1 Introduction

Classical epidemic models assume that the total population size is constant. More recent models consider a variable population size in order to take into account a longer time scale with disease causing death and reduced reproduction, see [3, 4].

SIRS epidemic models have been studied by many authors, see [2, 5]. It is our aim to analyze a variable population SIRS epidemic model with a delay. The total (host) population size $N(t)$ is divided into susceptible, infective, and recovered with temporary immunity individuals. The respective numbers are denoted by $S$, $I$ and $R$. The flow of individuals can schematically be described as

$$
\begin{array}{c}
B(N)N \\
\rightarrow \\
S \xrightarrow{\frac{B(t)I}{N}} I \\
\xrightarrow{\lambda} R \\
\xrightarrow{\tau} S
\end{array}
$$

We assume that everybody is born as susceptible. $B(N)N$ is a birth rate function with $B(N)$ satisfying the following assumptions for $N \in (0, \infty)$:

- (A1) $B(N) > 0$;
- (A2) $B(N)$ is continuously differentiable with $B'(N) < 0$;
- (A3) $B(0^+) > \mu + \alpha$ and $\mu > B(\infty)$.

Note that (A2) and (A3) imply that $B^{-1}(N)$ exists for $N \in (B(\infty), B(0^+))$, and (A3) assures that $N$ does not go to extinction and cannot blow up. The parameter $\mu > 0$ is the natural death rate constant, $\alpha \geq 0$ is the disease-related death rate constant, and $\lambda \geq 0$ is rate constant for recovery. The force of infection is assumed to be of standard type, namely $\beta I/N$, with $\beta > 0$, the effective per capita contact rate constant of infective individuals. The time delay $\tau$ denotes a constant immune period.
Our model thus take the following form:

\[ N(t) = S(t) + I(t) + R(t), \tag{1.1} \]

\[ S'(t) = B(N(t))N(t) - \mu S(t) - \frac{\beta S(t)I(t)}{N(t)} + \lambda I(t - \tau)e^{-\mu\tau}, \tag{1.2} \]

\[ I'(t) = \frac{\beta S(t)I(t)}{N(t)} - (\mu + \lambda + \alpha)I(t), \tag{1.3} \]

\[ R'(t) = \lambda I(t) - \lambda I(t - \tau)e^{-\mu\tau} - \mu R(t), \tag{1.4} \]

with initial conditions

\[ S(\theta) > 0, \ I(\theta) > 0, \ R(\theta) > 0 \text{ on } [-\tau, 0]. \tag{1.5} \]

In order to assure continuity of solutions at time 0, we assume that

\[ R(0) = \int_{-\tau}^{0} \lambda I(u)e^{\mu u}du. \tag{1.6} \]

System (1.1)–(1.4) always has the disease-free equilibrium \( E_0 = (B^{-1}(\mu), 0, 0) \). Furthermore, if the basic reproduction number \( R_0 := \frac{1}{\mu + \lambda + \alpha} > 1 \), then it also has the unique endemic equilibrium \( E_+ = (S^*, I^*, R^*) \) where

\[
S^* = \frac{\mu + \lambda + \alpha}{\beta}N^*, \quad I^* = \left(1 - \frac{\mu + \lambda + \alpha}{\beta}\right)N^*/\left(1 + \frac{\lambda(1 - e^{-\mu\tau})}{\mu}\right), \quad R^* = \frac{\lambda(1 - e^{-\mu\tau})}{\mu}I^*
\]

and \( N^* = B^{-1}\left(\mu + \alpha\left(1 - \frac{\mu + \lambda + \alpha}{\beta}\right)\right)/\left(1 + \frac{\lambda(1 - e^{-\mu\tau})}{\mu}\right)\).

2 Main result

The following basic result for solutions of system is given. The proof is omitted.

**Theorem 2.1.** Let \( S(t), I(t), R(t) \) be a solution of the delay differential system (1.2)–(1.4) with \( N(t) \) given by (1.1), and initial conditions given by (1.5). In addition, suppose that (1.6) holds. For all \( t \geq 0 \), this solution exists, is unique and has \( S(t) > 0, I(t) > 0, R(t) > 0 \).

A linear analysis shows the following theorem for disease-free equilibrium.

**Theorem 2.2.** If \( R_0 < 1 \), then the disease-free equilibrium is locally asymptotically stable.

A global stability result can be given by using the following results. Consider the systems:

\[ x' = f(t, x) \tag{2.1} \]

\[ y' = g(y) \tag{2.2} \]
where \( f \) and \( g \) are continuous and locally Lipschitz in \( x \) in \( \mathbb{R}^n \) and solutions exist for all positive time. (2.1) is called asymptotically autonomous with limit equation in \( \mathbb{R}^n \).

**Lemma 2.1** ([6]). Let \( e \) be a locally asymptotically stable equilibrium of (2.2) and \( \omega \) be the \( \omega \)-limit set of a forward bounded solution \( x(t) \) of (2.1). If \( \omega \) contains a point \( y_0 \) such that the solution of (2.2) with \( y(0) = y_0 \) converges to \( e \) as \( t \to \infty \), then \( \omega = \{e\} \), i.e. \( x(t) \to e \) as \( t \to \infty \).

**Corollary 2.1.** If solutions of system (2.1) are bounded and the equilibrium \( e \) of the limit system (2.2) is globally asymptotically stable, then any solution \( x(t) \) of system (2.1) satisfies \( x(t) \to e \) as \( t \to \infty \).

**Theorem 2.3.** For \( R_0 < 1 \) all solutions of the system (1.2)-(1.4) with (1.1) approach the disease free equilibrium as \( t \to \infty \).

*Proof.* By (1.3), we have \( I' \leq (\beta - \mu - \lambda - \alpha)I \), hence \( I(t) \) has limit zero as \( t \to \infty \) if \( \beta - \mu - \lambda - \alpha < 0 \). Then \( R(t) \to 0 \) as \( t \to \infty \) from (1.4).

Add equations (1.2)-(1.4), and use (1.1) to obtain
\[
N' = (B(N) - \mu)N - \alpha I.
\]
This equation has the limit equation
\[
N' = (B(N) - \mu)N.
\]
By Corollary 2.1, \( N(t) \to B^{-1}(\mu) \) as \( t \to \infty \). Hence \( S(t) \to B^{-1}(\mu) \) as \( t \to \infty \).

A global property of the endemic equilibrium for a restricted set of parameter values can be given as follows.

**Theorem 2.4.** Suppose that \( \alpha = 0 \) and \( R_0 > 1 \). If \( \tau < \frac{1}{\lambda} \), all solutions of system (1.2)-(1.4) with (1.1) approach the endemic equilibrium as \( t \to \infty \).

*Proof.* Define \( i(t) = I(t)/N(t) \). Let \( \iota^* = I^*/N^* \). System (1.2)-(1.4) leads to the following system
\[
i'(t) = \beta \left\{ \iota^* - i(t) + \frac{\lambda}{\mu} (1 - e^{-\mu \tau}) \iota^* - \frac{\lambda}{N(t)} \int_{t-\tau}^{t} i(u)N(u)e^{-\mu(t-u)}du \right\} i(t)
\]
\[- (B(N) - \mu)i(t)
\]
\[
N'(t) = (B(N(t)) - \mu)N(t).
\]
This system has a unique internal equilibrium \((i^*, B^{-1}(\mu))\) corresponding to the endemic equilibrium \(E_+\).

By the second equation of (2.5), if \(N(0) \leq B^{-1}(\mu)\), \(N(t)\) is monotone increasing and \(N(t) \leq B^{-1}(\mu)\), whereas if \(N(0) > B^{-1}(\mu)\), \(N(t)\) is monotone decreasing and \(N(t) > B^{-1}(\mu)\).

Derivative of \(V_1\) along a solution is

\[
\dot{V}_1(t) = \beta \left\{ i^* - i(t) + \frac{\lambda}{\mu} (1 - e^{-\mu \tau}) i^* - \frac{\lambda}{N(t)} \int_{t-\tau}^{t} i(u) N(u) e^{-\mu(t-u)} du \right\} i(t) \left( 1 - \frac{i^*}{i(t)} \right) \\
- (B(N(t)) - \mu) (i(t) - i^*) \\
= -\beta (i(t) - i^*)^2 + \beta \lambda (i(t) - i^*) \int_{t-\tau}^{t} \left( \frac{i(u)}{N(u)} e^{-\mu(t-u)} - \frac{i(u)}{N(t)} e^{-\mu(t-u)} \right) du \\
- (B(N(t)) - \mu) (i(t) - i^*) \\
= -\beta (i(t) - i^*)^2 - \beta \lambda \int_{t-\tau}^{t} (i(t) - i^*) (i(u) - i^*) e^{-\mu(t-u)} du \\
+ \beta \lambda \int_{t-\tau}^{t} (i(t) - i^*) \left( 1 - \frac{N(u)}{N(t)} \right) i(u) e^{-\mu(t-u)} du - (B(N(t)) - \mu) (i(t) - i^*) \\
\leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} (i(u) - i^*)^2 e^{-2\mu(t-u)} du \\
+ \beta \lambda \int_{t-\tau}^{t} (i(t) - i^*) \left( 1 - \frac{N(u)}{N(t)} \right) i(u) e^{-\mu(t-u)} du - (B(N(t)) - \mu) (i(t) - i^*) \\
\leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} (i(u) - i^*)^2 du \\
+ \beta \lambda \int_{t-\tau}^{t} (i(t) - i^*) \left( 1 - \frac{N(u)}{N(t)} \right) i(u) e^{-\mu(t-u)} du - (B(N(t)) - \mu) (i(t) - i^*) \\
(2.6)
\]

If \(N(0) \leq B^{-1}(\mu)\), we have from (2.6),

\[
\dot{V}_1(t) \leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} (i(u) - i^*)^2 du \\
+ \beta \lambda \int_{t-\tau}^{t} \left( 1 - \frac{N(u)}{N(t)} \right) du + i^* (B(N(t)) - \mu). \\
(2.7)
\]

In addition, define

\[
V_2(t) := \frac{1}{2} \beta \lambda \int_{t-\tau}^{t} \int_{\theta}^{t} (i(\xi) - i^*)^2 d\xi d\theta + \beta \lambda \int_{t-\tau}^{t} \int_{\theta}^{t} \left( 1 - \frac{N(\xi)}{N(t)} \right) d\xi d\theta. \\
(2.8)
\]

Then (2.7) and (2.8) lead to

\[
\frac{d}{dt} (V_1 + V_2) \leq -\beta (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda (i(t) - i^*)^2 + \frac{1}{2} \beta \lambda (i(t) - i^*)^2
\]
\[
+ \beta \lambda \int_{t-\tau}^{t} \int_{0}^{t} \frac{N(\xi)N'(t)}{N^2(t)} d\xi d\theta + i^* (B(N(t)) - \mu) \\
\leq -\beta (1 - \lambda \tau) (i(t) - i^*)^2 \\
+ \beta \lambda \int_{t-\tau}^{t} \int_{t-\tau}^{t} \frac{N(\xi)}{N(t)} d\xi d\theta + i^* (B(N(t)) - \mu) \\
= -\beta (1 - \lambda \tau) (i(t) - i^*)^2 \\
+ \beta \lambda \int_{t-\tau}^{t} \frac{N'(t)}{N(t)} d\xi + i^* (B(N(t)) - \mu) \\
\leq -\beta (1 - \lambda \tau) (i(t) - i^*)^2 + \beta \lambda \tau^2 \frac{N'(t)}{N(t)} + i^* (B(N(t)) - \mu) \\
= -\beta (1 - \lambda \tau) (i(t) - i^*)^2 + (\beta \lambda \tau^2 + i^*) \frac{N'(t)}{N(t)}.
\]

Note that
\[
\int_{0}^{+\infty} \frac{N'(u)}{N(u)} du = \ln \frac{B^{-1}(\mu)}{N(0)}.
\]

If \(1 > \lambda \tau\), we have
\[
\int_{0}^{+\infty} (i(u) - i^*)^2 du < +\infty.
\] (2.9)

From (2.5), we see that \((i(t) - i^*)^2\) is uniformly continuous on \([0, \infty)\). It follows from the well-known Barbálat’s lemma (see [1]),
\[
\lim_{t \to +\infty} i(t) = i^*.
\]

From (1.4),
\[
\lim_{t \to +\infty} R(t) = R^*,
\]
which implies
\[
\lim_{t \to +\infty} S(t) = S^*.
\]

In a similar manner, we can show that \(E_+\) is globally attractive if \(N(0) > B^{-1}(\mu)\).

This completes the proof. \(\square\)

3 Summary

In this paper, we considered stability of the few variable population SIRS epidemic model with a delay. We showed that if \(R_0 < 1\), the disease-free equilibrium is globally asymptotically stable, whereas if \(R_0 > 1\), the endemic equilibrium is globally attractive for small delay.
References


