

1 Introduction

Classical epidemic models assume that the total population size is constant. More recent models consider a variable population size in order to take into account a longer time scale with disease causing death and reduced reproduction, see [3, 4].

SIRS epidemic models have been studied by many authors, see [2, 5]. It is our aim to analyze a variable population SIRS epidemic model with a delay. The total (host) population size \( N(t) \) is divided into susceptible, infective, and recovered with temporary immunity individuals. The respective numbers are denoted by \( S, I \) and \( R \). The flow of individuals can schematically be described as

\[
\begin{array}{c}
\downarrow B(N)N \\
S \xrightarrow{\beta SI/N} I \xrightarrow{\lambda I} R^\tau \xrightarrow{\mu R} S, \\
\mu S \xrightarrow{(\mu+\alpha)I} (\mu+\alpha)I \xrightarrow{\mu R} \mu R
\end{array}
\]

We assume that everybody is born as susceptible. \( B(N)N \) is a birth rate function with \( B(N) \) satisfying the following assumptions for \( N \in (0, \infty) \):

(A1) \( B(N) > 0 \);

(A2) \( B(N) \) is continuously differentiable with \( B'(N) < 0 \);

(A3) \( B(0^+) > \mu + \alpha \) and \( \mu > B(+\infty) \).

Note that (A2) and (A3) imply that \( B^{-1}(N) \) exists for \( N \in (B(\infty), B(0^+)) \), and (A3) assures that \( N \) does not go to extinction and cannot blow up. The parameter \( \mu > 0 \) is the natural death rate constant, \( \alpha \geq 0 \) is the disease-related death rate constant, and \( \lambda \geq 0 \) is rate constant for recovery. The force of infection is assumed to be of standard type, namely \( \beta I/N \), with \( \beta > 0 \), the effective per capita contact rate constant of infective individuals. The time delay \( \tau \) denotes a constant immune period.

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1Email: yoshida@ms.osakafu-u.ac.jp
2Email: hara@ms.osakafu-u.ac.jp
Our model thus take the following form:

\[
N(t) = S(t) + I(t) + R(t),
\]

(1.1)

\[
S'(t) = B(N(t))N(t) - \mu S(t) - \frac{\beta S(t)I(t)}{N(t)} + \lambda I(t-\tau)e^{-\mu\tau},
\]

(1.2)

\[
I'(t) = \frac{\beta S(t)I(t)}{N(t)} - (\mu + \lambda + \alpha)I(t),
\]

(1.3)

\[
R'(t) = \lambda I(t) - \lambda I(t-\tau)e^{-\mu\tau} - \mu R(t),
\]

(1.4)

with initial conditions

\[
S(\theta) > 0, \quad I(\theta) > 0, \quad R(\theta) > 0 \quad \text{on } [-\tau, 0].
\]

(1.5)

In order to assure continuity of solutions at time 0, we assume that

\[
R(0) = \int_{-\tau}^{0} \lambda I(u)e^{\mu u} du.
\]

(1.6)

System (1.1)-(1.4) always has the disease-free equilibrium \(E_0 = (B^{-1}(\mu), 0, 0)\). Furthermore, if the basic reproduction number \(R_0 := \frac{1}{\mu + \lambda + \alpha} > 1\), then it also has the unique endemic equilibrium \(E_+ = (S^*, I^*, R^*)\) where

\[
S^* = \frac{\mu + \lambda + \alpha}{\beta} N^*, \quad I^* = \left(1 - \frac{\mu + \lambda + \alpha}{\beta}\right) N^*/\left(1 + \frac{\lambda(1-e^{-\mu\tau})}{\mu}\right), \quad R^* = \frac{\lambda(1-e^{-\mu\tau})}{\mu} I^*
\]

and

\[
N^* = B^{-1}\left(\mu + \alpha \left(1 - \frac{\mu + \lambda + \alpha}{\beta}\right)/\left(1 + \frac{\lambda(1-e^{-\mu\tau})}{\mu}\right)\right).
\]

2 Main result

The following basic result for solutions of system is given. The proof is omitted.

**Theorem 2.1.** Let \(S(t), I(t), R(t)\) be a solution of the delay differential system (1.2) – (1.4) with \(N(t)\) given by (1.1), and initial conditions given by (1.5). In addition, suppose that (1.6) holds. For all \(t \geq 0\), this solution exists, is unique and has \(S(t) > 0, I(t) > 0, R(t) > 0\).

A linear analysis shows the following theorem for disease-free equilibrium.

**Theorem 2.2.** If \(R_0 < 1\), then the disease-free equilibrium is locally asymptotically stable.

A global stability result can be given by using the following results. Consider the systems:

\[
x' = f(t, x)
\]

(2.1)

\[
y' = g(y)
\]

(2.2)
where $f$ and $g$ are continuous and locally Lipschitz in $x$ in $\mathbb{R}^n$ and solutions exist for all positive time. (2.1) is called asymptotically autonomous with limit equation in $\mathbb{R}^n$.

**Lemma 2.1 ([6]).** Let $e$ be a locally asymptotically stable equilibrium of (2.2) and $\omega$ be the $\omega$-limit set of a forward bounded solution $x(t)$ of (2.1). If $\omega$ contains a point $y_0$ such that the solution of (2.2) with $y(0) = y_0$ converges to $e$ as $t \to \infty$, then $\omega = \{e\}$, i.e. $x(t) \to e$ as $t \to \infty$.

**Corollary 2.1.** If solutions of system (2.1) are bounded and the equilibrium $e$ of the limit system (2.2) is globally asymptotically stable, then any solution $x(t)$ of system (2.1) satisfies $x(t) \to e$ as $t \to \infty$.

**Theorem 2.3.** For $R_0 < 1$ all solutions of the system (1.2)-(1.4) with (1.1) approach the disease free equilibrium as $t \to \infty$.

**Proof.** By (1.3), we have $I' \leq (\beta - \mu - \lambda - \alpha)I$, hence $I(t)$ has limit zero as $t \to \infty$ if $\beta - \mu - \lambda - \alpha < 0$. Then $R(t) \to 0$ as $t \to \infty$ from (1.4).

Add equations (1.2)-(1.4), and use (1.1) to obtain

$$N' = (B(N) - \mu)N - \alpha I.$$

This equation has the limit equation

$$N' = (B(N) - \mu)N.$$

By Corollary 2.1, $N(t) \to B^{-1}(\mu)$ as $t \to \infty$. Hence $S(t) \to B^{-1}(\mu)$ as $t \to \infty$. $\square$

A global property of the endemic equilibrium for a restricted set of parameter values can be given as follows.

**Theorem 2.4.** Suppose that $\alpha = 0$ and $R_0 > 1$. If $\tau < \frac{1}{\lambda}$, all solutions of system (1.2)-(1.4) with (1.1) approach the endemic equilibrium as $t \to \infty$.

**Proof.** Define $i(t) = I(t)/N(t)$. Let $i^* = I^*/N^*$. System (1.2)-(1.4) leads to the following system

$$i'(t) = \beta \left\{i^* - i(t) + \frac{\lambda}{\mu}(1 - e^{-\mu \tau})i^* - \frac{\lambda}{N(t)} \int_{t-\tau}^{t} i(u)N(u)e^{-\mu(t-u)}du \right\} i(t)$$

$$- (B(N) - \mu)i(t)$$

$$N'(t) = (B(N(t)) - \mu)N(t).$$

(2.5)
This system has a unique internal equilibrium \((i^*, B^{-1}(-\mu))\) corresponding to the endemic equilibrium \(E_+\).

By the second equation of (2.5), if \(N(0) \leq B^{-1}(-\mu)\), \(N(t)\) is monotone increasing and \(N(t) \leq B^{-1}(-\mu)\), whereas if \(N(0) > B^{-1}(-\mu)\), \(N(t)\) is monotone decreasing and \(N(t) > B^{-1}(-\mu)\).

Derivative of \(V_1\) along a solution is

\[
\dot{V}_1(t) = \beta \left\{ i^* - i(t) + \frac{\lambda}{\mu} (1 - e^{-\mu\tau}) i^* - \frac{\lambda}{N(t)} \int_{t-\tau}^{t} i(u) N(u) e^{-\mu(t-u)} du \right\} i(t) \left( 1 - \frac{i^*}{i(t)} \right)
- (B(N(t)) - \mu) (i(t) - i^*)
= -\beta (i(t) - i^*)^2 + \beta \lambda (i(t) - i^*) \int_{t-\tau}^{t} \left( i^* e^{-\mu(t-u)} - i(u) \frac{N(u)}{N(t)} e^{-\mu(t-u)} \right) du
- (B(N(t)) - \mu) (i(t) - i^*)
= -\beta (i(t) - i^*)^2 - \beta \lambda \int_{t-\tau}^{t} (i(t) - i^*) (i(u) - i^*) e^{-\mu(t-u)} du
+ \beta \lambda \int_{t-\tau}^{t} (i(t) - i^*) \left( 1 - \frac{N(u)}{N(t)} \right) i(u) e^{-\mu(t-u)} du - (B(N(t)) - \mu) (i(t) - i^*)
\leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} (i(t) - i^*)^2 + (i(u) - i^*)^2 e^{-2\mu(t-u)} du
+ \beta \lambda \int_{t-\tau}^{t} (i(t) - i^*) \left( 1 - \frac{N(u)}{N(t)} \right) i(u) e^{-\mu(t-u)} du - (B(N(t)) - \mu) (i(t) - i^*)
\leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} (i(t) - i^*)^2 + \beta \lambda \int_{t-\tau}^{t} (i(t) - i^*) \left( 1 - \frac{N(u)}{N(t)} \right) i(u) e^{-\mu(t-u)} du - (B(N(t)) - \mu) (i(t) - i^*)
\]

(2.6)

If \(N(0) \leq B^{-1}(-\mu)\), we have from (2.6),

\[
\dot{V}_1(t) \leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \tau (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} (i(u) - i^*)^2 du
+ \beta \lambda \int_{t-\tau}^{t} \left( 1 - \frac{N(u)}{N(t)} \right) du + i^* (B(N(t)) - \mu).
\]

(2.7)

In addition, define

\[
V_2(t) := \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} \int_{0}^{1} (i(\xi) - i^*)^2 d\xi d\theta + \beta \lambda \int_{t-\tau}^{t} \int_{0}^{1} \left( 1 - \frac{N(\xi)}{N(t)} \right) d\xi d\theta.
\]

(2.8)

Then (2.7) and (2.8) lead to

\[
\frac{d}{dt} (V_1 + V_2) \leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \tau (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \tau (i(t) - i^*)^2
\]
\[
\begin{align*}
&+ \beta \lambda \int_{t-\tau}^{t} \int_{\theta}^{t} \frac{N(\xi)N'(t)}{N^2(t)} d\xi d\theta + i^* (B(N(t)) - \mu) \\
&\leq -\beta(1 - \lambda \tau) (i(t) - i^*)^2 \\
&+ \beta \lambda \frac{N'(t)}{N(t)} \int_{t-\tau}^{t} \frac{N(\xi)}{N(t)} d\xi + i^* (B(N(t)) - \mu) \\
&= -\beta(1 - \lambda \tau) (i(t) - i^*)^2 \\
&+ \beta \lambda \tau \frac{N'(t)}{N(t)} \int_{t-\tau}^{t} \frac{N(\xi)}{N(t)} d\xi + i^* (B(N(t)) - \mu) \\
&\leq -\beta(1 - \lambda \tau) (i(t) - i^*)^2 + \beta \lambda \tau^2 \frac{N'(t)}{N(t)} + i^* (B(N(t)) - \mu) \\
&= -\beta(1 - \lambda \tau) (i(t) - i^*)^2 + (\beta \lambda \tau^2 + i^*) \frac{N'(t)}{N(t)}.
\end{align*}
\]

Note that

\[
\int_{0}^{+\infty} \frac{N'(u)}{N(u)} du = \ln \frac{B^{-1}(\mu)}{N(0)}.
\]

If \(1 > \lambda \tau\), we have

\[
\int_{0}^{+\infty} (i(u) - i^*)^2 du < +\infty.
\]

From (2.5), we see that \((i(t) - i^*)^2\) is uniformly continuous on \([0, \infty)\). It follows from the well-known Barbálat's lemma (see [1]),

\[
\lim_{t \to +\infty} i(t) = i^*.
\]

From (1.4),

\[
\lim_{t \to +\infty} R(t) = R^*,
\]

which implies

\[
\lim_{t \to +\infty} S(t) = S^*.
\]

In a similar manner, we can show that \(E_+\) is globally attractive if \(N(0) > B^{-1}(\mu)\).

This completes the proof. \(\square\)

3 Summary

In this paper, we considered stability of the few variable population SIRS epidemic model with a delay. We showed that if \(R_0 < 1\), the disease-free equilibrium is globally asymptotically stable, whereas if \(R_0 > 1\), the endemic equilibrium is globally attractive for small delay.
References


