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Holomorphic Solutions of a Functional Equation

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1 Introduction

We consider a functional equation

$$\Psi(X(x,\Psi(x)))=Y(x,\Psi(x)), \tag{1.1}$$

where $X(x,y)$ and $Y(x,y)$ are holomorphic functions in $|x|<\delta_1$, $|y|<\delta_1$. In Theorem 1 and Theorem 2, we suppose that $X(x,y)$ and $Y(x,y)$ are expanded there as

$$\begin{cases} X(x,y)=\lambda x+y+\sum_{i+j\geq 2} c_{ij}x^iy^j=\lambda x+X_1(x,y), \\ Y(x,y)=\lambda y+\sum_{i+j\geq 2} d_{ij}x^iy^j=\lambda y+Y_1(x,y). \end{cases} \tag{1.2}$$

On the other hand, in Theorem 3 and Theorem 4, we suppose that $X(x,y)$ and $Y(x,y)$ are expanded there as

$$\begin{cases} X(x,y)=\lambda x+\sum_{i+j\geq 2} c_{ij}x^iy^j=\lambda x+X_1(x,y), \\ Y(x,y)=\lambda y+\sum_{i+j\geq 2} d_{ij}x^iy^j=\lambda y+Y_1(x,y). \end{cases} \tag{1.3}$$

Our aim in this paper is to show the following 4 theorems.

Theorem 1 *Suppose $X(x,y)$ and $Y(x,y)$ be holomorphic in $|x|<\delta_1$, $|y|<\delta_1$, and expanded as shown in (1.2) with $|\lambda|>1$. There exists uniquely a function $\Psi(x)$, holomorphic in a disc $|x|<\delta$ and satisfying the equation (1.1):*

$$\Psi(X(x,\Psi(x)))=Y(x,\Psi(x)). \tag{1.1}$$

Theorem 2 *Suppose that $0<|\lambda|<1$ in (1.2). There exists uniquely a function $\Psi(x)$, holomorphic in a disc $|x|<\delta$ and satisfying the equation (1.1).*

Theorem 3 *Suppose $X(x,y)$ and $Y(x,y)$ be holomorphic in $|x|<\delta_1$, $|y|<\delta_1$, and expanded as shown in (1.3) with $|\lambda|>1$. There exists uniquely a function $\Psi(x)$, holomorphic in a disc $|x|<\delta$ and satisfying the equation (1.1).*

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Theorem 4 *Suppose that $0 < |\lambda| < 1$ in (1.3). There exists uniquely a function $\Psi(x)$, holomorphic in a disc $|x| < \delta$ and satisfying the equation (1.1).*

In the papers [4] and [7], we considered the functional equation (1.1), in which X and Y were expanded as

$$\begin{cases} X(x,y) = \lambda x + \sum_{i+j\geq 2} c_{ij}x^i y^j = \lambda x + X_1(x,y), \\ Y(x,y) = \mu y + \sum_{i+j\geq 2} d_{ij}x^i y^j = \mu y + Y_1(x,y), \end{cases} \tag{1.4}$$

with the condition $\lambda \neq \mu$. In the paper [6], we consider it with the condition $\lambda = \mu = 1$. and in the paper [7] we consider it with the condition $\lambda = 1$ $|\mu| = 1, (\mu \neq 1)$. (Furthermore in [5] we considered systems involving n functions $X_1(x_1, \cdots, x_n), \cdots,$ $X_n(x_1, \cdots, x_n)$, $n \geqq 2$ with the conditions $\lambda_1 \neq \lambda_i$, $i = 2, \cdots, n$). In the present paper, (we restrict to $n = 2$ and) consider the cases $\lambda = \mu$ and also the cases where the coefficient matrix are not diagonalizable, as shown in (1.2). Thus our results of this paper may be applied to other results.

Now we will consider the meaning of the equation (1.1).
Consider a simultaneous system of difference equations:

$$\begin{cases} x(t+1) = X(x(t), y(t)), \\ y(t+1) = Y(x(t), y(t)). \end{cases} \tag{1.5}$$

Suppose (1.5) admits a solution $(x(t), y(t))$. If $\frac{dx}{dt} \neq 0$, then we can write $t = \psi(x)$ with a function ψ in a neighborhood of $x_0 = x(t_0)$, and we can write

$$y = y(t) = y(\psi(x)) = \Psi(x), \tag{1.6}$$

as far as $\frac{dx}{dt} \neq 0$. Then the function Ψ satisfies equation (1.1).

Conversely we assume that a function Ψ is a solution of the functional equation (1.1). If the first order difference equation

$$x(t+1) = X(x(t), \Psi(x(t))), \tag{1.7}$$

has a solution $x(t)$, then we put $y(t) = \Psi(x(t))$ and have a solution $(x(t), y(t))$ of (1.5).

This relation is important to derive general solutions of nonlinear second order difference equations which are written such that,

$$\begin{cases} u(t+1) = U(u(t), v(t)), \\ v(t+1) = V(u(t), v(t)), \end{cases} \tag{1.8}$$

where $U(u,v)$ and $V(u,v)$ are entire functions for u and v. We give an example later in this paper. We studied analytic solutions of the nonlinear second order difference equation (1.8) in [8], where, we have treated cases in which the coefficient matrix of linear terms of (1.8) has two different eigenvalues. But in the example of the present paper, we consider the case where the coefficient matrix of linear terms of (1.8) has only one eigenvalue.

2 Proof of Theorem 1

2.1 A formal solution

At first, we put a formal solution to (1.1) $\Psi(x) = \sum_{m=1}^{\infty} a_m x^m$. To determine coefficients a_m, we substitute $\Psi(x) = \sum_{m=1}^{\infty} a_m x^m$ into (1.1) with (1.2). We compare the coefficients of x^m, $(m = 1, 2, \cdots)$, then we have

$$\begin{cases} a_1^2 = 0, \\ a_2(\lambda^2 - \lambda) = d_{20}, \\ a_3(\lambda^3 - \lambda) = a_2\{-2\lambda(a_2 + c_{20}) + d_{11}\} + d_{30}, \\ a_4(\lambda^4 - \lambda) = -a_2\{2\lambda(a_3 + c_{11}a_2) + (a_2 + c_{20})^2\} - a_3\{3\lambda^2(a_2 + c_{20})\} \\ \qquad\qquad + d_{11}a_3 + d_{02}a_2^2 + d_{21}a_2 + d_{40} \\ \qquad \cdots \quad , \\ a_k(\lambda^k - \lambda) = C_k(\lambda, a_1, \cdots, a_{k-1}, c_{i,j}, d_{ij}), \end{cases}$$

where $C_k(\lambda, a_1, \cdots, a_{k-1}, c_{i,j}, d_{ij})$ are polynomials for $\lambda, a_1, \cdots, a_{k-1}, c_{i,j}, d_{ij}$, $2 \leqq i + j \leqq k$, $i \geqq 0, j \geqq 0$.

Since we assume $|\lambda| > 1$, we have $\lambda^k - \lambda \neq 0$ for any $k \geqq 2$. Thus we can determine the coefficients a_k, $(k = 1, 2, \cdots)$ with $\lambda, a_1, \cdots, a_{k-1}$, $c_{i,j}, d_{ij}$, $2 \leqq i+j \leqq k$, $i \geqq 0, j \geqq 0$. Especially we have $a_1 = 0$. Therefore we can determine formal solution $\Psi(x)$ of (1.1) as follows

$$\Psi(x) = \sum_{m=2}^{\infty} a_m x^m. \tag{2.1}$$

2.2 A map T_1 and its fixed point

Take an integer N so large that $|\lambda^{1-N}| < \frac{1}{2}$. Put $\Psi_N(x) = a_2x^2 + a_3x^3 + \cdots + a_Nx^N$ and define a family F to be

$$F = \{\phi(x) : \text{holomorphic and } |\phi(x)| \leqq K|x|^{N+1} \quad \text{in } |x| \leqq \delta\},$$

where δ and K are positive constants to be determined later.

Take $\phi(x) \in F$ and put

$$z = \lambda x + X_1(x, \Psi_N(x) + \phi(x)) = X(x, \Psi_N(x) + \phi(x)). \tag{2.2}$$

Since the expansion of $\Psi(x)$ begins with x^2, if we take δ sufficiently small, then we have that for $|x| \leqq \delta$

$$|X_1(x, \Psi_N(x) + \phi(x))| = |\Psi_N(x) + \phi(x) + \sum_{i+j\geqq 2} c_{ij}x^i(\Psi_N(x) + \phi(x))^j| < \frac{|x|}{2},$$

and $|\frac{d}{dx}X_1(x, \Psi_N(x)+\phi(x))| < 1$, where δ can be chosen independently of $\phi(x)$. Since $|\lambda| > 1$, we have $\frac{dz}{dx} = \lambda+\frac{d}{dx}X_1(x, \Psi_N(x)+\phi(x)) \neq 0$ in $|x| \leqq \delta$. Thus we obtain inverse function ζ such that $x = \zeta(z)$ for $|z| \leqq \delta_1$, where δ_1 can be chosen independently of $\phi(x)$. We also have that

$$|z| \geqq |\lambda x| - |X_1(x, \Psi_N(x) + \phi(x))| > \lambda_1|x|, \tag{2.3}$$

for a λ_1, $1 < \lambda_1 < |\lambda|$. Hence $\phi(\zeta(z))$ is defined if $\zeta(z)$ is defined for $|z| \leqq \delta_1$. Furthermore, we assume that $\alpha = |\frac{\lambda}{\lambda_1^N}| < 1$.

For $\phi(x) \in F$, we put

$$T_1[\phi](z) = Y\Big(\zeta(z), \Psi_N(\zeta(z)) + \phi(\zeta(z))\Big) - \Psi_N(z). \tag{2.4}$$

We will prove the existence of a fixed point $\phi_N(x) \in F$ for the map T_1. If it should be done, then Theorem 1 would be proved, since $\Psi(x) = \Psi_N(x) + \phi_N(x)$ would be a solution of (1.1). Then we can have constants K_1, K_2 and K_3 such that

$$|T_1[\phi](z)| < \left(K_1 + \Big(\alpha + (K_2 + K_3)\delta_1\Big)K\right)|z|^{N+1}.$$

We can take δ_1 to be sufficiently small such that $0 < \alpha + (K_2 + K_3)\delta_1 = A < 1$. Then we take K so large such that

$$K > \frac{K_1}{1-A},$$

and δ is taken as $\delta \leqq \delta_1$. If the family F is defined by means of thus determined numbers K and δ, then the operator T_1 in (2.4) maps F into itself. F is clearly convex, and a normal family by the theorem of Montel. Since T_1 is obviously continuous, we obtain a fixed point $\phi_N(x)$ by Schauder's fixed point Theorem [3].

The fixed point $\phi(x)$ of T_1 is holomorphic function in F on $|x| \leqq \delta$. Therefore the fixed point exists uniquely. Therefore we have a solution $\Psi(x) = \Psi_N(x) + \phi_N(x)$ of (1.1) such that

$$\Psi(x) = \sum_{m=2}^{\infty} a_m x^m. \tag{2.5}$$

□

3 Proofs of Theorem 2 and Theorem 3

In this note, we omit the proofs of Theorem 2 and Theorem 3.

4 Proof of Theorem 4

4.1 A formal solution

At first, we put a formal solution to (1.1) $\Psi(x) = \sum_{m=1}^{\infty} a_m x^m$. To determine coefficients a_m, we substitute $\Psi(x) = \sum_{m=1}^{\infty} a_m x^m$ into (1.1) with (1.3). We compare the coefficients of x^m, $(m = 1, 2, \cdots)$, then we have

$$\begin{cases} a_1\lambda = a_1\lambda, \\ a_2(\lambda^2 - \lambda) = -a_1(c_{20} + c_{11}a_1 + c_{02}a_1^2) + (d_{20} + d_{11}a_1 + d_{02}a_1^2), \\ a_3(\lambda^3 - \lambda) = -a_1(c_{30} + c_{21}a_1 + c_{12}a_1^2 + c_{03}a_1^3 + c_{11}a_2 + c_{02}2a_1a_2) \\ \qquad - a_2 \cdot 2\lambda(c_{20} + c_{11}a_1 + c_{02}a_1^2 \\ \qquad + (d_{30} + d_{21}a_1 + d_{12}a_1^2 + d_{03}a_1^3) + (d_{11}a_2 + d_{02}2a_1a_2), \\ \cdots \quad , \\ a_k(\lambda^k - \lambda) = D_k(\lambda, a_1, \cdots, a_{k-1}, c_{i,j}, d_{ij}), \end{cases}$$

where $D_k(\lambda, a_1, \cdots, a_{k-1}, c_{i,j}, d_{ij})$ are polynomials for $\lambda, a_1, \cdots, a_{k-1}, c_{i,j}, d_{ij}$, $2 \leqq i + j \leqq k$, $i \geqq 0, j \geqq 0$.

Since we assume $0 < |\lambda| < 1$, we have $\lambda^k - \lambda \neq 0$ for any $k \geqq 2$. Thus we can determine the coefficients a_k, $(k = 2, \cdots)$ with $\lambda, a_1, \cdots, a_{k-1}$, $c_{i,j}, d_{ij}$, $2 \leqq i + j \leqq k$, $i \geqq 0, j \geqq 0$. Especially we have a_1 to be arbitrary. Therefore we can determine formal solution $\Psi(x)$ of (1.1), which begins with x, as follows

$$\Psi(x) = \sum_{m=1}^{\infty} a_m x^m. \tag{4.1}$$

Put $z = Y(x, y) = \lambda y + Y_1(x, y)$ and $Q(x, y, z) = z - \lambda y - Y_1(x, y)$. Then $\frac{\partial Q(0,0,0)}{\partial y} = -\lambda \neq 0$ and $Q(0, 0, 0) = 0$. From implicit function theorem, we have a holomorphic function R such that

$$y = R(x, z) = (\frac{1}{\lambda})z + R_1(x, z), \quad \text{for} \quad |x|, |z| \leqq \delta_2,$$

where R_1 is higher order terms for x, z such that $R_1(x, z) = \sum_{i+j\geqq 2} d''_{ij} x^i z^j$, d''_{ij} are constants, δ_2 is a positive constant.

Thus the equation (1.1) is equivalent to

$$\Psi(x) = R(x, \Psi(X(x, \Psi(x)))). \tag{4.2}$$

Take integer N so large that $|\lambda^{N-1}| < \frac{1}{2}$, and put $\Psi_N(x) = a_1x + a_2x^2 + \cdots + a_Nx^N$. Further we define a family F to be

$$F = \{\phi(x) \, : \, \text{holomorphic and } |\phi(x)| \leqq K|x|^{N+1} \quad \text{in } |x| \leqq \delta\},$$

where δ and K are positive constants to be determined later.

For $\phi(x) \in F$, since $X_1(x, y) = \sum_{i+j\geqq 2} c_{ij}x^iy^j$, we have $|X(x, \Psi_N(x) + \phi(x))| \leqq \lambda_2|x|$, with a constant λ_2, $|\lambda| < \lambda_2 < 1$. Therefore $\phi\big(X(x, \Psi_N(x) + \phi(x))\big)$ can be defined at $|x| \leqq \delta_2$. Furthermore, we assume that $\beta = |\frac{\lambda_2^N}{\lambda}| < 1$.

Take $\phi(x) \in F$ and put

$$T_4[\phi](x) = R\Big(x, \Psi_N\Big(X\big(x, \Psi_N(x) + \phi(x)\big)\Big) + \phi\Big(X\big(x, \Psi_N(x) + \phi(x)\big)\Big)\Big) - \Psi_N(x) \tag{4.3}$$

As in the proof of Theorem 1, we see that a fixed point $\phi_N(x)$ of the map T_4 in F gives a solution of (1.1) as $\Psi(x) = \Psi_N(x) + \phi_N(x)$. Thus we will prove the existence a fixed point of T_4. Then we can have constants K_1, K_2, and K_3 such that

$$|T_4[\phi](x)| < \left(K_1 + \Big((K_2 + K_3)\delta + \beta \Big) K \right) \cdot |x|^{N+1}.$$

We take δ sufficiently small such that $A_4 = \beta + (K_2 + K_3)\delta < 1$, and K is taken so large that

$$K > \frac{K_1}{1 - A_4},$$

therefore the Map T_4 in (4.3) maps F into F, and we have the existence of the fixed point as in the proof of Theorem 1. Hence we have a solution $\Psi(x) = \Psi_N(x) + \phi_N(x)$ of (1.1). □

5 An example of nonlinear second order difference equations

We consider the following second order nonlinear difference equation,

$$u(t+2) = f(u(t), u(t+1)), \tag{5.1}$$

where $f(x, y)$ is an entire function of $(x, y) \in \mathbb{C}^2$.

We suppose that the equation (5.1) admits an equilibrium point $u^* : u^* = f(u^*, u^*)$. We can assume, without losing generality, that $u^* = 0$. Then $f(x, y)$ can be written as

$$f(x, y) = -\beta x - \alpha y + g(x, y), \tag{5.2}$$

in which $g(x, y) = \sum_{i+j \geq 2} b_{ij} x^i y^j$, b_{ij} are constants. We assume that $\beta \neq 0$. Our purpose is to obtain *analytic general solutions* of difference equation (5.1). Analytic solutions of nonlinear difference equation have been studied for a long time. For example, in [1], Harris derived analytic general solutions, which have asymptotic expansion with t, of nonlinear first order difference equation $u(t+1) = F(t, u(t))$ under some conditions. But for general nonlinear difference equations, we can not use the Harris's methods. Especially, for characteristic values λ^* of linear terms of the difference equation, if $|\lambda^*| = 1$ for all λ^*, then it is difficult to prove a existence of analytic solution of it. Kimura [2], and Yanagihara [9] studied the cases $|\lambda^*| = 1$ in nonlinear general first order difference equations. Here we seek analytic general solutions of nonlinear second order difference equation such that (5.1). In [8] we consider (5.1) under an assumption. But we seek general solutions of the system without the assumptions in this example.

The characteristic matrix of (5.1) is

$$M = \begin{pmatrix} 0 & 1 \\ f_x(0,0) & f_y(0,0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix}.$$

Let λ_1, λ_2 be roots of the following characteristic equation of M

$$D(\lambda) = \begin{vmatrix} -\lambda & 1 \\ -\beta & -\alpha - \lambda \end{vmatrix} = \lambda^2 + \alpha\lambda + \beta = 0. \tag{5.3}$$

In [8], we consider under the condition $\lambda_1 \neq \lambda_2$ making use of a theorem in [4], but could not treated under the condition $\lambda_1 = \lambda_2$ in it. But in this example, we can seek analytic general solutions of (5.1) under the condition $\lambda = \lambda_1 = \lambda_2$ making use of Theorem 1 and Theorem 2 in the present paper.

Hereafter we consider t to be a complex variable.

5.1 An analytic solution.

We consider following two cases, i) $|\lambda| > 1$ and ii) $|\lambda| < 1$.

In case i) we consider solutions such that $u(t+n) \to 0$, as $n \to -\infty$. In case ii) we consider solutions such that $u(t+n) \to 0$, as $n \to +\infty$.

In the both cases, we can determine a formal solution of (5.1),

$$u(t) = \sum_{n=1}^{\infty} \gamma_n \lambda^{nt}, \tag{5.4}$$

where $\gamma_1 \neq 0$ can be arbitrarily prescribed, and $\gamma_k, k \geq 2$, are determined by γ_1, see [8].

Similarly in [8], we have following Theorem 5.

Theorem 5 *Let λ_1 and λ_2 be roots of $D(\lambda) = 0$ in (2.1), with $\lambda = \lambda_1 = \lambda_2$. Suppose $0 < |\lambda| < 1$ or $|\lambda| > 1$. Then there is a $\eta > 0$ such that we have a holomorphic solution $u(t) = \sum_{n=1}^{\infty} \gamma_n \lambda^{nt}$ in $S(\eta) = \{t; |\lambda^t| < \eta\}$.*

When $|\lambda| > 1$, the solution $u(t)$ can be analytically continued to the whole plane, by making use of the equation (5.1).

When $0 < |\lambda| < 1$, the function $\phi(w, z)$ such that $u(t) = \phi(u(t+1), u(t+2))$ is defined only locally, though we can also analytically continue $u(t)$, keeping out of branch points. The solution obtained is multi-valued.

The analytic solution u obtained in Theorem 5 is "A Particular Solution" of (5.1).

5.2 Analytic General Solutions

Let $u(t)$ be an analytic solution of (5.1) which we have in Theorem 5, and $w(t) = u(t+1)$. Then (5.1) can be written as a system of simultaneous equations

$$\begin{pmatrix} u(t+1) \\ w(t+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix} \begin{pmatrix} u(t) \\ w(t) \end{pmatrix} + \begin{pmatrix} 0 \\ g(u(t), w(t)) \end{pmatrix}. \tag{5.5}$$

From the assumption that $\lambda = \lambda_1 = \lambda_2$, we can not transform the matrix $\begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix}$ into diagonal form. Let $P = \begin{pmatrix} 1 & 1 \\ \lambda & \lambda+1 \end{pmatrix}$, and put

$$\begin{pmatrix} u \\ w \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}. \tag{5.6}$$

We can transform the coefficient matrix of linear terms of (5.5) into Jordan normal form, i.e., (5.5) is transformed to a following system with respect to x, y :

$$\begin{cases} x(t+1) = \lambda x(t) + y(t) + \sum_{i+j\geq 2} c_{ij} x(t)^i y(t)^j = X(x(t), y(t)), \\ y(t+1) = \lambda y(t) + \sum_{i+j\geq 2} d_{ij} x(t)^i y(t)^j = Y(x(t), y(t)), \end{cases} \tag{5.7}$$

where c_{ij} and d_{ij} are constants.

At first we consider the case i) $|\lambda| > 1$. We suppose $\Upsilon(t)$ be a solution of (5.1) such that $\Upsilon(t+n) \to 0$ as $n \to -\infty$ uniformly on any compact subset of t-plane. Then we have following Lemma 6 from Theorem 1.

Lemma 6 *Let λ_1, λ_2 be roots of the characteristic equation of (5.3) and $\lambda = \lambda_1 = \lambda_2$. Furthermore we assume that $|\lambda| > 1$ (case ii). Suppose that $\Upsilon(t)$ be an analytic solution of (5.1) such that $\Upsilon(t+n) \to 0$ as $n \to -\infty$ uniformly on any compact subsets of the t-plane, then we have $\frac{\Upsilon(t+1+n)}{\Upsilon(t+n)} \to \lambda$, as $n \to -\infty$.*

From Lemma 6, we have following Theorem 7 and we obtain analytic general solution of (5.1).

Theorem 7 *Let λ_1, λ_2 be roots of the characteristic equation of (5.3) and $\lambda = \lambda_1 = \lambda_2$. We assume $|\lambda| > 1$, and $u(\tau)$ is the solution of (5.1) which has the expansion $u(t) = \sum_{n=1}^{\infty} \gamma_n \lambda^{nt}$ in $S(\eta) = \{t; |\lambda^t| < \eta\}$ with some constant $\eta > 0$. Further suppose that $\Upsilon(t)$ is an analytic solution of (5.1) such that $\Upsilon(t+n) \to 0$ as $n \to -\infty$, uniformly on any compact subsets of the t-plane. Then there is a periodic entire function $\pi(t), (\pi(t+1) = \pi(t))$, such that*

$$\Upsilon(t) = \sum_{n=1}^{\infty} \gamma_n (1 + \lambda - \lambda^n) \lambda^{n(t+\pi(t))} + \Psi\Big(\sum_{n=1}^{\infty} \gamma_n (1 + \lambda - \lambda^n) \lambda^{n(t+\pi(t))}\Big), \tag{5.8}$$

in $S(\eta)$, where Ψ is a solution of

$$\Psi(X(x, \Psi(x))) = Y(x, \Psi(x)), \tag{1.1}$$

and X, Y are defined in (5.6) and (5.7). Furthermore we have $\frac{\Upsilon(t+1+n)}{\Upsilon(t+n)} \to \lambda$ as $n \to -\infty$.

Conversely, a function $\Upsilon(t)$ which is represented as (5.8) in $S(\eta)$ for some $\eta > 0$, where $\pi(t)$ is a periodic function with the period one, is a solution of (5.1) such that $\Upsilon(t+n) \to 0$ and $\frac{\Upsilon(t+1+n)}{\Upsilon(t+n)} \to \lambda$ as $n \to -\infty$.

Proof. Let $u(t)$ be the analytic solution of (5.1) which we have in Theorem 5. And suppose $\Upsilon(t)$ be a solution of (5.1) such that $\Upsilon(t+n) \to 0$ as $n \to -\infty$ uniformly on any compact subsets of t-plane.

As above arguments in Section 1, if a solution (x, y) of (5.7) exists, then we can put $t = \psi(x)$ for a function ψ and we can write

$$y(t) = y(\psi(x)) = \Psi(x), \tag{1.6}$$

when $\frac{dx}{dt} \neq 0$. Then the function Ψ satisfies equation (1.1).

Conversely we assume that a function Ψ is a solution of the functional equation (1.1). If the first order difference equation

$$x(t+1) = X(x(t), \Psi(x(t))), \tag{1.7}$$

has a solution $x(t)$, then we put $y(t) = \Psi(x(t))$ and have a solution $(x(t), y(t))$ of (5.7).

Put $\omega(t) = \Upsilon(t+1)$, from (5.6), then we have $\chi(t) = (1+\lambda)\Upsilon(t) - \omega(t))$. Since $\Upsilon(t+n) \to 0$ and $\omega(t+n) \to 0$ as $n \to -\infty$, we have $\chi(t+n) \to 0$ as $n \to -\infty$.

Since the solution $u(t) = \sum_{n=1}^{\infty} \gamma_n \lambda^{nt}$ of (5.7) is a function of λ^t,

$$x(t) = (1+\lambda)u(t) - u(t+1) = (1+\lambda)\sum_{n=1}^{\infty} \gamma_n(1+\lambda-\lambda^n)(\lambda^t)^n = U(\lambda^t). \tag{5.9}$$

where $\zeta = U(\tau)$ is a function of $\tau = \lambda^t$ and $U'(0) = a_1 \neq 0$ and $U(0) = 0$. Since $U(\tau)$ is an open map, for any $\eta_1 > 0$ there is an $\eta_2 > 0$ such that $U(\{|\tau| < \eta_1\}) \supset \{|\zeta| < \eta_2\}$.

Since $\chi(t+n) \to 0$ as $n \to \infty$, supposed that t belongs to a compact set K, there is a $n_0 \in \mathbb{N}$ such that $|\chi(t'+n)| < \eta_2 \quad (n \geqq n_0)$ for $t' \in K$. Thus there is a τ' such that $\chi(t'+n) = U(\tau')$. We can write $\tau' = \lambda^\sigma$, and

$$\chi(t'+n) = U(\tau') = U(\lambda^\sigma). \tag{5.10}$$

Since $U'(0) = \gamma_1 \neq 0$, using the theorem on implicit function we have the U^{-1} such that $\lambda^\sigma = U^{-1}(\chi(t'+n))$. Put $t = t'+n$, then $\lambda^\sigma = U^{-1}(\chi(t))$, and we write

$$\sigma = \log_\lambda U^{-1}(\chi(t)) = l(t). \tag{5.11}$$

When there is a solution $\chi(t)$ of (5.7), from (1.7), (5.9) and (5.10) we have

$$\chi(t+1) = X(\chi(t), \Psi(\chi(t))) = X(x(\sigma), \Psi(x(\sigma))) = x(\sigma+1) = U(\lambda^{\sigma+1}).$$

Hence $\sigma + 1 = l(t+1)$, $l(t)+1 = l(t+1)$. If we put $\pi(t) = \ell(t) - t$, then we obtain $\pi(t+1) = \ell(t+1) - (t+1) = \ell(t) - t = \pi(t)$. and we can write as

$$l(t) = t + \pi(t), \tag{5.12}$$

$\pi(t)$ defined for a compact set K with $\Re[t]$ sufficiently large, which we can continue analytically as a periodic function with the period 1. Then $\sigma = t + \pi(t)$. Thus we have $\sigma = t + \pi(t)$. From (5.9), (5.10), (5.11) and (5.12), $\chi(t)$ can be written as

$$\chi(t) = U(\lambda^{t+\pi(t)}) = x(t + \pi(t)) = \sum_{n=1}^{\infty} \gamma_n (1 + \lambda - \lambda^n)(\lambda^{t+\pi(t)})^n. \tag{5.13}$$

From (5.6) and (5.13), we have

$$\Upsilon(t) = \chi(t) + \nu(t) = \sum_{n=1}^{\infty} \gamma_n (1 + \lambda - \lambda^n)\lambda^{n(t+\pi(t))} + \Psi\Big(\sum_{n=1}^{\infty} \gamma_n (1 + \lambda - \lambda^n)\lambda^{n(t+\pi(t))}\Big),$$

where $\pi(t)$ is defined for $t \in \cup_{n\in\mathbb{Z}}(K + n)$ with a compact set K. Since K is arbitrary, we can continue $\pi(t)$ analytically to a periodic entire function with period 1, and Ψ is a solution of (1.1), as in (2.5),

$$\Psi(x) = \sum_{m=2}^{\infty} a_m x^m, \tag{2.6}$$

From Lemma 6, we obtain $\frac{\Upsilon(t+n+1)}{\Upsilon(t+n)} \to \lambda$, as $n \to -\infty$.

Conversely, if we put $\Upsilon(t)$ such that in (5.8), where π is an arbitrary periodic entire function, and Ψ is a solution of (5.1). Then we can have $\Upsilon(t)$ is a solution of (1.1) such that $\Upsilon(t + n) \to 0$ as $n \to -\infty$. Hence, from Lemma 6, we have $\frac{\Upsilon(t+1+n)}{\Upsilon(t+n)} \to \lambda$ as $n \to -\infty$. □

When $0 < |\lambda| < 1$, we have following similar results. Here we omit the proofs.

Lemma 8 *Let λ_1, λ_2 be roots of the characteristic equation of (5.3) and $\lambda = \lambda_1 = \lambda_2$ (case ii). And we assume that $|\lambda| < 1$. Let $\Upsilon(t)$ be an analytic solution of (5.1) such that $\Upsilon(t+n) \to 0$ as $n \to +\infty$ uniformly on any compact subsets of the t-plane, then we have $\frac{\Upsilon(t+1+n)}{\Upsilon(t+n)} \to \lambda$, as $n \to +\infty$.*

Theorem 9 *Let λ_1, λ_2 be roots of the characteristic equation of (5.3) and $\lambda = \lambda_1 = \lambda_2$. We assume that $|\lambda| < 1$ and $u(t)$ is a solution of (5.1) which has the expansion $u(t) = \sum_{n=1}^{\infty} \gamma_n \lambda^{nt}$ in $S(\eta) = \{t; |\lambda^t| < \eta\}$ with some constant $\eta > 0$.*

Suppose that $\Upsilon(t)$ is an analytic solution of (5.1) such that $\Upsilon(t + n) \to 0$ as $n \to +\infty$, uniformly on any compact subsets of the t-plane. Then there is a periodic entire function $\pi(t), (\pi(t + 1) = \pi(t))$, such that

$$\Upsilon(t) = \sum_{n=1}^{\infty} \gamma_n (1 + \lambda - \lambda^n)\lambda^{n(t+\pi(t))} + \Psi\Big(\sum_{n=1}^{\infty} \gamma_n (1 + \lambda - \lambda^n)\lambda^{n(t+\pi(t))}\Big), \tag{5.14}$$

in $S(\eta)$, where Ψ is a solution of (1.1) and X, Y are defined in (5.6) and (5.7). Furthermore we have $\frac{\Upsilon(t+1+n)}{\Upsilon(t+n)} \to \lambda$ as $n \to +\infty$.

Conversely, a function $\Upsilon(t)$ which is represented as shown in (5.14) in $S(\eta)$ for some $\eta > 0$, where $\pi(t)$ is a periodic function with the period one, is a solution of (5.1) such that $\Upsilon(t + n) \to 0$ and $\frac{\Upsilon(t+1+n)}{\Upsilon(t+n)} \to \lambda$ as $n \to +\infty$.

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