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Asymptotic Stability for the Linear Integro-Differential Equation $\dot{x}(t) = ax(t) - b\int_{t-h}^{t} x(s)\,ds$

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1 Introduction

In this paper, we will discuss the uniform asymptotic stability of the zero solution of the linear integro-differential equation

$$\dot{x}(t) = ax(t) - b\int_{t-h}^{t} x(s)\,ds, \tag{E}$$

where a and b are real and $h > 0$. As a special case, for $a = 0$, (E) becomes

$$\dot{x}(t) = -b\int_{t-h}^{t} x(s)\,ds \tag{I}$$

and in [4] it is shown that the zero solution of (I) is uniformly asymptotically stable if and only if

$$0 < bh^2 < \frac{\pi^2}{2}.$$

There are also some stability results for (I) with a generalized continuously distributed delay which is expressed in Stieltjes integral [3]. In case $a < 0$, some sufficient stability conditions for $a < 0$ are obtained by using Liapunov functionals in [1].

But, there exist no results on the stability of (E) for the case $a > 0$ as far as the authors know. So, we will study (E) for $a > 0$ and give results on the uniform asymptotic stability of (E).

2 Main results

We obtain the following theorems on the uniform asymptotic stability of

$$\dot{x}(t) = ax(t) - b\int_{t-h}^{t} x(s)\,ds, \tag{E}$$

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where $a > 0$, b is real and $h > 0$.

Theorem 2.1. *Let $a^2 < 2b$. Then, the zero solution of* (E) *is uniformly asymptotically stable if and only if*

$$\frac{a}{b} < h < \frac{1}{\sqrt{2b - a^2}} \operatorname{Cos}^{-1} \frac{a^2 - b}{b} \tag{2.1}$$

is satisfied.

Theorem 2.2. *Let $a^2 > 2b$. Then, the zero solution of* (E) *is not uniformly asymptotically stable for all $h > 0$.*

To give the proofs of Theorems 2.1 and 2.2 with the root analysis, we need to consider the characteristic equation of (E) which is expressed in the form

$$\lambda = a - b \int_{-h}^{0} e^{\lambda s}\, ds \tag{C}$$

and we introduce the following results which are used without proofs.

Theorem A. [2] *The zero solution of* (E) *is uniformly asymptotically stable if and only if any root of* (C) *has a negative real part.*

Now, let $\nu(h)$ be the number of roots of (C) including multiplicity whose real parts are positive at h. Then, the following property holds.

Theorem B. [5] *Let $h_i\,(i = 0, 1, 2, \cdots)$ be constants at which* (C) *has a root on the imaginary axis of the complex plane. Then, the number $\nu(h)$ is constant on each interval $h_i < h < h_{i+1}$.*

Lemma C. [6] *If $a - bh \geq 0$, then* (C) *has a nonnegative real root.*

Lemma C implies that if $0 < h \leq a/b$ then the zero solution of (E) is not uniformly asymptotically stable by Theorem A. In case $b \leq 0$, (C) has a nonnegative real root for all $h > 0$ because the condition $a - bh > 0$ is satisfied. Thus, in case $b \leq 0$, we also see easily that the zero solution of (E) is not uniformly asymptotically stable. Hereafter, we assume $b > 0$.

3 The proof of main results

In this section, we will prove Theorems 2.1 and 2.2 regarding a and b are fixed constants and h is a variable. At first, we give a proof of Theorem 2.1.

Proof of Theorem 2.1. (*Sufficiency.*) Our purpose is to show $\nu(h) = 0$ when h satisfies the condition (2.1). However, it is difficult to get $\nu(h) = 0$ directly only by using the condition (2.1). So at first we will discuss the case $h \in (0, a/b)$.

Before proving sufficiency of Theorem 2.1, we give four lemmas. Now, we consider the case where h increases minutely from zero. Then, we have

Lemma 3.1. *For a sufficiently small $h > 0$, (C) has no pairs of imaginary roots $\lambda = x \pm iy$ such that $x > 0, y > 0$.*

Proof. Suppose that there exists a pair of imaginary roots $\lambda = x \pm iy\,(x > 0, y > 0)$. Here, we note that (C) has a root of complex conjugate. Thus, it is sufficient to discuss $\lambda = x + iy$ only. Then, substituting $\lambda = x + iy$ for (C), we have

$$x = a - b\int_{-h}^{0} e^{xs}\cos ys\,ds, \tag{3.1}$$

$$y = -b\int_{-h}^{0} e^{xs}\sin ys\,ds. \tag{3.2}$$

From (3.2),

$$y \le b\left|\int_{-h}^{0} e^{xs}\sin ys\,ds\right| \le b\int_{-h}^{0} e^{xs}|\sin ys|\,ds.$$

Also, from $|\sin ys| \le y|s|$,

$$y \le by\int_{-h}^{0} e^{xs}|s|\,ds < bhy\int_{-h}^{0} e^{xs}\,ds = bhy \times \frac{1}{x}(1 - e^{-xh}) < \frac{bhy}{x}.$$

Hence,

$$0 < x < bh,$$

so that $x \to +0$ as $h \to +0$. However, from (3.1), we have $x \to a - 0$ as $h \to +0$, which is a contradiction. This completes the proof of Lemma 3.1. $\square$

Lemma 3.2. *Suppose $a^2 < 2b$ and $0 < h < a/b$. Then, (C) has a positive real root. Moreover, the positive real root is simple.*

Proof. Suppose $\lambda = x$ is a root of (C). Now, we difine the characteristic function of (E) expressed as

$$p(\lambda, a, b, h) := \lambda - a + b\int_{-h}^{0} e^{\lambda s}\,ds. \tag{3.3}$$

Then from (3.3),

$$p(x, a, b, h) = x - a + b\int_{-h}^{0} e^{xs}\,ds = 0. \tag{3.4}$$

(3.4) is reduced to

$$\frac{1}{x}\{x^2 - ax + b - be^{-xh}\} = 0.$$

Here, we define the function $q(x)$ as

$$q(x) := x^2 - ax + b - be^{-xh}. \tag{3.5}$$

Thus, it is sufficient to show that there exists a root $x > 0$ which satisfies $q(x) = 0$. From (3.5),

$$q^{'}(x) = 2x - a + bhe^{-xh}, \tag{3.6}$$

$$q^{''}(x) = 2 - bh^2 e^{-xh}. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$q(0) = 0, \quad q^{'}(0) = -a + bh < 0, \quad q(a) = b(1 - e^{-ah}) > 0.$$

Then, there exists the root $x^* \in (0, a)$ which satisfies $q(x^*) = 0$. Moreover, we note that $q^{''}(x)$ is monotone increasing for all $x \geq 0$ and

$$q^{''}(0) = 2 - bh^2 > 2 - b\left(\frac{a}{b}\right)^2 = 2 - \frac{a^2}{b} > 0,$$

so we have $q^{''}(x) > 0$ on $[0, \infty)$. Therefore, $q^{'}(x)$ is monotone increasing on $[0, \infty)$ and x^* is determined uniquely.

Here, by Rolle's theorem we also see that there exists the root $\tilde{x} \in (0, x^*)$ which satisfies $q'(\tilde{x}) = 0$. Since $q'(x)$ is monotone increasing on $[0, \infty)$, we have $q'(x^*) > 0$, which implies that $x = x^* > 0$ is a simple root of $q(x) = 0$. □

Now, we consider a case where (C) has a root on the imaginary axis of the complex plane. At first, we assume that (C) has a pair of purely imaginary roots $\lambda = \pm i\omega\,(\omega > 0; \mathit{constant})$ at the first time $h = h^*$ when h is increases from zero. Then for $\lambda \neq 0$, we rewrite (C) as follows:

$$\lambda^2 = a\lambda - b(1 - e^{-\lambda h}), \tag{C*}$$

where $a > 0$, $b > 0$ and $h > 0$. Substituting $\lambda = i\omega$ and $h = h^*$ for (C*),

$$-\omega^2 = ia\omega - b(1 - e^{-i\omega h^*}).$$

From the above,

$$-\omega^2 = -b + b\cos\omega h^*, \tag{3.8}$$

$$a\omega = b\sin\omega h^*. \tag{3.9}$$

From (3.8) and (3.9),

$$\left(\frac{b-\omega^2}{b}\right)^2 + \left(\frac{a\omega^2}{b}\right)^2 = 1.$$

Because of $\omega > 0$, we have $a^2 < 2b$ and

$$\omega = \sqrt{2b - a^2}. \tag{3.10}$$

Substituting (3.10) for (3.8) and (3.9),

$$\cos\sqrt{2b-a^2}h^* = \frac{a^2-b}{b}, \tag{3.11}$$

$$\sin\sqrt{2b-a^2}h^* = \frac{a\sqrt{2b-a^2}}{b} > 0. \tag{3.12}$$

From (3.11) and (3.12),

$$h^* = \frac{1}{\sqrt{2b-a^2}} \operatorname{Cos}^{-1} \frac{a^2-b}{b}.$$

We see that (C) has a pair of purely imaginary roots $\lambda = \pm i\omega$ at $h = h^*$. However, we have not yet made reference to the case where (C) has the root $\lambda = 0$. Next, we prove the following lemma on the root $\lambda = 0$.

Lemma 3.3. (C) *has the root* $\lambda = 0$ *at* $h = a/b$. *Moreover, there exist no positive real roots at* $h = a/b$.

Proof. We will give a proof with the characteristic function $p(x, a, b, h)$ expressed by (3.3), where $x \in \mathbf{R}$. We have

$$p(0, a, b, \frac{a}{b}) = -a + b\left\{0 - \left(-\frac{a}{b}\right)\right\} = 0,$$

then (C) has the root $\lambda = 0$ at $h = a/b$.

Now, we assume that (C) has a positive real root $x = x^*$ at $h = a/b$, that is, $p(x^*, a, b, a/b) = 0$. Then,

$$\frac{\partial}{\partial x}p(x, a, b, \frac{a}{b}) = \frac{1}{x^2}\{x^2 + axe^{-\frac{a}{b}x} - b + be^{-\frac{a}{b}x}\}.$$

Here, we define the function $f(x)$ as

$$f(x) := x^2 + axe^{-\frac{a}{b}x} - b + be^{-\frac{a}{b}x}.$$

Then,

$$f'(x) = 2x\left(1 - \frac{a^2}{2b}e^{-\frac{a}{b}x}\right).$$

Because of $a^2 < 2b$, $f'(x) > 0$ holds for all $x > 0$. Hence, $f(x)$ is monotone increasing for all $x > 0$. Moreover, from $f(0) = 0$, we have $f(x) > 0$ for all $x > 0$. This implies that $\frac{\partial}{\partial x}p(x, a, b, a/b) > 0$ for all $x > 0$. Since $p(0, a, b, a/b) = 0$ and $\frac{\partial}{\partial x}p(x, a, b, a/b) > 0$ is satisfied for all $x > 0$, so we obtain $p(x^*, a, b, a/b) > 0$, which contradicts the initial assumption. Therefore, (C) has no positive real root at $h = a/b$. It is clear that (C) has the root $\lambda = 0$ at $h = a/b$. Thus, the proof of Lemma 3.3 is complete. □

Now, we show that $h^* > a/b$ if $0 < a^2 < 2b$. Let b be fixed and difine the function $g(a)$ as

$$g(a) := \sqrt{2b - a^2}\left(\frac{1}{\sqrt{2b - a^2}}\mathrm{Cos}^{-1}\frac{a^2 - b}{b} - \frac{a}{b}\right).$$

Then,

$$g(a) = \mathrm{Cos}^{-1}\frac{a^2 - b}{b} - \frac{a\sqrt{2b - a^2}}{b}.$$

It is easily seen that $g(0) = \pi$, $g(\sqrt{2b}) = 0$ and $g'(a) = -2\sqrt{2b - a^2}/b < 0$ for all $a \in (0, \sqrt{2b})$. Therefore, we have that $g(a) > 0$ for all $a \in (0, \sqrt{2b})$, so $h^* > a/b$ holds.

By Lemmas C and 3.2, (C) has a nonnegative and simple real root for $h \in (0, a/b]$. We also see that (C) has no imaginary roots for a sufficiently small $h > 0$ by Lemma 3.1. Moreover, (C) has the zero root but no positive real roots by Lemma 3.3. Here, noting that (C) has no pair of purely imaginary roots during h moves from zero to h^*, we see that a root of whose real part is nonnegative for $h \in (0, a/b]$ is unique and the real root determined by Lemma C. Thus, we have $\nu(h) = 1$ for $h \in (0, a/b)$.

Finally, we investigate $\nu(h)$ for the case $h > a/b$. Now, we will investigate the behavior of the root $\lambda(h)$ with $\lambda(a/b) = 0$. Then, we show the following lemma for the behavior of $\lambda(h)$.

Lemma 3.4. *For the root $\lambda(h)$ with $\lambda(a/b) = 0$, $\mathrm{Re}\,(d\lambda/dh)|_{\substack{\lambda=0\\h=a/b}} < 0$ holds.*

Proof. We again use the characteristic function $p(\lambda, a, b, h)$ given by (3.3). Then, we obtain

$$\frac{\partial}{\partial\lambda}p(\lambda, a, b, h) = 1 + b\int_{-h}^{0} se^{\lambda s}\,ds, \quad \frac{\partial}{\partial h}p(\lambda, a, b, h) = be^{-\lambda h}.$$

By the theorem on implicit function, we obtain

$$\frac{d\lambda}{dh} = -\frac{be^{-\lambda h}}{1 + b\int_{-h}^{0} se^{\lambda s}\,ds}. \tag{3.13}$$

Here, we investigate the behavior of the root $\lambda(a/b)$ when h increases minutely from $h = a/b$. Then, from (3.13),

$$\mathrm{Re}\,\frac{d\lambda}{dh}\bigg|_{\substack{\lambda=0\\h=\frac{a}{b}}} = -\frac{b}{1 - \frac{a^2}{2b}} < 0.$$

This completes the proof of Lemma 3.4. □

Lemma 3.4 shows that the root $\lambda(h)$ moves into the left half-plane of the complex plane when h increases minutely from a/b. Since $\nu(h) = 1$ for $h \in (0, a/b)$, we have $\nu(h) = 0$ for $h \in (a/b, h^*)$. Thus, the zero solution of (E) is uniformly asymptotically stable on the interval $a/b < h < h^*$ by Theorem A. From the above, the proof of sufficiency of Theorem 2.1 is complete.

(*Necessity.*) We will show that the uniform asymptotic stability of (E) implies the condition (2.1). We consider the contraposition of this statement, that is,

Proposition 3.1. *Suppose $a^2 < 2b$. If $h \le a/b$ or $h \ge h^*$. Then the zero solution of* (E) *is not uniformly asymptotically stable.*

Proof. First, we consider a case of $h \le a/b$. Then by Lemma C, (C) has a nonnegative real root, which implies $\nu(h) > 0$. Hence, the zero solution of (E) is not uniformly asymptotically stable.

Next, we consider a case of $h \ge h^*$. We proved that the (C) had a pair of purely imaginary roots $\lambda = \pm i\omega$ at $h = h^*$ in the proof of sufficiency. Here, we investigate the behavior of the root $\lambda(h)$ with $\lambda(h^*) = \pm i\omega$ when h increases minutely from h^*.

From (C*), the characteristic function of (E) is defined as follows:

$$p^*(\lambda, a, b, h) := \lambda^2 - a\lambda + b(1 - e^{-\lambda h}). \tag{3.14}$$

From (3.14),

$$\frac{\partial}{\partial \lambda} p^*(\lambda, a, b, h) = 2\lambda - a + bhe^{-\lambda h}, \quad \frac{\partial}{\partial h} p^*(\lambda, a, b, h) = b\lambda e^{-\lambda h}.$$

Then,

$$\left.\frac{d\lambda}{dh}\right|_{\substack{\lambda = i\omega \\ h = h^*}} = \frac{-ib\omega e^{-i\omega h^*}}{2i\omega - a + bh^* e^{-i\omega h^*}} = \frac{-ib\omega e^{-i\omega h^*}(-2i\omega - a + bh^* e^{i\omega h^*})}{|2i\omega - a + bh^* e^{-i\omega h^*}|^2} \tag{3.15}$$

by the theorem on implicit function. Now, we define the function $h_1(\omega)$ as follows:

$$h_1(\omega) := -ib\omega e^{-i\omega h^*}(-2i\omega - a + bh^* e^{i\omega h^*}).$$

Then, $h_1(\omega)$ is reduced to

$$h_1(\omega) = b\omega\{(-2\omega \cos \omega h^* + a \sin \omega h^*) + i(2\omega \sin \omega h^* + a \cos \omega h^* - bh^*)\}.$$

Considering the real part of $h_1(\omega)$, we have

$$\begin{aligned} \operatorname{Re} h_1(\omega) &= b\omega(-2\omega \cos \omega h^* + a \sin \omega h^*) \\ &= b\omega(-2\omega \frac{b - \omega^2}{b} + a\frac{a\omega}{b}) \\ &= \omega^4 > 0 \end{aligned}$$

from (3.8)–(3.10). This implies that the real part of the numerator of (3.15) is positive. We also define the function $h_2(\omega)$ as follows:

$$h_2(\omega) := 2i\omega - a + bh^* e^{-i\omega h^*}.$$

Then, $h_2(\omega)$ becomes

$$h_2(\omega) = (-a + bh^* \cos \omega h^*) + i(2\omega - bh^* \sin \omega h^*) \tag{3.16}$$

Here, we consider a case where both the real and imaginary part of $h_2(\omega)$ are zero. Then from (3.16),

$$-a + bh^* \cos \omega h^* = 0, \tag{3.17}$$

$$2\omega - bh^* \sin \omega h^* = 0. \tag{3.18}$$

By (3.17) $\times \sin \omega h^*$ + (3.18) $\times \cos \omega h^*$, we obtain

$$-a \sin \omega h^* + 2\omega \cos \omega h^* = 0.$$

Therefore, from (3.8)–(3.10), we have $\omega = 0$ only, which contradicts $\omega > 0$. Hence, we showed $\frac{\partial}{\partial \lambda} p^*(i\omega, a, b, h^*) \neq 0$.

Thus, $\mathrm{Re}\,(d\lambda/dh)|_{\substack{\lambda = i\omega \\ h = h^*}} > 0$ holds. This means that a pair of purely imaginary roots $\lambda(h)$ move into the right half-plane of the complex plane when h increases minutely from h^*. Therefore, since we have $\nu(h) > 0$, the zero solution of (E) is not uniformly asymptotically stable by Theorem A. This completes the proof of Proposition 3.1. □

From Proposition 3.1, we can show the necessity of Theorem 2.1. Thus, the proof of Theorem 2.1 is finished completely. □

Proof of Theorem 2.2. Next, we give a proof of Theorem 2.2. we note that $\nu(h) = 1$ for $h \in (0, a/b)$ and (C) has the root $\lambda = 0$ at $h = a/b$ in the same way as Theorem 2.1. Thus, it is sufficient to investigate the behavior of the root $\lambda(h)$ with $\lambda(a/b) = 0$ and $\nu(h)$ for all $h > a/b$. Here, for all $h > a/b$, the following lemmas holds.

Lemma 3.5. *Suppose $a^2 \geq 2b$. Then,* (C) *has no roots on the imaginary axis for all $h > a/b$.*

Lemma 3.5 implies that $\nu(h)$ is constant for all $h > a/b$. Since (C) has no pairs of purely imaginary roots and no the zero root for all $h > a/b$ by using the characteristic function $p(\lambda, a, b, h)$, it is easy to give a proof of Lemma 3.5. In this paper, we omit the details of this proof.

Now, we will investigate the behavior of $\lambda(h)$ with $\lambda(a/b) = 0$, which it is quiet important on $\nu(h)$ for all $h > a/b$. Then, we must prove the following property.

Lemma 3.6. *Suppose $a^2 > 2b$. Then, we have $\nu(h) > 0$ for all $h > a/b$.*

Proof. In case $a^2 > 2b$, we consider a sign of $\mathrm{Re}\,(d\lambda/dh)|_{\substack{\lambda=0\\h=a/b}}$ in the same way as Lemma 3.4. Then,

$$\mathrm{Re}\,\frac{d\lambda}{dh}\bigg|_{\substack{\lambda=0\\h=\frac{a}{b}}} = -\frac{b}{1-\frac{a^2}{2b}} > 0.$$

It implies that $\lambda(h)$ moves to the right half-plane when h is increased from a/b minutely and the root which exists in the right half-plane for $h \in (0, a/b)$ remains in the right half-plane. Hence, we have $\nu(h) = 2 > 0$ for all $h > a/b$ from Lemma 3.5. □

Thus, by Lemmas 3.5 and 3.6 we show that the zero solution of (E) is not uniformly asymptotically stable for all $h > 0$, so the proof of Theorem 2.2 is complete. □

4 Critical case $a^2 = 2b$ and conjectures

In this section, we consider the case $a^2 = 2b$. Here, in the same way as case $a^2 > 2b$, we can see easily that $\nu(h) = 1$ for $h \in (0, a/b)$ and (C) has the root $\lambda = 0$ at $h = a/b$. However, if we introduce the characteristic function $p(\lambda, a, b, h)$. Then,

$$p(0, a, b, \frac{a}{b}) = 0, \quad \frac{\partial}{\partial\lambda}p(0, a, b, \frac{a}{b}) = 0, \quad \frac{\partial^2}{\partial\lambda^2}p(0, a, b, \frac{a}{b}) = \frac{a^3}{3b^2} \neq 0.$$

Therefore, we see that the root $\lambda = 0$ is a double root of (C), and so we cannot analyze the behavior of $\lambda(h)$ with $\lambda(a/b) = 0$ by using the derivative

$$\mathrm{Re}\,\frac{d\lambda}{dh}\bigg|_{\substack{\lambda=0\\h=\frac{a}{b}}} = -\frac{b}{1-\frac{a^2}{2b}}.$$

Thus, we need to discuss the case $a^2 = 2b$ by another method. But, to our negret we cannot find the new method now. By the numerical examples (Figures 1 through 3), we are convinced that the zero solution of (E) is not uniformly asymptotically stable in case $a^2 = 2b$. Then, we have the following conjecture.

Conjecture 4.1. *Let $a^2 = 2b$. Then, the zero solution of* (E) *is not uniformly asymptotically stable for all $h > 0$.*

If we can prove Conjecture 4.1, then we can show immediately the following statement by Theorems 2.1 and 2.2.

Conjecture 4.2. *The zero solution of* (E) *is uniformly asymptotically stable if and only if conditions* $a^2 < 2b$ *and*

$$\frac{a}{b} < h < \frac{1}{\sqrt{2b - a^2}} \operatorname{Cos}^{-1} \frac{a^2 - b}{b}$$

are satisfied.

Finally, we will show the behavior of solutions numerically for the case $a^2 = 2b$ which illustrate Conjecture 4.2. Then, we fix $a = 4$ and $b = 8$ and take the initial function as $\phi(t) = 100 + 20 \sin t$. We put the parameter h as follows and illustrate Conjecture 4.2 with drawing the solution curves.

$$\begin{aligned} &\text{Figure 1}: \ h = 0.45 \ \ (0 < h < a/b) \\ &\text{Figure 2}: \ h = 0.5 \ \ (h = a/b) \\ &\text{Figure 3}: \ h = 0.55 \ \ (h > a/b) \end{aligned}$$

Figures 1 through 3 suggest that the zero solution of (E) is not uniformly asymptotically stable for all $h > 0$ in case $a^2 = 2b$.

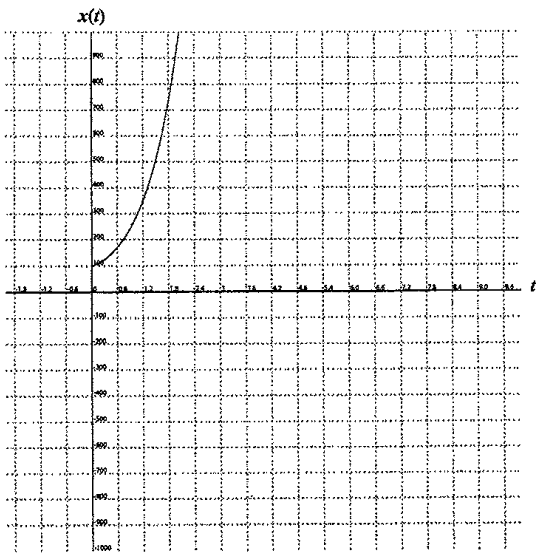

Figure 1: $h = 0.45\,(h < a/b)$

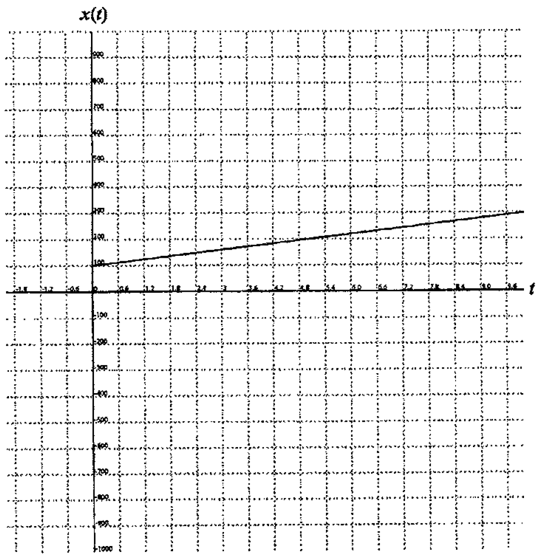

Figure 2: $h = 0.5\,(h = a/b)$

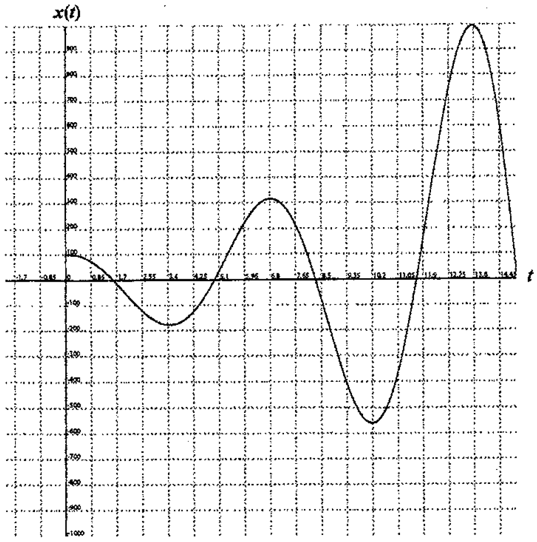

Figure 3: $h = 0.55\,(h > a/b)$

References

[1] T. A. Burton, "Volterra Integral and Differential Equations", Academic Press, New York, 1983.

[2] J. K. Hale and S. M. V. Lunel, "Introduction to Functional Differential Equations," Springer-Verlag, New York, 1993.

[3] R. Miyazaki, Characteristic equation and asymptotic behavior of delay-differential equation, *Funkcial. Ekvac.*, **40** (1997), 471–481.

[4] J. C. F. de Oliveira and L. A. V. Carvalho, A Lyapunov functional for a retarded differential equation, *SIAM. J. Math. Anal.*, **16** (1985), 1925–1305.

[5] S. Sakata, Stability sets for linear differential-difference equations with two delays, *Dynam. Systems Appl.*, **9** (2000), 569–594.

[6] S. Sakata and T. Hara, Stability regions for linear differential equations with two kinds of time delays, *Kyoto Univ. RIMS Kokyuroku*, **1216** (2001), 133–144.