<table>
<thead>
<tr>
<th>Title</th>
<th>Mathematical Analysis of an SIRS Epidemic Model with Delay (Functional Equations and Complex Systems)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Yoshida, Naoki; Hara, Tadayuki</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2005), 1445: 40-45</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2005-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/47640">http://hdl.handle.net/2433/47640</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
1 Introduction

Classical epidemic models assume that the total population size is constant. More recent models consider a variable population size in order to take into account a longer time scale with disease causing death and reduced reproduction, see [3, 4].

SIRS epidemic models have been studied by many authors, see [2, 5]. It is our aim to analyze a variable population SIRS epidemic model with a delay. The total (host) population size $N(t)$ is divided into susceptible, infective, and recovered with temporary immunity individuals. The respective numbers are denoted by $S$, $I$ and $R$. The flow of individuals can schematically be described as

$$
\begin{align*}
B(N)N & \downarrow \\
S & \xrightarrow{\beta S I/N} I \\
\mu S & \xrightarrow{(\mu+\alpha)I} \mu R \\
\lambda I & \downarrow \\
R & \xrightarrow{\mu R} S.
\end{align*}
$$

We assume that everybody is born as susceptible. $B(N)N$ is a birth rate function with $B(N)$ satisfying the following assumptions for $N \in (0, \infty)$:

(A1) $B(N) > 0$;

(A2) $B(N)$ is continuously differentiable with $B'(N) < 0$;

(A3) $B(0^+) > \mu + \alpha$ and $\mu > B(+\infty)$.

Note that (A2) and (A3) imply that $B^{-1}(N)$ exists for $N \in (B(\infty), B(0^+))$, and (A3) assures that $N$ does not go to extinction and cannot blow up. The parameter $\mu > 0$ is the natural death rate constant, $\alpha \geq 0$ is the disease-related death rate constant, and $\lambda \geq 0$ is rate constant for recovery. The force of infection is assumed to be of standard type, namely $\beta I/N$, with $\beta > 0$, the effective per capita contact rate constant of infective individuals. The time delay $\tau$ denotes a constant immune period.

---

1 Email: yoshida@ms.osakafu-u.ac.jp

2 Email: hara@ms.osakafu-u.ac.jp
Our model thus take the following form:

\[
N(t) = S(t) + I(t) + R(t), \quad (1.1)
\]

\[
S'(t) = B(N(t))N(t) - \mu S(t) - \frac{\beta S(t)I(t)}{N(t)} + \lambda I(t - \tau)e^{-\mu\tau}, \quad (1.2)
\]

\[
I'(t) = \frac{\beta S(t)I(t)}{N(t)} - (\mu + \lambda + \alpha)I(t), \quad (1.3)
\]

\[
R'(t) = \lambda I(t) - \lambda I(t - \tau)e^{-\mu\tau} - \mu R(t), \quad (1.4)
\]

with initial conditions

\[
S(\theta) > 0, \ I(\theta) > 0, \ R(\theta) > 0 \text{ on } [-\tau, 0]. \quad (1.5)
\]

In order to assure continuity of solutions at time 0, we assume that

\[
R(0) = \int_{-\tau}^{0} \lambda I(u)e^\mu du. \quad (1.6)
\]

System (1.1)–(1.4) always has the disease-free equilibrium \( E_0 = (B^{-1}(\mu), 0, 0) \). Furthermore, if the basic reproduction number \( R_0 := \frac{1}{\mu + \lambda + \alpha} > 1 \), then it also has the unique endemic equilibrium \( E_+ = (S^*, I^*, R^*) \) where

\[
S^* = \frac{\mu + \lambda + \alpha}{\beta}N^*, \quad I^* = \left(1 - \frac{\mu + \lambda + \alpha}{\beta}\right)N^*/\left(1 + \frac{\lambda(1 - e^{-\mu\tau})}{\mu}\right), \quad R^* = \frac{\lambda(1 - e^{-\mu\tau})}{\mu}I^*
\]

and

\[
N^* = B^{-1}\left(\mu + \alpha\left(1 - \frac{\mu + \lambda + \alpha}{\beta}\right)\right)/\left(1 + \frac{\lambda(1 - e^{-\mu\tau})}{\mu}\right).
\]

## 2 Main result

The following basic result for solutions of system is given. The proof is omitted.

**Theorem 2.1.** Let \( S(t), I(t), R(t) \) be a solution of the delay differential system (1.2)–(1.4) with \( N(t) \) given by (1.1), and initial conditions given by (1.5). In addition, suppose that (1.6) holds. For all \( t \geq 0 \), this solution exists, is unique and has \( S(t) > 0, I(t) > 0 \), \( R(t) > 0 \).

A linear analysis shows the following theorem for disease-free equilibrium.

**Theorem 2.2.** If \( R_0 < 1 \), then the disease-free equilibrium is locally asymptotically stable.

A global stability result can be given by using the following results. Consider the systems:

\[
x' = f(t, x) \quad (2.1)
\]

\[
y' = g(y) \quad (2.2)
\]
where \( f \) and \( g \) are continuous and locally Lipschitz in \( x \) in \( \mathbb{R}^n \) and solutions exist for all positive time. (2.1) is called asymptotically autonomous with limit equation in \( \mathbb{R}^n \).

**Lemma 2.1** ([8]). Let \( e \) be a locally asymptotically stable equilibrium of (2.2) and \( \omega \) be the \( \omega \)-limit set of a forward bounded solution \( x(t) \) of (2.1). If \( \omega \) contains a point \( y_0 \) such that the solution of (2.2) with \( y(0) = y_0 \) converges to \( e \) as \( t \to \infty \), then \( \omega = \{ e \} \), i.e. \( x(t) \to e \) as \( t \to \infty \).

**Corollary 2.1.** If solutions of system (2.1) are bounded and the equilibrium \( e \) of the limit system (2.2) is globally asymptotically stable, then any solution \( x(t) \) of system (2.1) satisfies \( x(t) \to e \) as \( t \to \infty \).

**Theorem 2.3.** For \( R_0 < 1 \) all solutions of the system (1.2)-(1.4) with (1.1) approach the disease free equilibrium as \( t \to \infty \).

**Proof.** By (1.3), we have \( I' \leq (\beta - \mu - \lambda - \alpha)I \), hence \( I(t) \) has limit zero as \( t \to \infty \) if \( \beta - \mu - \lambda - \alpha < 0 \). Then \( R(t) \to 0 \) as \( t \to \infty \) from (1.4).

Add equations (1.2)-(1.4), and use (1.1) to obtain

\[
N' = (B(N) - \mu)N - \alpha I.
\]  

This equation has the limit equation

\[
N' = (B(N) - \mu)N.
\]  

By Corollary 2.1, \( N(t) \to B^{-1}(\mu) \) as \( t \to \infty \). Hence \( S(t) \to B^{-1}(\mu) \) as \( t \to \infty \). \( \square \)

A global property of the endemic equilibrium for a restricted set of parameter values can be given as follows.

**Theorem 2.4.** Suppose that \( \alpha = 0 \) and \( R_0 > 1 \). If \( \tau < \frac{1}{\lambda} \), all solutions of system (1.2)-(1.4) with (1.1) approach the endemic equilibrium as \( t \to \infty \).

**Proof.** Define \( i(t) = I(t)/N(t) \). Let \( i^* = I^*/N^* \). System (1.2)-(1.4) leads to the following system

\[
i'(t) = \beta \left\{ i^* - i(t) + \frac{\lambda}{\mu} (1 - e^{-\mu \tau}) i^* - \frac{\lambda}{N(t)} \int_{t-\tau}^{t} i(u)N(u)e^{-\mu(t-u)}du \right\} i(t) \\
- (B(N) - \mu)i(t)
\]

\[
N'(t) = (B(N(t)) - \mu)N(t).
\]  

(2.5)
This system has a unique internal equilibrium \((i^*, B^{-1}(\mu))\) corresponding to the endemic equilibrium \(E_+\).

By the second equation of (2.5), if \(N(0) \leq B^{-1}(\mu)\), \(N(t)\) is monotone increasing and \(N(t) \leq B^{-1}(\mu)\), whereas if \(N(0) > B^{-1}(\mu)\), \(N(t)\) is monotone decreasing and \(N(t) > B^{-1}(\mu)\).

Derivative of \(V_1\) along a solution is

\[
\dot{V}_1(t) = \beta \left\{ i^* - i(t) + \frac{\lambda}{\mu} (1 - e^{-\mu t}) i^* - \frac{\lambda}{N(t)} \int_{t-\tau}^{t} i(u) N(u) e^{-\mu (t-u)} du \right\} i(t) \left( 1 - \frac{i^*}{i(t)} \right) \\
- (B(N(t)) - \mu) (i(t) - i^*) \\
= -\beta (i(t) - i^*)^2 + \beta \lambda \int_{t-\tau}^{t} (i(t) - i^*) (i(u) - i^*) e^{-\mu (t-u)} du \\
+ \beta \lambda \int_{t-\tau}^{t} (i(t) - i^*) \left( 1 - \frac{N(u)}{N(t)} \right) i(u) e^{-\mu (t-u)} du - (B(N(t)) - \mu) (i(t) - i^*) \\
\leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} \left\{ (i(t) - i^*)^2 + (i(u) - i^*)^2 e^{-2\mu (t-u)} \right\} du \\
+ \beta \lambda \int_{t-\tau}^{t} (i(t) - i^*) \left( 1 - \frac{N(u)}{N(t)} \right) i(u) e^{-\mu (t-u)} du - (B(N(t)) - \mu) (i(t) - i^*) \\
\leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \tau (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} (i(u) - i^*)^2 du \\
+ \beta \lambda \int_{t-\tau}^{t} (i(t) - i^*) \left( 1 - \frac{N(u)}{N(t)} \right) i(u) e^{-\mu (t-u)} du - (B(N(t)) - \mu) (i(t) - i^*)
\]

(2.6)

If \(N(0) \leq B^{-1}(\mu)\), we have from (2.6),

\[
\dot{V}_1(t) \leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \tau (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} (i(u) - i^*)^2 du \\
+ \beta \lambda \int_{t-\tau}^{t} \left( 1 - \frac{N(u)}{N(t)} \right) i(u) e^{-\mu (t-u)} du + i^* (B(N(t)) - \mu).
\]

(2.7)

In addition, define

\[
V_2(t) := \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} \int_{\theta}^{t} (i(\xi) - i^*)^2 d\xi d\theta + \beta \lambda \int_{t-\tau}^{t} \int_{\theta}^{t} \left( 1 - \frac{N(\xi)}{N(t)} \right) d\xi d\theta.
\]

(2.8)

Then (2.7) and (2.8) lead to

\[
\frac{d}{dt} (V_1 + V_2) \leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \tau (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \tau (i(t) - i^*)^2
\]
44

\begin{align*}
&+ \beta \lambda \int_{t-	au}^{t} \int_{t-	au}^{t} \frac{N(\xi)N'(t)}{N^2(t)} d\xi d\theta + i^*(B(N(t)) - \mu) \\
&\leq -\beta(1 - \lambda \tau) (i(t) - i^*)^2 \\
&+ \beta \lambda \frac{N'(t)}{N(t)} \int_{t-	au}^{t} \int_{t-	au}^{t} \frac{N(\xi)}{N(t)} d\xi d\theta + i^*(B(N(t)) - \mu) \\
&= -\beta(1 - \lambda \tau) (i(t) - i^*)^2 \\
&+ \beta \lambda \frac{N'(t)}{N(t)} \int_{t-	au}^{t} \frac{N(\xi)}{N(t)} d\xi + i^*(B(N(t)) - \mu) \\
&\leq -\beta(1 - \lambda \tau) (i(t) - i^*)^2 + \beta \lambda \tau \frac{N'(t)}{N(t)} + i^*(B(N(t)) - \mu) \\
&= -\beta(1 - \lambda \tau) (i(t) - i^*)^2 + (\beta \lambda \tau + i^*) \frac{N'(t)}{N(t)}. \\
\end{align*}

Note that
\[ \int_{0}^{+\infty} \frac{N'(u)}{N(u)} du = \ln \frac{B^{-1}(\mu)}{N(0)}. \]

If \(1 > \lambda \tau\), we have
\[ \int_{0}^{+\infty} (i(u) - i^*)^2 du < +\infty. \quad (2.9) \]

From (2.5), we see that \((i(t) - i^*)^2\) is uniformly continuous on \([0, \infty)\). It follows from the well-known Barb"{a}lat's lemma (see [1]),
\[ \lim_{t \to +\infty} i(t) = i^*. \]

From (1.4),
\[ \lim_{t \to +\infty} R(t) = R^*, \]
which implies
\[ \lim_{t \to +\infty} S(t) = S^*. \]

In a similar manner, we can show that \(E_+\) is globally attractive if \(N(0) > B^{-1}(\mu)\).

This completes the proof. \(\square\)

3 Summary

In this paper, we considered stability of the few variable population \(SIRS\) epidemic model with a delay. We showed that if \(R_0 < 1\), the disease-free equilibrium is globally asymptotically stable, whereas if \(R_0 > 1\), the endemic equilibrium is globally attractive for small delay.
References


