Periodicizing Functions

1 Introduction

We denote by $\mathbb{R}$ and by $\mathbb{C}$ the set of real numbers and complex numbers, respectively. Let us consider a linear differential equation of the form

$$\frac{dx}{dt} = Ax + f(t), \quad x(0) = w \in \mathbb{C}^d,$$

where $A \in M_d(\mathbb{C})$, the set of all $d \times d$ complex matrices, and $f : \mathbb{R} \rightarrow \mathbb{C}^d$ is a nontrivial continuous $\tau$-periodic function.

The purpose of this paper is to find a periodicizing function for Equation (1). It is well known that the solution of the above equation is expressed as

$$x(t) := x(t; 0, w) = e^{At}w + \int_0^t e^{A(t-s)}f(s)ds.$$

However, from this representation it is not easy to see asymptotic behaviors of solutions of Equation (1). In the paper [2], we gave a new representation of the solution of Equation (1), from which asymptotic behaviors of solutions are seen.

Its representation is essentially related to a periodicizing function for Equation (1), as stated below. We take a continuous function $z(t)$ such that the function

$$h(t) := z(t) + \int_0^t e^{A(t-s)}f(s)ds, \quad t \in \mathbb{R},$$

becomes a continuous $\tau$-periodic function. Then the solution $x(t)$ of Equation (1) is rewritten as

$$x(t) = (e^{At}w - z(t)) + h(t),$$

the first term of the right hand side in which is well known. We call such a function $z(t)$ a "periodicizing" function (for Equation (1)). Therefore, to find a periodicizing function is very important in obtaining the representation of solutions and in
studying asymptotic behaviors of solutions for Equation (1). In this paper we will construct a periodicizing function $z(t)$ as follows. Put
\[ b_f = \int_0^\tau e^{A(\tau-s)} f(s) ds. \]

In the first step, we prove that a periodicizing function $z(t)$ satisfies
\[ \Delta_\tau z(t) := z(t + \tau) - z(t) = -e^{tA}b_f, \quad t \in [0, \infty). \]

In the next step, we calculate an indefinite sum of $-e^{tA}b_f$; that is, $z(t) = \Delta_\tau^{-1}(-e^{At}b_f)$ by using a new representation of the solution obtained in [2] for the linear difference equation of the form $x_{n+1} = e^{\tau A}x_n + b_f$.

## 2 Discrete linear difference equations and the indefinite sum

### 2.1 Discrete linear difference equations

Let $\sigma(A)$ be the set of all eigenvalues of $A$ and $m$ the index of $\lambda \in \sigma(A)$. Let $M_\lambda = \mathcal{N}((A - \lambda E)^m)$ be the generalized eigenspace of $\lambda \in \sigma(A)$, where $E \in M_d(\mathbb{C})$ stands for the unite matrix. Then we have the direct sum decomposition
\[ \mathbb{C}^d = \bigoplus_{\lambda \in \sigma(A)} M_\lambda. \]

Let $P_\lambda$ be the projection on $\mathbb{C}^d$ to $M_\lambda$ induced from this decomposition. Set $\mathbb{N} := \{1, 2, 3, \cdots\}$.

Now, we solve the discrete linear difference equation of the form
\[ x_{n+1} = e^{\tau A}x_n + b, \quad x_0 = w, \tag{3} \]
where $n \in \mathbb{N} \cup \{0\}$. For the simplicity of the description, we set
\[ \varepsilon(z) = \frac{1}{e^z - 1}, \quad \varepsilon^{(i)}(z) = \frac{d^i}{dz^i} \frac{1}{e^z - 1}. \]

Moreover, we define $X_\lambda(A)$ and $Y_\lambda(A)$ for $\lambda \in \sigma(A)$ as
\[ X_\lambda(A) = \sum_{i=0}^{m-1} \varepsilon^{(i)}(\tau\lambda) \frac{\tau^i}{i!} (A - \lambda E)^i \quad \text{if} \quad e^{\tau\lambda} \neq 1 \]
and
\[ Y_\lambda(A) = \sum_{i=0}^{m-1} B_i \frac{\tau^i}{i!} (A - \lambda E)^i \quad \text{if} \quad e^{\tau\lambda} = 1, \]
where $B_i, i \in \mathbb{N} \cup \{0\}$, stand for Bernoulli’s numbers, refer to [3].

The following result is found in [2].
Theorem 1 [2] Let $\lambda \in \sigma(A)$. The component $P_\lambda x_n$ of the solution $x_n, n \in \mathbb{N}$, of Equation (3) is given as follows:

1) If $e^{\tau \lambda} \neq 1$, then

$$P_\lambda x_n = e^{n \tau \lambda} \sum_{i=0}^{m-1} \frac{\tau^i}{i!} (A - \lambda E)^i [P_\lambda w + X_\lambda(A) P_\lambda b] - X_\lambda(A) P_\lambda b$$

2) If $e^{\tau \lambda} = 1$, then

$$P_\lambda x_n = \sum_{i=0}^{m-1} \frac{\tau^i}{i+1} (A - \lambda E)^i [\tau(A - \lambda E) P_\lambda w + Y_\lambda(A) P_\lambda b] + P_\lambda w.$$ 

2.2 The indefinite sum

We prepare fundamental results on the indefinite sum. Let $\tau > 0$ and $h : [0, \infty) \to \mathbb{C}^d$ be a continuous function.

First, we consider the problem of finding a continuous solution of the following equation

$$\Delta_\tau z(t) := z(t + \tau) - z(t) = h(t), \quad t \in [0, \infty),$$

(4)

that is, the indefinite sum $z(t) = \Delta_\tau^{-1} h(t)$. If $z_0(t)$ is one of solutions of Equation (4), then any other solution $z(t)$ is given by

$$z(t) = z_0(t) + c(t)$$

with an arbitrary continuous $\tau$-periodic function $c(t)$ (it is called the periodic constant). The following lemma is easily proved, refer to [3].

Lemma 2.1

1) Let $\varphi : [0, \tau] \to \mathbb{C}^d$ be a continuous function such that

$$\varphi(\tau) = \varphi(0) + h(0).$$

(5)

Then a continuous solution $z(t)$ of Equation (4) satisfying the the initial condition $z(s) = \varphi(s), s \in [0, \tau]$, exists uniquely on $[0, \infty)$. Moreover, it is given by

$$z(s + n\tau) = \varphi(s) + \sum_{i=0}^{n-1} h(s + i\tau), \quad (s \in [0, \tau), \quad n = 1, 2, \cdots).$$

(6)

2) Conversely, if a continuous function $z(t)$ is a solution of Equation (4), then $\varphi(t) := z(t), t \in [0, \tau]$, satisfies the condition (5) and $z(t)$ is given by (6).
Next, we consider a special case of Equation (4); that is,

\[ z(t + \tau) - z(t) = -B(t)b, \quad t \in [0, \infty), \tag{7} \]

where \( B(t), t \in [0, \infty) \), is a continuous matrix function such that

\[ B(s + k\tau) = B(s)B^k(\tau), \quad k \in \mathbb{N}. \tag{8} \]

In this case the continuous variable \( t \) in Equation (7) is reduced to the discrete variable.

**Lemma 2.2**

1) Let \( \varphi: [0, \tau] \to \mathbb{C}^d \) be a continuous function such that

\[ \varphi(\tau) = \varphi(0) - B(0)b. \tag{9} \]

Then a continuous solution \( z(t) \) of Equation (7) satisfying the initial condition \( z(t) = \varphi(t), t \in [0, \tau] \), exists uniquely on \([0, \infty)\). Moreover, it is given by

\[ z(s + n\tau) = \varphi(s) - B(s)x_n(0), \quad (s \in [0, \tau), \ n \in \mathbb{N}), \tag{10} \]

where \( x_n(0) \) is the solution of the difference equation of the form

\[ x_{m+1} = B(\tau)x_m + b, \ x_0 = 0, \tag{11} \]

2) Conversely, if a continuous function \( z(t) \) is a solution of Equation (7), then \( \varphi(t) := z(t), \ t \in [0, \tau] \), satisfies the condition (9) and \( z(t) \) is given by (10).

**Proof** 1) If \( s \in [0, \tau) \) and \( n \in \mathbb{N} \), then from (6) in Lemma 2.1 and (8) it follows that

\[
\begin{align*}
z(s + n\tau) &= z(s) - \sum_{i=0}^{n-1} B(s + i\tau)b \\
&= z(s) - B(s) \sum_{i=0}^{n-1} B^i(\tau)b.
\end{align*}
\]

Clearly, we have that \( \sum_{i=0}^{n-1} B^i(\tau)b = x_n(0) \). 2) is obvious. \( \square \)

We note that \( B(t) = e^{tA}, A \in M_d(\mathbb{C}) \), satisfies the condition (8).

3  A periodicizing function

In this section we construct a periodicizing function for Equation (1); that is,

\[
\frac{dx}{dt} = Ax(t) + f(t), \quad x(0) = w \in \mathbb{C}^d.
\]
Let \( \lambda \in \sigma(A) \). If an \( M_\lambda \) valued function \( y(t) \) satisfies the equation

\[
\frac{dy}{dt} = Ay(t) + P_\lambda f(t),
\]
we say that \( y(t) \) is a solution of Equation (1) in \( M_\lambda \). Clearly, if \( x(t) \) is a solution of Equation (1), then \( P_\lambda x(t) \) is a solution of Equation (1) in \( M_\lambda \).

To apply our idea for Equation (1), we will translate the solution \( x(t) := x(t; 0, w) \) of Equation (1) as follows:

\[
x(t) = e^{At}w - z(t) + h(t),
\]
where

\[
h(t) = z(t) + \int_{0}^{t} e^{A(t-s)} f(s) ds. \tag{12}
\]

The condition that \( h(t) \) is \( \tau \)-periodic is equivalent to the condition that

\[
z(t + \tau) + \int_{0}^{t+\tau} e^{A(t+\tau-s)} f(s) ds = z(t) + \int_{0}^{t} e^{A(t-s)} f(s) ds.
\]

Since

\[
\int_{0}^{t+\tau} e^{A(t+\tau-s)} f(s) ds = e^{At}b_{f} + \int_{0}^{t} e^{A(t-s)} f(s) ds,
\]
we have

\[
\Delta_{\tau}z(t) := z(t + \tau) - z(t) = -e^{At}b_{f}. \tag{13}
\]

Therefore \( z(t) \) is an indefinite sum of \( -e^{At}b_{f} \); that is, \( z(t) = \Delta_{\tau}^{-1}(-e^{At}b_{f}) \). Summarizing these, we obtain the following result.

**Lemma 3.1** A periodicizing function for Equation (1) is an indefinite sum of \( -e^{At}b_{f} \). Moreover, the solution \( x(t) \) of Equation (1) is expressed as follows:

\[
x(t) = e^{At}w - \Delta_{\tau}^{-1}(-e^{At}b_{f}) + h(t),
\]
where

\[
h(t) = \Delta_{\tau}^{-1}(-e^{At}b_{f}) + \int_{0}^{t} e^{A(t-s)} f(s) ds
\]
is a \( \tau \)-periodic function.

Since \( h(t) \) is a \( \tau \)-periodic function and the second term of the right hand side in (12) is defined on \( \mathbb{R} \), the periodicizing function \( z(t) \) is well defined on \( \mathbb{R} \) provided \( z(t) \) is defined on \([0, \infty)\).

Now, we are in a position to state the main theorem in this paper.
Theorem 2 Let $\lambda \in \sigma(A)$.
1) If $e^{\tau \lambda} \neq 1$, then

\[ \Delta_\tau^{-1}(-e^{At}P_\lambda b) = -e^{tA}X_\lambda(A)P_\lambda b + c(t), \quad t \geq 0, \]

where $c(t)$ is periodic constant.

2) If $e^{\tau \lambda} = 1$, then

\[ \Delta_\tau^{-1}(-e^{At}P_\lambda b) = -\frac{e^{\lambda t}}{\tau} \sum_{j=0}^{m-1} \frac{t^{j+1}}{(j+1)!} (A - \lambda E)^j Y_\lambda(A)P_\lambda b + d(t), \quad t \geq 0, \]

where $d(t)$ is periodic constant.

Proof Let us consider the equation

\[ P_\lambda z(t + \tau) - P_\lambda z(t) = -P_\lambda e^{tA}b. \quad (14) \]

It follows from Lemma 2.2 that there exists a continuous solution $P_\lambda z(t)$ of Equation (14), which satisfies the relation

\[ P_\lambda z(s + n\tau) = P_\lambda z(s) - P_\lambda e^{sA}x_n(0), \quad (s \in [0, \tau), n = 0, 1, 2, \cdots), \quad (15) \]

where $x_n(0)$ is the solution of Equation (3) with $w = 0$.

1) Assume that $e^{\tau \lambda} \neq 1$. Put $X = X_\lambda(A)P_\lambda b$. Using Theorem 1 we have

\[ P_\lambda x_n(0) = e^{n\tau A}X - X, \]

from which yields that

\[ P_\lambda e^{sA}x_n(0) = e^{sA} (e^{n\tau A}X - X) = -e^{sA}X + e^{(s+n\tau)A}X. \]

Hence the relation (15) is reduced to

\[ P_\lambda z(s + n\tau) = (P_\lambda z(s) + e^{sA}X) - e^{(s+n\tau)A}X. \]

Since

\[ P_\lambda z(s + n\tau) + e^{(s+n\tau)A}X = P_\lambda z(s) + e^{sA}X, \]

$c(t) := P_\lambda z(t) + e^{tA}X$ is $\tau$-periodic. Therefore we obtain

\[ P_\lambda z(t) = -e^{-tA}X + c(t). \]
2) Assume that $e^{\lambda \tau} = 1$. Put $Y = Y_\lambda(A)P_\lambda b$. Using Theorem 1 again, we have

\[ P_\lambda e^{sA}x_n(0) \]

\[ = e^{\lambda s} \sum_{k=0}^{m-1} \frac{s^k}{k!} (A - \lambda E)^k \sum_{j=0}^{m-1} \frac{\tau^j}{j+1} (A - \lambda E)^j Y \]

\[ = e^{\lambda s} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \frac{s^k \tau^{j+1}}{k!(j+1)!} (A - \lambda E)^j Y \]

\[ = \frac{e^{\lambda s}}{\tau} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \frac{s^k \tau^{j+1}}{k!(j+1)!} (A - \lambda E)^j Y \]

\[ = \frac{e^{\lambda s}}{\tau} \sum_{i=0}^{m-1} \sum_{k=0}^{i} \frac{s^k (\tau)^{i+1-k}}{k!(i+1-k)!} (A - \lambda E)^i Y \]

Thus the relation (15) becomes

\[ P_\lambda z(s + n\tau) = P_\lambda z(s) + \frac{e^{\lambda s}}{\tau} \sum_{j=0}^{m-1} \frac{s^j}{(j+1)!} (A - \lambda E)^j Y \]

\[ - \frac{e^{\lambda(s+n\tau)}}{\tau} \sum_{j=0}^{m-1} \frac{(s+n\tau)^j}{(j+1)!} (A - \lambda E)^j Y. \]

Since

\[ d(t) := P_\lambda z(t) + \frac{e^{\lambda t}}{\tau} \sum_{j=0}^{m-1} \frac{t^j}{(j+1)!} (A - \lambda E)^j Y \]

is $\tau$-periodic, we obtain

\[ P_\lambda z(t) = - \frac{e^{\lambda t}}{\tau} \sum_{j=0}^{m-1} \frac{t^j}{(j+1)!} (A - \lambda E)^j Y + d(t). \]

\[ \square \]

Combining Lemma 3.1 and Theorem 2, we can obtain the following result, which slightly modifies the one given in [2].
Theorem 3  Let $\lambda \in \sigma(A)$ and $x(t) := x(t; 0, w)$ be the solution of Equation (1).

1) If $e^{\tau \lambda} \neq 1$, then

$$P_{\lambda}x(t) = e^{At}[P_{\lambda}w + X_{\lambda}(A)P_{\lambda}b_f] + u_{\lambda}(t, b_f)$$

$$= e^{\lambda t} \sum_{j=0}^{m-1} \frac{t^j}{j!} (A - \lambda E)^j [P_{\lambda}w + X_{\lambda}(A)P_{\lambda}b_f] + u_{\lambda}(t, b_f),$$

where

$$u_{\lambda}(t, b_f) = -e^{At}X_{\lambda}(A)P_{\lambda}b_f + \int_0^t e^{(t-s)A}P_{\lambda}f(s)ds$$

is a $\tau$-periodic solution of Equation (1) in $M_{\lambda}$.

2) If $e^{\tau \lambda} = 1$, then

$$P_{\lambda}x(t) = e^{At}P_{\lambda}w - \Delta_{\tau}^{-1}(-e^{At}P_{\lambda}b_f) + v_{\lambda}(t, b_f)$$

$$= e^{\lambda t} P_{\lambda}w + v_{\lambda}(t, b_f),$$

where $e^{At}P_{\lambda}w$ and

$$v_{\lambda}(t, b_f) := -e^{At}X_{\lambda}(A)P_{\lambda}b_f + \int_0^t e^{(t-s)A}P_{\lambda}f(s)ds$$

are $\tau$-periodic functions, which are not necessarily a solution of Equation (1) in $M_{\lambda}$.

Proof 1) Assume that $e^{\lambda \tau} \neq 1$. Combining Lemma 3.1 and Theorem 2, we have

$$P_{\lambda}x(t) = e^{At}P_{\lambda}w - \Delta_{\tau}^{-1}(-e^{At}P_{\lambda}b_f) + u_{\lambda}(t, b_f)$$

$$= e^{\lambda t} P_{\lambda}w + u_{\lambda}(t, b_f),$$

where

$$u_{\lambda}(t, b_f) = -e^{At}X_{\lambda}(A)P_{\lambda}b_f + \int_0^t e^{(t-s)A}P_{\lambda}f(s)ds.$$

Notice that the periodic constant $c(t)$ is canceled. It is easy to see that $u_{\lambda}(t, b_f)$ is a $\tau$-periodic solution of Equation (1) in $M_{\lambda}$.

2) Assume that $e^{\lambda \tau} = 1$. In view of Lemma 3.1 we have

$$P_{\lambda}x(t) = e^{At}P_{\lambda}w - \Delta_{\tau}^{-1}(-e^{At}P_{\lambda}b_f) + v_{\lambda}(t, b_f),$$

where

$$v_{\lambda}(t, b_f) = \Delta_{\tau}^{-1}(-e^{At}P_{\lambda}b_f) + \int_0^t e^{(t-s)A}P_{\lambda}f(s)ds.$$
Furthermore, from Theorem 2 we have

$$
e^{tA}P_{\lambda}w - \Delta_{\tau}^{-1}(-e^{At}P_{\lambda}b_{f})$$

$$= e^{t\lambda}P_{\lambda}w + e^{t\lambda} \sum_{j=1}^{m-1} \frac{t^{j}}{j!}(A - \lambda E)^{j}P_{\lambda}w$$

$$+ \frac{e^{\lambda l}}{\tau} \sum_{j=0}^{m-1} \frac{t^{j+1}}{(j+1)!}(A - \lambda E)^{j}Y_{\lambda}(A)P_{\lambda}b_{f} - d(t)$$

$$= e^{t\lambda}P_{\lambda}w + \frac{e^{t\lambda}}{\tau} \sum_{j=0}^{m-1} \frac{t^{j+1}}{(j+1)!}(A - \lambda E)^{j}(\tau(A - \lambda E)P_{\lambda}w + Y_{\lambda}(A)P_{\lambda}b_{f}) - d(t).$$

Therefore

$$P_{\lambda}x(t) = e^{t\lambda}P_{\lambda}w + \frac{e^{t\lambda}}{\tau} \sum_{j=0}^{m-1} \frac{t^{j+1}}{(j+1)!}(A - \lambda E)^{j}(\tau(A - \lambda E)P_{\lambda}w + Y_{\lambda}(A)P_{\lambda}b_{f})$$

$$+ e^{t\lambda}P_{\lambda}w + v_{\lambda}(t, b_{f}).$$

We note that the periodic constant $d(t)$ is canceled. □

Notice that from this result we can easily obtain asymptotic behaviors of solutions of Equation (1), for details, refer to [2].

**Example** We will explain Theorem 2 and Theorem 3 through a simple one dimensional linear differential equation

$$\frac{dx}{dt} = ax(t) + f(t), \quad x(0) = w \in \mathbb{C}, \quad (16)$$

where $a \in \mathbb{C}$ and $f$ is a continuous $\tau$-periodic scalar function. Then (13) is reduced to

$$\Delta_{\tau}z(t) := z(t + \tau) - z(t) = -e^{at}b_{f}.$$

Using Theorem 2 with $B_{0} = 1$, we have

$$z(t) := \Delta_{\tau}^{-1}(-e^{at}b_{f}) = \left\{ \begin{array}{ll}
\frac{e^{at}}{1 - e^{at}}b_{f}, & (e^{at} \neq 1) \\
\frac{t}{-\tau}b_{f}, & (e^{at} = 1). \end{array} \right. \quad (17)$$

Therefore, by Theorem 3 the solution $x(t)$ of Equation (16) is expressed as follows.

1) If $e^{at} \neq 1$, then

$$x(t; 0, w) = e^{at} \left( w - \frac{1}{1 - e^{at}}b_{f} \right) + u(t, b_{f}),$$
where
\[ u(t, b_f) = e^{at} \frac{1}{1 - e^{a\tau}} b_f + \int_0^t e^{a(t-s)} f(s) ds \]
is a \( \tau \)-periodic solution of Equation (16).

2) If \( e^{a\tau} = 1 \), then
\[ x(t; 0, w) = \frac{e^{at}}{\tau} tb_f + e^{at} w + v(t, b_f), \]
where
\[ v(t, b_f) = -e^{at} \frac{t}{\tau} b_f + \int_0^t e^{a(t-s)} f(s) ds \]
is a \( \tau \)-periodic function, however, which is not necessary a solution of of Equation (16).

References

