\in\Omega\,.$$

It can be shown that $\mathcal{F}_3$ is a continuous mapping and that the set $\mathcal{F}_3(\Omega)$ is precompact in $C[T,\infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem, there exists an element $x\in\Omega$ such that $\mathcal{F}_3\,x=x$, which is equivalent to

$$x(t)=X(T)+\int_T^t\left\{\int_s^\infty\frac{1}{\xi^{\alpha+1}}\Big(\alpha\,E(\alpha)+g(x(\xi))\Big)x^\alpha(\xi)\,d\xi\right\}^{1/\alpha}ds\,.$$

The function $x(t)$ is in fact a positive solution of the equation (B). This completes the proof. $\triangle$

4. Examples

Our main results developed in the previous sections will be illustrated by the following two examples.

Example 4.1. Consider the equation

$$(E_1)\qquad \Big(\varphi(x')\Big)'+\frac{E(\alpha)}{t^{\alpha+1}}\left[\alpha+\frac{\lambda}{(\log t)^\beta}\right]\varphi(x)=0,\quad t\geq a\,,$$

where α, β, λ are positive constants. By Theorem A we conclude that:

(i) if $\beta<2$, then eq. (E_1) is oscillatory for all $\lambda>0$;
(ii) if $\beta=2$, then eq. (E_1) is oscillatory for all $\lambda>\frac{\alpha+1}{2}$ and nonoscillatory for all $\lambda\leq\frac{\alpha+1}{2}$;
(iii) if $\beta>2$, then eq. (E_1) is nonoscillatory for all $\lambda>0$.

Accordingly oscillatory properties of the equation (E_1) is expressed by the following table:

(E1)	$0<\lambda\leq\frac{\alpha+1}{2}$	$\lambda>\frac{\alpha+1}{2}$
$\beta<2$	Osc.	Osc.
$\beta=2$	NonOsc.	Osc.
$\beta>2$	NonOsc.	NonOsc.

Now, by Theorems 3.1 and 3.2 we can develop oscillation and nonoscillation behaviour of solutions of the nonlinear differential equation

$$(N_1)\qquad \Big(\varphi(x')\Big)'+\frac{E(\alpha)}{t^{\alpha+1}}\left[\alpha+\frac{\mu}{(\log|x|)^\beta}\right]\varphi(x)=0,\quad t\geq a\,.$$

(i) for $\beta > 2$, Theorem 3.1 implies that the equation (N_1) has a nonoscillatory solution for all $\mu > 0$;

(ii) for $\beta = 2$, Theorem 3.1 implies that the equation (N_1) has a nonoscillatory solution for all $\mu < \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$;

(iii) for $\beta = 2$, we claim that all solutions of the equation (N_1) are oscillatory for every $\mu > \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$. Assume the contrary, that the equation (N_1) has a positive solution for some $\mu > \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$. Applying Theorem 3.2, we see that the half-linear equation

$$\left(\varphi(x')\right)' + \frac{E(\alpha)}{t^{\alpha+1}}\left(\alpha + \frac{\mu}{\delta^2}\frac{1}{(\log t)^2}\right)\varphi(x) = 0 \tag{4.18}$$

is nonoscillatory for all $\delta > \frac{\alpha}{\alpha+1}$ and some $\mu > \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$. But this is impossible, since for all $\lambda = \mu/\delta^2 > \frac{\alpha+1}{2}$, or for all

$$\mu > \delta^2\frac{\alpha+1}{2} > \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2,$$

eq. (4.18) must be oscillatory by Theorem A.

In the similar way we have that

(iv) for $\beta < 2$, every solution of eq. (N_1) are oscillatory for all $\mu > 0$.

Therefore, oscillatory properties of the equation (N_1) is expressed by the following table:

(N1)	$0 < \mu < \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$	$\mu > \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$
$\beta < 2$	Osc.	Osc.
$\beta = 2$	$\exists$ NonOsc. Sol.	Osc.
$\beta > 2$	$\exists$ NonOsc Sol.	$\exists$ NonOsc. Sol.

Example 4.2. Consider the nonlinear differential equation

$$\left(\varphi(x')\right)' + \frac{E(\alpha)}{t^{\alpha+1}}\left[\alpha + \frac{\mu}{(\log|x|)^2} + \frac{\nu}{(\log|x|\cdot\log\log|x|)^2}\right]\varphi(x) = 0 \tag{N}$$

where α, μ, ν are positive constants.

Using oscillatory properties of solutions of the nonlinear differential equation (N_1), for $\beta = 2$, established in the previous example and Comparison Theorem 3.3, we can conclude that:

(A) Let $\mu < \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$. Then there exists a nonoscillatory solution of the equation (N) for all $\nu > 0$;

(B) Let $\mu > \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$. Then all solutions of the equation (N) are oscillatory for all $\nu > 0$;

Indeed, let $\mu < \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$. Choose μ_0 such that

$$\mu < \mu_0 < \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$$

and

$$f(\xi) = \frac{\mu_0}{(\log\xi)^2}, \quad g(\xi) = \frac{\mu}{(\log\xi)^2} + \frac{\nu}{(\log\xi\cdot\log\log\xi)^2}.$$

The equation (N_1), for $\beta = 2$ and $\mu = \mu_0$, has a nonoscillatory solution, which has been established in the previous Example. Moreover, for large enough ξ, we have that $g(\xi) \le f(\xi)$, $g(\xi)$ is nonicreasing and $\xi^\alpha g(\xi)$ is nondecreasing function. By Comparison Theorem 3.3, we conclude that the equation (N) is nonoscillatory.

Let $\mu > \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$. We claim that all solutions of the equation (N) are oscillatory for every $\nu > 0$. Assume the contrary, that the equation (N) has a positive solution for some $\nu > 0$. By Comparison Theorem 3.2, we have that that the half-linear equation

$$\Big(\varphi(x')\Big)' + \frac{E(\alpha)}{t^{\alpha+1}}\left(\alpha + \frac{\mu}{\delta^2}\frac{1}{(\log t)^2} + \frac{\nu}{(\log t^\delta \cdot \log\log t^\delta)^2}\right)\varphi(x) = 0$$

is nonoscillatory for all $\delta > \frac{\alpha}{\alpha+1}$. Then, since $\nu > 0$ by Sturm Comparison Theorem for the half-linear differential equation, we have that the half-linear differential equation (4.18) is nonoscillatory for all $\delta > \frac{\alpha}{\alpha+1}$. But, as we saw in the Example 4.1 it is impossible.

References

[1] W.A. Coppel, *Stability and Asymptotic Behavior of Differential equations*, D.C. Heath and Company, Boston, 1965

[2] A. Elbert, *A half-linear second order differential equations*, Colloquia Math. Soc. Janos Bolyai 30: Qualitative Theory of Differential Equations, Szeged (1979), 153–180

[3] A. Elbert, *Asymptotic behaviour of autonomous half-linear differential system on the plain*, Studia Sci. Math. Hungarica **19** (1984), 447–464

[4] A. Elbert, T. Kusano, *Principal solution of non-oscillatory half-linear differential equations*, Advances Math. Sci. Applic. **8** (1998), 745–759

[5] A. Elbert, A. Schneider, *Perturbation of the half-linear Euler differential equation*, Results in Math. **37** (2000), 56–83

[6] H. Hoshino, R. Imabayashi, T. Kusano, T. Tanigawa, *On second-order half-linear oscillation*, Advances Math. Sci. Applic. **8** (1998), 199–216

[7] J. Jaros, T. Kusano, T. Tanigawa, *Nonoscillation theory for second order half-linear differential equations in the framework of regular variation*, Results in Math. **43** (2003), 129–149

[8] T. Kusano, Y. Naito, A. Ogata, *Strong oscillation and nonoscillation of quasilinear differential equations of second order*, Diff. Equa. Dynamic Systems **2** (1994), 1–10

[9] H.J.Li, C.C.Yeh, *Sturmian comparison theorem for half-linear second-order differential equations*, Proc. Royal Soc. Edinburgh **125A** (1995), 1193–1204

[10] H.J.Li, C.C.Yeh, *Nonoscillation criteria for second-order half-linear differential equations*, Appl. Math. Lett. **8** (1995), 63–70

[11] J. V. Manojlović, *Oscillation criteria for second-order half-linear differential equations*, Math. Comp. Modelling **30** (1999) 109–119