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タイムラグをもつ SIRS 伝染病モデルの数理解析
Mathematical Analysis of an SIRS Epidemic Model with Delay

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1 Introduction

Classical epidemic models assume that the total population size is constant. More recent models consider a variable population size in order to take into account a longer time scale with disease causing death and reduced reproduction, see [3, 4].

SIRS epidemic models have been studied by many authors, see [2, 5]. It is our aim to analyze a variable population SIRS epidemic model with a delay. The total (host) population size $N(t)$ is divided into susceptible, infective, and recovered with temporary immunity individuals. The respective numbers are denoted by S, I and R. The flow of individuals can schematically be described as

$$\begin{array}{ccccccc} B(N)N\downarrow & & & & & & \\ S & \xrightarrow{\beta SI/N} & I & \xrightarrow{\lambda I} & R^{\tau} & \longrightarrow & S. \\ \mu S\downarrow & & (\mu+\alpha)I\downarrow & & \mu R\downarrow & & \end{array}$$

We assume that everybody is born as susceptible. $B(N)N$ is a birth rate function with $B(N)$ satisfying the following assumptions for $N \in (0, \infty)$:

(A1) $B(N) > 0$;

(A2) $B(N)$ is continuously differentiable with $B'(N) < 0$;

(A3) $B(0^+) > \mu + \alpha$ and $\mu > B(+\infty)$.

Note that (A2) and (A3) imply that $B^{-1}(N)$ exists for $N \in (B(\infty), B(0^+))$, and (A3) assures that N does not go to extinction and cannot blow up. The parameter $\mu > 0$ is the natural death rate constant, $\alpha \geq 0$ is the disease-related death rate constant, and $\lambda \geq 0$ is rate constant for recovery. The force of infection is assumed to be of standard type, namely $\beta I/N$, with $\beta > 0$, the effective per capita contact rate constant of infective individuals. The time delay τ denotes a constant immune period.

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Our model thus take the following form:

$$N(t) = S(t) + I(t) + R(t), \tag{1.1}$$

$$S'(t) = B(N(t))N(t) - \mu S(t) - \frac{\beta S(t)I(t)}{N(t)} + \lambda I(t-\tau)e^{-\mu\tau}, \tag{1.2}$$

$$I'(t) = \frac{\beta S(t)I(t)}{N(t)} - (\mu + \lambda + \alpha)I(t), \tag{1.3}$$

$$R'(t) = \lambda I(t) - \lambda I(t-\tau)e^{-\mu\tau} - \mu R(t), \tag{1.4}$$

with initial conditions

$$S(\theta) > 0,\ I(\theta) > 0,\ R(\theta) > 0 \text{ on } [-\tau, 0]. \tag{1.5}$$

In order to assure continuity of solutions at time 0, we assume that

$$R(0) = \int_{-\tau}^{0} \lambda I(u)e^{\mu u}du. \tag{1.6}$$

System (1.1)–(1.4) always has the disease-free equilibrium $E_0 = (B^{-1}(\mu), 0, 0)$. Furthermore, if the basic reproduction number $\mathcal{R}_0 := \frac{1}{\mu+\lambda+\alpha} > 1$, then it also has the unique endemic equilibrium $E_+ = (S^*, I^*, R^*)$ where

$$S^* = \frac{\mu+\lambda+\alpha}{\beta}N^*, I^* = \left(1 - \frac{\mu+\lambda+\alpha}{\beta}\right)N^* \Big/ \left(1 + \frac{\lambda(1-e^{-\mu\tau})}{\mu}\right), R^* = \frac{\lambda(1-e^{-\mu\tau})}{\mu}I^*$$
$$\text{and} \quad N^* = B^{-1}\left(\mu + \alpha\left(1 - \frac{\mu+\lambda+\alpha}{\beta}\right) \Big/ \left(1 + \frac{\lambda(1-e^{-\mu\tau})}{\mu}\right)\right).$$

2 Main result

The following basic result for solutions of system is given. The proof is omitted.

Theorem 2.1. *Let $S(t)$, $I(t)$, $R(t)$ be a solution of the delay differential system (1.2) – (1.4) with $N(t)$ given by (1.1), and initial conditions given by (1.5). In addition, suppose that (1.6) holds. For all $t \geq 0$, this solution exists, is unique and has $S(t) > 0$, $I(t) > 0$, $R(t) > 0$.*

A linear analysis shows the following theorem for disease-free equilibrium.

Theorem 2.2. *If $\mathcal{R}_0 < 1$, then the disease-free equilibrium is locally asymptotically stable.*

A global stability result can be given by using the following results. Consider the systems:

$$x' = f(t, x) \tag{2.1}$$

$$y' = g(y) \tag{2.2}$$

where f and g are continuous and locally Lipschitz in x in $\mathbb{R}^n$ and solutions exist for all positive time. (2.1) is called asymptotically autonomous with limit equation in $\mathbb{R}^n$.

Lemma 2.1 ([6]). *Let e be a locally asymptotically stable equilibrium of (2.2) and ω be the ω-limit set of a forward bounded solution $x(t)$ of (2.1). If ω contains a point y_0 such that the solution of (2.2) with $y(0) = y_0$ converges to e as $t \to \infty$, then $\omega = \{e\}$, i.e. $x(t) \to e$ as $t \to \infty$.*

Corollary 2.1. *If solutions of system (2.1) are bounded and the equilibrium e of the limit system (2.2) is globally asymptotically stable, then any solution $x(t)$ of system (2.1) satisfies $x(t) \to e$ as $t \to \infty$.*

Theorem 2.3. *For $\mathcal{R}_0 < 1$ all solutions of the system (1.2)–(1.4) with (1.1) approach the disease free equilibrium as $t \to \infty$.*

Proof. By (1.3), we have $I' \leq (\beta - \mu - \lambda - \alpha)I$, hence $I(t)$ has limit zero as $t \to \infty$ if $\beta - \mu - \lambda - \alpha < 0$. Then $R(t) \to 0$ as $t \to \infty$ from (1.4).

Add equations (1.2)–(1.4), and use (1.1) to obtain

$$N' = (B(N) - \mu)N - \alpha I. \tag{2.3}$$

This equation has the limit equation

$$N' = (B(N) - \mu)N. \tag{2.4}$$

By Corollary 2.1, $N(t) \to B^{-1}(\mu)$ as $t \to \infty$. Hence $S(t) \to B^{-1}(\mu)$ as $t \to \infty$. □

A global property of the endemic equilibrium for a restricted set of parameter values can be given as follows.

Theorem 2.4. *Suppose that $\alpha = 0$ and $\mathcal{R}_0 > 1$. If $\tau < \frac{1}{\lambda}$, all solutions of system (1.2)–(1.4) with (1.1) approach the endemic equilibrium as $t \to \infty$.*

Proof. Define $i(t) = I(t)/N(t)$. Let $i^* = I^*/N^*$. System (1.2)–(1.4) leads to the following system

$$\begin{aligned} i'(t) &= \beta\left\{i^* - i(t) + \frac{\lambda}{\mu}(1 - e^{-\mu\tau})i^* - \frac{\lambda}{N(t)}\int_{t-\tau}^{t} i(u)N(u)e^{-\mu(t-u)}du\right\} i(t) \\ &\quad - (B(N) - \mu)i(t) \\ N'(t) &= (B(N(t)) - \mu)N(t). \end{aligned} \tag{2.5}$$

This system has a unique internal equilibrium $(i^*, B^{-1}(\mu))$ corresponding to the endemic equilibrium E_+.

By the second equation of (2.5), if $N(0) \le B^{-1}(\mu)$, $N(t)$ is monotone increasing and $N(t) \le B^{-1}(\mu)$, whereas if $N(0) > B^{-1}(\mu)$, $N(t)$ is monotone decreasing and $N(t) > B^{-1}(\mu)$.

Derivative of V_1 along a solution is

$$
\begin{aligned}
\dot{V}_1(t) = {} & \beta\left\{i^* - i(t) + \frac{\lambda}{\mu}(1-e^{-\mu\tau})i^* - \frac{\lambda}{N(t)}\int_{t-\tau}^{t} i(u)N(u)e^{-\mu(t-u)}du\right\} i(t)\left(1-\frac{i^*}{i(t)}\right) \\
& - \left(B(N(t))-\mu\right)\left(i(t)-i^*\right) \\
= {} & -\beta\left(i(t)-i^*\right)^2 + \beta\lambda\left(i(t)-i^*\right)\int_{t-\tau}^{t}\left(i^*e^{-\mu(t-u)} - i(u)\frac{N(u)}{N(t)}e^{-\mu(t-u)}\right)du \\
& - \left(B(N(t))-\mu\right)\left(i(t)-i^*\right) \\
= {} & -\beta\left(i(t)-i^*\right)^2 - \beta\lambda\int_{t-\tau}^{t}\left(i(t)-i^*\right)\left(i(u)-i^*\right)e^{-\mu(t-u)}du \\
& + \beta\lambda\int_{t-\tau}^{t}\left(i(t)-i^*\right)\left(1-\frac{N(u)}{N(t)}\right)i(u)e^{-\mu(t-u)}du - \left(B(N(t))-\mu\right)\left(i(t)-i^*\right) \\
\le {} & -\beta\left(i(t)-i^*\right)^2 + \frac{1}{2}\beta\lambda\int_{t-\tau}^{t}\left\{\left(i(t)-i^*\right)^2 + \left(i(u)-i^*\right)^2e^{-2\mu(t-u)}\right\}du \\
& + \beta\lambda\int_{t-\tau}^{t}\left(i(t)-i^*\right)\left(1-\frac{N(u)}{N(t)}\right)i(u)e^{-\mu(t-u)}du - \left(B(N(t))-\mu\right)\left(i(t)-i^*\right) \\
\le {} & -\beta\left(i(t)-i^*\right)^2 + \frac{1}{2}\beta\lambda\tau\left(i(t)-i^*\right)^2 + \frac{1}{2}\beta\lambda\int_{t-\tau}^{t}\left(i(u)-i^*\right)^2du \\
& + \beta\lambda\int_{t-\tau}^{t}\left(i(t)-i^*\right)\left(1-\frac{N(u)}{N(t)}\right)i(u)e^{-\mu(t-u)}du - \left(B(N(t))-\mu\right)\left(i(t)-i^*\right)
\end{aligned}
\tag{2.6}
$$

If $N(0) \le B^{-1}(\mu)$, we have from (2.6),

$$
\begin{aligned}
\dot{V}_1(t) \le {} & -\beta\left(i(t)-i^*\right)^2 + \frac{1}{2}\beta\lambda\tau\left(i(t)-i^*\right)^2 + \frac{1}{2}\beta\lambda\int_{t-\tau}^{t}\left(i(u)-i^*\right)^2du \\
& + \beta\lambda\int_{t-\tau}^{t}\left(1-\frac{N(u)}{N(t)}\right)du + i^*\left(B(N(t))-\mu\right).
\end{aligned}
\tag{2.7}
$$

In addition, define

$$
V_2(t) := \frac{1}{2}\beta\lambda\int_{t-\tau}^{t}\int_{\theta}^{t}\left(i(\xi)-i^*\right)^2 d\xi d\theta + \beta\lambda\int_{t-\tau}^{t}\int_{\theta}^{t}\left(1-\frac{N(\xi)}{N(t)}\right)d\xi d\theta. \tag{2.8}
$$

Then (2.7) and (2.8) lead to

$$
\frac{d}{dt}(V_1+V_2) \le -\beta\left(i(t)-i^*\right)^2 + \frac{1}{2}\beta\lambda\tau\left(i(t)-i^*\right)^2 + \frac{1}{2}\beta\lambda\tau(i(t)-i^*)^2
$$

$$
\begin{aligned}
&+\beta\lambda\int_{t-\tau}^{t}\int_{\theta}^{t}\frac{N(\xi)N'(t)}{N^2(t)}d\xi d\theta+i^*\left(B(N(t))-\mu\right)\\
&\le -\beta(1-\lambda\tau)\left(i(t)-i^*\right)^2\\
&\quad+\beta\lambda\frac{N'(t)}{N(t)}\int_{t-\tau}^{t}\int_{t-\tau}^{t}\frac{N(\xi)}{N(t)}d\xi d\theta+i^*\left(B(N(t))-\mu\right)\\
&=-\beta(1-\lambda\tau)\left(i(t)-i^*\right)^2\\
&\quad+\beta\lambda\tau\frac{N'(t)}{N(t)}\int_{t-\tau}^{t}\frac{N(\xi)}{N(t)}d\xi+i^*\left(B(N(t))-\mu\right)\\
&\le -\beta(1-\lambda\tau)\left(i(t)-i^*\right)^2+\beta\lambda\tau^2\frac{N'(t)}{N(t)}+i^*\left(B(N(t))-\mu\right)\\
&=-\beta(1-\lambda\tau)\left(i(t)-i^*\right)^2+(\beta\lambda\tau^2+i^*)\frac{N'(t)}{N(t)}.
\end{aligned}
$$

Note that

$$\int_0^{+\infty}\frac{N'(u)}{N(u)}du=\ln\frac{B^{-1}(\mu)}{N(0)}.$$

If $1>\lambda\tau$, we have

$$\int_0^{+\infty}(i(u)-i^*)^2du<+\infty. \tag{2.9}$$

From (2.5), we see that $(i(t)-i^*)^2$ is uniformly continuous on $[0,\infty)$. It follows from the well-known Barbălat's lemma (see [1]),

$$\lim_{t\to+\infty} i(t)=i^*.$$

From (1.4),

$$\lim_{t\to+\infty} R(t)=R^*,$$

which implies

$$\lim_{t\to+\infty} S(t)=S^*.$$

In a similar manner, we can show that E_+ is globally attractive if $N(0)>B^{-1}(\mu)$. This completes the proof. □

3 Summary

In this paper, we considered stability of the few variable population $SIRS$ epidemic model with a delay. We showed that if $R_0<1$, the disease-free equilibrium is globally asymptotically stable, whereas if $R_0>1$, the endemic equilibrium is globally attractive for small delay.

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