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1 Introduction

Classical epidemic models assume that the total population size is constant. More recent models consider a variable population size in order to take into account a longer time scale with disease causing death and reduced reproduction, see [3, 4].

SIRS epidemic models have been studied by many authors, see [2, 5]. It is our aim to analyze a variable population SIRS epidemic model with a delay. The total (host) population size $N(t)$ is divided into susceptible, infective, and recovered with temporary immunity individuals. The respective numbers are denoted by $S$, $I$ and $R$. The flow of individuals can schematically be described as

$$
\begin{array}{c}
B(N)N \\
\downarrow S
\end{array} \xrightarrow{\beta SI/N} I \xrightarrow{\lambda} R \xrightarrow{\tau} S
\downarrow \mu R
\begin{array}{c}
\mu S \\
\downarrow (\mu + \alpha)I
\end{array}
$$

We assume that everybody is born as susceptible. $B(N)N$ is a birth rate function with $B(N)$ satisfying the following assumptions for $N \in (0, \infty)$:

(A1) $B(N) > 0$;

(A2) $B(N)$ is continuously differentiable with $B'(N) < 0$;

(A3) $B(0^+) > \mu + \alpha$ and $\mu > B(+\infty)$.

Note that (A2) and (A3) imply that $B^{-1}(N)$ exists for $N \in (B(\infty), B(0^+))$, and (A3) assures that $N$ does not go to extinction and cannot blow up. The parameter $\mu > 0$ is the natural death rate constant, $\alpha \geq 0$ is the disease-related death rate constant, and $\lambda \geq 0$ is rate constant for recovery. The force of infection is assumed to be of standard type, namely $\beta I/N$, with $\beta > 0$, the effective per capita contact rate constant of infective individuals. The time delay $\tau$ denotes a constant immune period.

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Our model thus take the following form:

\begin{align}
N(t) &= S(t) + I(t) + R(t), \quad (1.1) \\
S'(t) &= B(N(t))N(t) - \mu S(t) - \frac{\beta S(t)I(t)}{N(t)} + \lambda I(t - \tau)e^{-\mu\tau}, \quad (1.2) \\
I'(t) &= \frac{\beta S(t)I(t)}{N(t)} - (\mu + \lambda + \alpha)I(t), \quad (1.3) \\
R'(t) &= \lambda I(t) - \lambda I(t - \tau)e^{-\mu\tau} - \mu R(t), \quad (1.4)
\end{align}

with initial conditions

\begin{align}
S(\theta) > 0, \quad I(\theta) > 0, \quad R(\theta) > 0 \text{ on } [-\tau, 0]. \quad (1.5)
\end{align}

In order to assure continuity of solutions at time 0, we assume that

\begin{align}
R(0) = \int_{-\tau}^{0} \lambda I(u)e^{\mu u}du. \quad (1.6)
\end{align}

System (1.1)-(1.4) always has the disease-free equilibrium $E_0 = (B^{-1}(\mu), \ 0, \ 0)$. Furthermore, if the basic reproduction number $R_0 := \frac{1}{\mu + \lambda + \alpha} > 1$, then it also has the unique endemic equilibrium $E_+ = (S^*, I^*, R^*)$ where

\begin{align}
S^* &= \frac{\mu + \lambda + \alpha}{\beta}N^*, \quad I^* = \left(1 - \frac{\mu + \lambda + \alpha}{\beta}\right)N^*/\left(1 + \frac{\lambda(1 - e^{-\mu\tau})}{\mu}\right), \quad R^* = \frac{\lambda(1 - e^{-\mu\tau})}{\mu}I^* \\
\text{and} \quad N^* &= B^{-1}\left(\mu + \alpha\left(1 - \frac{\mu + \lambda + \alpha}{\beta}\right)/\left(1 + \frac{\lambda(1 - e^{-\mu\tau})}{\mu}\right)\right).
\end{align}

2 Main result

The following basic result for solutions of system is given. The proof is omitted.

**Theorem 2.1.** Let $S(t)$, $I(t)$, $R(t)$ be a solution of the delay differential system (1.2) – (1.4) with $N(t)$ given by (1.1), and initial conditions given by (1.5). In addition, suppose that (1.6) holds. For all $t \geq 0$, this solution exists, is unique and has $S(t) > 0$, $I(t) > 0$, $R(t) > 0$.

A linear analysis shows the following theorem for disease-free equilibrium.

**Theorem 2.2.** If $R_0 < 1$, then the disease-free equilibrium is locally asymptotically stable.

A global stability result can be given by using the following results. Consider the systems:

\begin{align}
x' &= f(t, x) \quad (2.1) \\
y' &= g(y) \quad (2.2)
\end{align}
where $f$ and $g$ are continuous and locally Lipschitz in $x$ in $\mathbb{R}^n$ and solutions exist for all positive time. (2.1) is called asymptotically autonomous with limit equation in $\mathbb{R}^n$.

**Lemma 2.1** ([6]). Let $e$ be a locally asymptotically stable equilibrium of (2.2) and $\omega$ be the $\omega$-limit set of a forward bounded solution $x(t)$ of (2.1). If $\omega$ contains a point $y_0$ such that the solution of (2.2) with $y(0) = y_0$ converges to $e$ as $t \to \infty$, then $\omega = \{e\}$, i.e. $x(t) \to e$ as $t \to \infty$.

**Corollary 2.1.** If solutions of system (2.1) are bounded and the equilibrium $e$ of the limit system (2.2) is globally asymptotically stable, then any solution $x(t)$ of system (2.1) satisfies $x(t) \to e$ as $t \to \infty$.

**Theorem 2.3.** For $R_0 < 1$ all solutions of the system (1.2)-(1.4) with (1.1) approach the disease free equilibrium as $t \to \infty$.

**Proof.** By (1.3), we have $I' \leq (\beta - \mu - \lambda - \alpha)I$, hence $I(t)$ has limit zero as $t \to \infty$ if $\beta - \mu - \lambda - \alpha < 0$. Then $R(t) \to 0$ as $t \to \infty$ from (1.4).

Add equations (1.2)-(1.4), and use (1.1) to obtain

$$N' = (B(N) - \mu)N - \alpha I.$$  \hspace{1cm} (2.3)

This equation has the limit equation

$$N' = (B(N) - \mu)N.$$  \hspace{1cm} (2.4)

By Corollary 2.1, $N(t) \to B^{-1}(\mu)$ as $t \to \infty$. Hence $S(t) \to B^{-1}(\mu)$ as $t \to \infty$. \hfill \Box

A global property of the endemic equilibrium for a restricted set of parameter values can be given as follows.

**Theorem 2.4.** Suppose that $\alpha = 0$ and $R_0 > 1$. If $\tau < \frac{1}{\lambda}$, all solutions of system (1.2)-(1.4) with (1.1) approach the endemic equilibrium as $t \to \infty$.

**Proof.** Define $i(t) = I(t)/N(t)$. Let $i^* = I^*/N^*$. System (1.2)-(1.4) leads to the following system

$$i'(t) = \beta \left( i^* - i(t) + \frac{\lambda}{\mu} (1 - e^{-\mu \tau}) i^* - \frac{\lambda}{N(t)} \int_{t-\tau}^{t} i(u) N(u) e^{-\mu (t-u)} du \right) i(t)$$

$$- (B(N) - \mu) i(t) \hspace{1cm} (2.5)$$

$$N'(t) = (B(N(t)) - \mu)N(t).$$
This system has a unique internal equilibrium \((i^*, B^{-1}(\mu))\) corresponding to the endemic equilibrium \(E_+\).

By the second equation of (2.5), if \(N(0) \leq B^{-1}(\mu)\), \(N(t)\) is monotone increasing and \(N(t) \leq B^{-1}(\mu)\), whereas if \(N(0) > B^{-1}(\mu)\), \(N(t)\) is monotone decreasing and \(N(t) > B^{-1}(\mu)\).

Derivative of \(V_1\) along a solution is

\[
\dot{V}_1(t) = \beta \left\{ i^* - i(t) + \frac{\lambda}{\mu} (1 - e^{-\mu \tau}) i^* - \frac{\lambda}{N(t)} \int_{t-\tau}^{t} i(u) e^{-\mu(t-u)} du \right\} i(t) \left( 1 - \frac{i^*}{i(t)} \right)
- (B(N(t)) - \mu)(i(t) - i^*)
= -\beta (i(t) - i^*)^2 + \beta \lambda (i(t) - i^*) \int_{t-\tau}^{t} \left( i^* e^{-\mu(t-u)} - i(u) \frac{N(u)}{N(t)} e^{-\mu(t-u)} \right) du
- (B(N(t)) - \mu)(i(t) - i^*)
= -\beta (i(t) - i^*)^2 + \beta \lambda \int_{t-\tau}^{t} \left( i(t) - i^* \right) (i(u) - i^*) e^{-\mu(t-u)} du
+ \beta \lambda \int_{t-\tau}^{t} \left( i(t) - i^* \right) \left( 1 - \frac{N(u)}{N(t)} \right) i(u) e^{-\mu(t-u)} du
- (B(N(t)) - \mu)(i(t) - i^*)
\leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} (i(u) - i^*)^2 e^{-2\mu(t-u)} du
+ \beta \lambda \int_{t-\tau}^{t} \left( i(t) - i^* \right) \left( 1 - \frac{N(u)}{N(t)} \right) i(u) e^{-\mu(t-u)} du
- (B(N(t)) - \mu)(i(t) - i^*)
\leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \tau (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} (i(u) - i^*)^2 du
+ \beta \lambda \int_{t-\tau}^{t} \left( i(t) - i^* \right) \left( 1 - \frac{N(u)}{N(t)} \right) i(u) e^{-\mu(t-u)} du
- (B(N(t)) - \mu)(i(t) - i^*)
\}
(2.6)

If \(N(0) \leq B^{-1}(\mu)\), we have from (2.6),

\[
\dot{V}_1(t) \leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \tau (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} (i(u) - i^*)^2 du
+ \beta \lambda \int_{t-\tau}^{t} \left( 1 - \frac{N(u)}{N(t)} \right) du + i^* (B(N(t)) - \mu).
(2.7)

In addition, define

\[
V_2(t) := \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} \int_{\theta}^{\xi} (i(\xi) - i^*)^2 d\xi d\theta + \beta \lambda \int_{t-\tau}^{t} \int_{\theta}^{\xi} \left( 1 - \frac{N(\xi)}{N(t)} \right) d\xi d\theta.
(2.8)
\]

Then (2.7) and (2.8) lead to

\[
\frac{d}{dt} (V_1 + V_2) \leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \tau (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \tau (i(t) - i^*)^2
\]
\[ + \beta \lambda \int_{t-\tau}^{t} \int_{t}^{t-\tau} \frac{N(\xi)N'(t)}{N^2(t)} d\xi d\theta + i^* (B(N(t)) - \mu) \]
\[ \leq -\beta(1 - \lambda\tau) (i(t) - i^*)^2 \]
\[ + \beta \lambda \int_{t-\tau}^{t} \int_{t-\tau}^{t} \frac{N(\xi)}{N(t)} d\xi d\theta + i^* (B(N(t)) - \mu) \]
\[ = -\beta(1 - \lambda\tau) (i(t) - i^*)^2 \]
\[ + \beta \lambda \frac{N'(t)}{N(t)} \int_{t-\tau}^{t} \frac{N(\xi)}{N(t)} d\xi + i^* (B(N(t)) - \mu) \]
\[ \leq -\beta(1 - \lambda\tau) (i(t) - i^*)^2 + \beta \lambda \tau^2 \frac{N'(t)}{N(t)} \]
\[ \leq -\beta(1 - \lambda\tau) (i(t) - i^*)^2 + (\beta \lambda \tau^2 + i^*) \frac{N'(t)}{N(t)}. \]

Note that
\[ \int_{0}^{+\infty} \frac{N'(u)}{N(u)} du = \ln \frac{B^{-1}(\mu)}{N(0)} . \]

If \( 1 > \lambda\tau \), we have
\[ \int_{0}^{+\infty} (i(u) - i^*)^2 du < +\infty. \quad (2.9) \]

From (2.5), we see that \((i(t) - i^*)^2\) is uniformly continuous on \([0, \infty)\). It follows from the well-known Barbálat’s lemma (see [1]),
\[ \lim_{t \to +\infty} i(t) = i^*. \]

From (1.4),
\[ \lim_{t \to +\infty} R(t) = R^*, \]

which implies
\[ \lim_{t \to +\infty} S(t) = S^*. \]

In a similar manner, we can show that \(E_+\) is globally attractive if \(N(0) > B^{-1}(\mu)\).

This completes the proof. \(\square\)

3 Summary

In this paper, we considered stability of the few variable population SIRS epidemic model with a delay. We showed that if \(R_0 < 1\), the disease-free equilibrium is globally asymptotically stable, whereas if \(R_0 > 1\), the endemic equilibrium is globally attractive for small delay.
References


