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Holomorphic classification of 2-dimensional quadratic maps tangent to the identity

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0. Introduction

One of the main open problems in the field of local holomorphic dynamics in several variables is the classification (either formal, holomorphic, or topological) of germs tangent to the identity. To briefly survey what is known up to now, let us introduce some notations.

We shall denote by $\mathrm{End}(\mathbb{C}^n, O)$ the set of germs at the origin of holomorphic self-maps of $\mathbb{C}^n$ fixing the origin. Every $f \in \mathrm{End}(\mathbb{C}^n, O)$ admits a unique *homogeneous expansion* of the form

$$f(z) = P_1(z) + P_2(z) + \cdots,$$

where $P_j = (P_j^1, \ldots, P_j^n)$ is an n-uple of homogeneous polynomials of degree $j \geq 1$.

Definition 0.1: A germ $f \in \mathrm{End}(\mathbb{C}^n, O)$ is *tangent to the identity* if $P_1 = \mathrm{id}$. In this case we shall write

$$f(z) = z + P_\nu(z) + P_{\nu+1}(z) + \cdots, \tag{0.1}$$

where $\nu \geq 2$, the *order* of f, is the least $j \geq 2$ such that $P_j \not\equiv O$.

The formal classification of holomorphic germs tangent to the identity in one variable is obtained by an easy computation (see, e.g., [M]):

Proposition 0.1: *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic germ tangent to the identity of order $\nu \geq 2$. Then f is formally conjugated to the map*

$$z \mapsto z + z^\nu + \beta z^{2\nu-1},$$

where β is a formal (and holomorphic) invariant given by

$$\beta = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - f(z)},$$

where the integral is taken over a small positive loop γ about the origin.

The topological classification is simple too, though the proof is not trivial:

Theorem 0.2: (Camacho, 1978 [C]) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic germ tangent to the identity of order $\nu \geq 2$. Then f is topologically locally conjugated to the map*

$$z \mapsto z + z^\nu.$$

The holomorphic classification is much more complicated: as shown by Voronin [V] and Écalle [É1] in 1981, it depends on functional invariants. See also [I] and [K] for details.

In several variables, a major role in the study of the dynamics of maps tangent to the identity is played by characteristic directions. As a matter of notation, we shall denote by $v \mapsto [v]$ the canonical projection of $\mathbb{C}^n \setminus \{O\}$ onto $\mathbb{P}^{n-1}$.

Definition 0.2: Let $P: \mathbb{C}^n \to \mathbb{C}^n$ be a homogeneous polynomial map. A *characteristic direction* for P is the projection in $\mathbb{P}^{n-1}$ of any $v \in \mathbb{C}^n \setminus \{O\}$ such that $P(v) = \lambda v$ for some $\lambda \in \mathbb{C}$; the characteristic direction is *degenerate* if $\lambda = 0$, *non-degenerate* if $\lambda \neq 0$.

The set of characteristic directions is an algebraic subset of $\mathbb{P}^{n-1}$, which is either infinite or composed by $(d^n - 1)/(d - 1)$ points, counting with respect to a suitable multiplicity, where d is the degree of the map (see [AT]). In particular, if $n = d = 2$ either all directions are characteristic or there are at most 3 distinct characteristic directions.

Definition 0.3: Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be tangent to the identity, $f \neq \mathrm{id}$. A *characteristic direction* for f is a characteristic direction for P_ν, where $\nu \geq 2$ is the order of f. We shall say that the origin is *dicritical* for f if all directions are characteristic for P_ν.

Remark 0.1: There is an equivalent definition of characteristic direction. The n-uple of ν-homogeneous polynomial P_ν induces a meromorphic self-map of $\mathbb{P}^{n-1}(\mathbb{C})$, still denoted by P_ν. Then the non-degenerate characteristic directions are exactly fixed points of P_ν, and the degenerate characteristic directions are exactly indeterminacy points of P_ν.

Definition 0.4: Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be tangent to the identity, $f \neq \mathrm{id}$, of order $\nu \geq 2$. Let $[v] \in \mathbb{P}^{n-1}$ be a non-degenerate characteristic direction of f. The *directors* of $[v]$ are the eigenvalues of the linear operator $d(P_\nu)_{[v]} - \mathrm{id}\colon T_{[v]}\mathbb{P}^{n-1} \to T_{[v]}\mathbb{P}^{n-1}$.

The characteristic directions are strictly connected to the dynamics. To describe exactly how, we need two more definitions.

Definition 0.5: Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be tangent to the identity. We say that an orbit $\{f^k(z_0)\}$ converges to the origin *tangentially* to a direction $[v] \in \mathbb{P}^{n-1}$ if $f^k(z_0) \to O$ in $\mathbb{C}^n$ and $[f^k(z_0)] \to [v]$ in $\mathbb{P}^{n-1}$ as $k \to +\infty$.

Definition 0.6: A *parabolic curve* for a germ $f \in \mathrm{End}(\mathbb{C}^n, O)$ tangent to the identity is an injective holomorphic map $\varphi\colon \Delta \to \mathbb{C}^n \setminus \{O\}$ satisfying the following properties:

(a) Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$;

(b) φ is continuous at the origin, and $\varphi(0) = O$;

(c) $\varphi(\Delta)$ is f-invariant, and $(f|_{\varphi(\Delta)})^k \to O$ uniformly on compact subsets as $k \to +\infty$.

Furthermore, if $[\varphi(\zeta)] \to [v]$ in $\mathbb{P}^{n-1}$ as $\zeta \to 0$ in Δ, we shall say that the parabolic curve φ is *tangent* to the direction $[v] \in \mathbb{P}^{n-1}$.

Then the relationships between dynamics and characteristic directions can be stated as follows:

Theorem 0.3: (Écalle, 1985 [É2]; Hakim, 1998 [H1, 2]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a germ tangent to the identity of order $\nu \geq 2$. Then:*

(i) *if there is an orbit of f converging to the origin tangentially to a direction $[v] \in \mathbb{P}^{n-1}$, then $[v]$ is a characteristic direction of f;*

(ii) *conversely, for any non-degenerate characteristic direction $[v] \in \mathbb{P}^{n-1}$ there exist (at least) $\nu - 1$ parabolic curves for f tangent to $[v]$;*

(iii) *if the real part of all directors of a non-degenerate characteristic direction $[v] \in \mathbb{P}^{n-1}$ are positive, then there exist (at least) $\nu - 1$ open domains formed by orbits converging to the origin tangentially to $[v]$.*

This result leaves a priori open what happens at degenerate characteristic directions. This has been clarified in [A2], at least in dimension two. I do not want to enter here in too many technical details, but the idea is that not all characteristic directions can admit an orbit converging tangentially to them; only *singular* directions can (see [A2] and [ABT] for the definition of singular direction; it turns out that all singular directions are characteristic, and all non-degenerate characteristic directions are singular, but the reverse implications are both false).

Furthermore, to any tangent direction $[v] \in \mathbb{P}^1$ can be associated an *index* $\iota_f([v]) \in \mathbb{C}$. Again, the general definition is quite technical, and so it will be omitted here; but if $[v] \in \mathbb{P}^1$ is a non-degenerate characteristic direction with non-zero director α, then it turns out that $\iota_f([v]) = 1/\alpha$.

Then in [A2] the following result is proved:

Theorem 0.4: (Abate, 2001 [A2]) *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be a germ tangent to the identity of order $\nu \geq 2$ and with an isolated fixed point at the origin. Let $[v] \in \mathbb{P}^1$ be a singular direction such that $\iota_f([v]) \notin \mathbb{Q}^+$. Then there exist (at least) $\nu - 1$ parabolic curves for f tangent to $[v]$.*

Actually, probably in the previous statement it suffices to assume $\iota_f([v]) \neq 0$ to get the existence of parabolic curves tangent to $[v]$; Molino [Mo] has proved such a result under a mild technical assumption on f.

Now, in his monumental work [É2] Écalle has given a complete set of formal invariants for holomorphic local dynamical systems tangent to the identity with *at least one* non-degenerate characteristic direction. For instance, he has proved the following

Theorem 0.5: (Écalle, 1985 [É2]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic germ tangent to the identity of order $\nu \geq 2$. Assume that*

(a) *f has exactly $(\nu^n - 1)/(\nu - 1)$ distinct non-degenerate characteristic directions and no degenerate characteristic directions;*

(b) *the directors of any non-degenerate characteristic direction are irrational and mutually independent over $\mathbb{Z}$.*

Choose a non-degenerate characteristic direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$, and let $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$ be its directors. Then there exist a unique $\rho \in \mathbb{C}$ and unique (up to dilations) formal series $R_1, \ldots, R_n \in \mathbb{C}[[z_1, \ldots, z_n]]$, where each R_j contains only monomial of total degree at least $\nu + 1$ and of partial degree in z_j at most $\nu - 2$, such that f is formally conjugated to the time-1 map of the formal vector field

$$X = \frac{1}{(\nu - 1)(1 + \rho z_n^{\nu-1})} \left\{ [-z_n^\nu + R_n(z)] \frac{\partial}{\partial z_n} + \sum_{j=1}^{n-1} [-\alpha_j z_n^{\nu-1} z_j + R_j(z)] \frac{\partial}{\partial z_j} \right\}.$$

Another approach to the formal classification, at least in dimension 2, is described in [BM]; his results are particularly complete when the origin is dicritical.

Furthermore, using his theory of resurgence, and always assuming the existence of at least one non-degenerate characteristic direction, Écalle has also provided a set of holomorphic invariants for holomorphic local dynamical systems tangent to the identity, in terms of differential operators with formal power series as coefficients. Moreover, if the directors of all non-degenerate characteristic direction are irrational and satisfy a suitable diophantine condition, then these invariants become a complete set of invariants. See [É3] for a description of his results, and [É2] for the details.

Of course, Écalle's results do not apply to germs with no non-degenerate characteristic directions. It is easy to construct examples of such germs; for instance, for any $b \in \mathbb{C}^*$ all the characteristic directions of the map

$$\begin{cases} f_1(z) = z_1 + b z_1 z_2 + z_2^2, \\ f_2(z) = z_2 - b^2 z_1 z_2 - b z_2^2 + z_1^3, \end{cases}$$

are degenerate, and it not difficult to build similar examples of any order.

Finally, as far as I know, the topological classification of germs tangent to the identity is completely open, even in dimension 2.

In this paper we shall present a holomorphic classification of quadratic germs tangent to the identity in two dimensions, with an eye toward the topological classification. Our aim is to obtain a preliminary list of possible models for 2-dimensionals germs tangent to the identity of order two; we shall also briefly discuss what is known of the dynamics in most cases, and present several open questions.

The list is the following:

- (0) $f(z, w) = (z, w)$;
- (∞) $f(z, w) = (z + z^2, w + zw)$;
- (1_{00}) $f(z, w) = (z, w + z^2)$;
- (1_{10}) $f(z, w) = (z + z^2, w + z^2 + zw)$;
- (1_{11}) $f(z, w) = (z + zw, w + z^2 + w^2)$;
- (2_{001}) $f(z, w) = (z, w + zw)$;
- (2_{011}) $f(z, w) = (z + zw, w + zw + w^2)$;
- $(2_{10\gamma})$ $f(z, w) = (z + z^2, w + \gamma zw)$, with $\gamma \neq 1$;
- $(2_{11\gamma})$ $f(z, w) = (z + z^2 + zw, w + \gamma zw + w^2)$, with $\gamma \neq 1$;
- (3_{100}) $f(z, w) = (z + z^2 - zw, w)$;
- $(3_{\alpha 10})$ $f(z, w) = (z + \alpha z^2 - \alpha zw, w - zw + w^2)$, with $\alpha \neq 0, -1$;
- $(3_{\alpha\beta 1})$ $f(z, w) = \big(z + \alpha z^2 + (1 - \alpha) zw, w + (1 - \beta) zw + \beta w^2\big)$, with $\alpha + \beta \neq 1$ and $\alpha, \beta \neq 0$.

In the literature, two other such classifications are already known: one due to Ueda (quoted in [W]), and another one due to Rivi ([R]). The latter is not based on characteristic directions, and as we shall see is considerably different from ours. Ueda's classification, on the other hand, is more or less equivalent to ours; the presentation described here might however make computations easier in some cases. Anyway, in the last section we shall discuss how our classification is related to the previous ones.

1. Generalities

We shall begin by discussing (in a generality slightly greater than the one we shall actually need) how the homogeneous expansion changes under a holomorphic change of coordinates.

First of all, every $\chi \in \mathrm{End}(\mathbb{C}^n, O)$ such that $d\chi_O$ is invertible is a local biholomorphism; in particular, all germs tangent to the identity are local biholomorphisms. Conversely, it is clear that any local biholomorphism can be obtained by composing a linear isomorphism and a local biholomorphism tangent to the identity; so let us first study the effect of local biholomorphisms tangent to the identity.

Lemma 1.1: *Let $\chi \in \mathrm{End}(\mathbb{C}^n, O)$ be a local biholomorphism tangent to the identity, and let*

$$\chi(z) = z + \sum_{k\geq 2} A_k(z), \qquad \chi^{-1}(z) = z + \sum_{h\geq 2} B_h(z)$$

be the homogeneous expansions of χ and its inverse. Then

$$B_2 = -A_2, \qquad \text{and} \qquad B_3 = \sum_{j=1}^n A_2^j \frac{\partial A_2}{\partial z^j} - A_3, \tag{1.1}$$

where $A_2^1, \ldots, A_2^n$ are the components of A_2.

Proof: Using the usual formula

$$A_k(z+w) = A_k(z) + \sum_{j=1}^n w^j \frac{\partial A_k}{\partial z^j}(z) + O(\|w\|^2)$$

we get

$$\begin{aligned} z = \chi \circ \chi^{-1}(z) &= z + \sum_{h\geq 2} B_h(z) + \sum_{k\geq 2} A_k \left(z + \sum_{h\geq 2} B_h(z) \right) \\ &= z + \sum_{h\geq 2} B_h(z) + \sum_{k\geq 2} \left[A_k(z) + \sum_{j=1}^n \frac{\partial A_k}{\partial z^j}(z) \sum_{h\geq 2} B_h^j(z) + O\left(\left\| \sum_{h\geq 2} B_h(z) \right\|^2 \right) \right], \end{aligned}$$

where, as usual, we have set $B_h = (B_h^1, \ldots, B_h^n)$. Replacing z by λz, with $\lambda \in \mathbb{C}^*$, and dividing by λ^2 we get

$$B_2(z) + \lambda B_3(z) + A_2(z) + \lambda A_3(z) + \lambda \sum_{j=1}^n B_2^j(z) \frac{\partial A_2}{\partial z^j}(z) + O(\lambda^2) = O; \tag{1.2}$$

therefore letting $\lambda \to 0$ we get $B_2 = -A_2$. Putting this in (1.2), dividing by λ and letting again $\lambda \to 0$ we get the second formula. □

Corollary 1.2: *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be tangent to the identity, and let*

$$f(z) = z + \sum_{\ell\geq 2} P_\ell(z)$$

be its homogeneous expansion. Then, for any local biholomorphism $\chi \in \mathrm{End}(\mathbb{C}^n, O)$ tangent to the identity, the quadratic term in the homogeneous expansion of $\chi^{-1} \circ f \circ \chi$ is still P_2, while the cubic term is

$$P_3 + \sum_{j=1}^n \left[A_2^j \frac{\partial P_2}{\partial z^j} - P_2^j \frac{\partial A_2}{\partial z^j} \right],$$

where A_2 is the quadratic term in the homogeneous expansion of χ.

Proof: Arguing as in the proof of the previous lemma (and with the same notations) we see that

$$\chi^{-1} \circ f \circ \chi(\lambda z) = \lambda z + \lambda^2 [A_2 + P_2 + B_2] + \lambda^3 \left[A_3 + P_3 + B_3 + \sum_{j=1}^n \left(A_2^j \frac{\partial P_2}{\partial z^j} + [A_2^j + P_2^j] \frac{\partial B_2}{\partial z^j} \right) \right] + O(\lambda^4),$$

where B_2 and B_3 are the quadratic and cubic terms of the homogeneous expansion of χ^{-1}. The assertion then follows from (1.1). □

The main consequence for us of these formulas is that for quadratic maps holomorphic conjugacy reduces to linear conjugacy:

Corollary 1.3: *Let f, $g \in \mathrm{End}(\mathbb{C}^n, O)$ be two quadratic maps fixing the origin and tangent to the identity. Then f and g are holomorphically conjugate if and only if they are linearly conjugate.*

Proof: Assume that $g = \varphi^{-1} \circ f \circ \varphi$ for a suitable local biholomorphism $\varphi \in \mathrm{End}(\mathbb{C}^n, O)$. We can write $\varphi = U \circ \chi$, where $U \in GL_n(\mathbb{C})$ is linear and $\chi \in \mathrm{End}(\mathbb{C}^n, O)$ is tangent to the identity. Therefore $g = \chi^{-1} \circ (U^{-1} \circ f \circ U) \circ \chi$, and the previous corollary says that g and $U^{-1} \circ f \circ U$ must have the same quadratic terms, and thus $g = U^{-1} \circ f \circ U$, as claimed. □

So the holomorphic classification of quadratic maps tangent to the identity (or, more generally, of the quadratic term in the homogeneous expansion of maps tangent to the identity) is a linear one.

As mentioned in the introduction, from our point of view a good way to attack this classification problem is via characteristic directions. *From now on, we shall restrict our attention to the 2-dimensional case.*

Let us fix a couple of notations for the rest of this paper. If $f \in \mathrm{End}(\mathbb{C}^2, O)$ is a quadratic map tangent to the identity, we shall write $f = \mathrm{id} + P_2$, with $P_2 = (P_2^1, P_2^2)$ and

$$P_2^j(z, w) = a_{11}^j z^2 + a_{12}^j zw + a_{22}^j w^2.$$

In particular, $[u : v] \in \mathbb{P}^1$ is a characteristic direction of f if and only if

$$\begin{cases} a_{11}^1 u^2 + a_{12}^1 uv + a_{22}^1 v^2 = \lambda u, \\ a_{11}^2 u^2 + a_{12}^2 uv + a_{22}^2 v^2 = \lambda v, \end{cases} \tag{1.3}$$

for a suitable $\lambda \in \mathbb{C}$. Notice that $[u_0 : v_0]$ is a characteristic direction of f if and only if it is a root of the equation $G(u, v) = 0$, where

$$G(z, w) = zP_2^2(z, w) - wP_2^1(z, w). \tag{1.4}$$

In particular, $G \equiv 0$ if and only if the origin is dicritical for f.

Definition 1.1: The *multiplicity* of a characteristic direction of a germ $f \in \mathrm{End}(\mathbb{C}^2, O)$ is its multiplicity as root of (1.4). In other words, $[u_0 : v_0] \in \mathbb{P}^1$ is a characteristic direction of f of multiplicity $\mu \geq 1$ if and only if there is a homogeneous polynomial $q(z, w)$ of degree $3 - \mu$ with $q(u_0, v_0) \neq 0$ such that

$$G(z, w) = (u_0 w - v_0 z)^\mu q(z, w).$$

In particular, either all directions are characteristic (dicritical case), or there are exactly 3 characteristic directions, counted with respect to their multiplicity (this is a particular case of a result of [AT]).

Furthermore, if given $U \in GL_2(\mathbb{C})$ and $f \in \mathrm{End}(\mathbb{C}^2, O)$ tangent to the identity, we set $\tilde{f} = U^{-1} \circ f \circ U$ and $\tilde{G} = z\tilde{P}_2^2 - w\tilde{P}_2^1$, where $\tilde{P}_2$ is the quadratic term of $\tilde{f}$, it is clear that $[v] \in \mathbb{P}^1$ is a characteristic direction for f if and only if $[U^{-1}v]$ is a characteristic direction for $\tilde{f}$. Furthermore, it is not difficult to check that G and $\tilde{G}$ are linked by the formula

$$G \circ U = (\det U)\tilde{G},$$

and from this it follows easily that the multiplicity of $[v]$ as characteristic direction of f is equal to the multiplicity of $[U^{-1}v]$ as characteristic direction of $\tilde{f}$.

As a consequence, it makes sense to try and classify quadratic maps tangent to the identity in terms of the number of characteristic directions; and this is the approach we shall follow in the sequel.

Finally, let us introduce in this particular case the index mentioned in the introduction. Let $[u_0 : v_0] \in \mathbb{P}^1$ be a characteristic direction of f (and assume that the origin is not dicritical for f). Assume first that $u_0 \neq 0$; notice that in this case $[u_0 : v_0]$ is non-degenerate if and only if $P_2^1(u_0, v_0) \neq 0$. Then the index of $[u_0 : v_0]$ is given by the residue at $t_0 = v_0/u_0$ of the meromorphic function

$$\frac{P_2^1(1, t)}{P_2^2(1, t) - tP_2^1(1, t)};$$

notice that the denominator vanishes at t_0 because $[u_0 : v_0]$ is a characteristic direction. Similarly, if $[0 : 1]$ is a characteristic direction for f, its index is given by the residue at $s_0 = 0$ of the meromorphic function

$$\frac{P_2^2(s, 1)}{P_2^1(s, 1) - sP_2^2(s, 1)}$$

(see [A2] and [ABT] for details). The main property of the index is the following particular case of the index theorem proved in [A2]:

Theorem 1.4: (Abate, 2001 [A2]) *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be a quadratic map tangent to the identity. Assume that the origin is not dicritical for f. Then the sum of the indices of the characteristic directions of f is always -1.*

A last few pieces of terminology:

Definition 1.2: Assume that our quadratic map $f \in \mathrm{End}(\mathbb{C}^2, O)$ tangent to the identity has a line L of fixed points, of equation $\ell(z,w) = 0$. Then we can write $P_2 = \ell^\nu Q$ for a suitable $\nu \geq 1$, and ℓ does not divide $Q = (Q^1, Q^2)$. We shall say that the origin is *singular* for L if $\nu = 1$, that is $Q(O) = O$, and we shall say that f is *tangential* to L if ℓ divides $\ell \circ Q$. If the origin is singular for L, then Q is a linear map; we shall say that the origin is a *$(\star_1)$-point* if both the eigenvalues of Q are non-zero and their quotient does not belong to $\mathbb{Q}^+$; that it is a *$(\star_2)$-point* if it has exactly one non-zero eigenvalue; that it is a *non-reduced* point otherwise. Finally, we shall say that O is a *corner* for f if $\mathrm{Fix}(f)$ splits in the union of two distinct lines.

We refer to [A2] and [ABT] for an explanation of the dynamical relevance of these notions.

2. Dicritical maps

Let us then start with the case of infinite characteristic directions, that is with the dicritical case. Apart from the identity, there is essentially only one possibility:

Proposition 2.1: *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be a quadratic map tangent to the identity. Assume that the origin is dicritical for f. Then either f is the identity, that is*

$$f(z,w) = (z,w), \tag{0}$$

or f is (linearly) conjugate to the map

$$f(z,w) = (z + z^2, w + zw). \tag{∞}$$

Proof: Since the only map conjugated to the identity is the identity, we can just assume that $f \not\equiv \mathrm{id}$. Then there exists a function $\lambda\colon \mathbb{C}^2 \to \mathbb{C}$ not identically zero such that $P_2(x) \equiv \lambda(x)x$ for all $x \in \mathbb{C}^2$, that is

$$\begin{cases} a^1_{11}u^2 + a^1_{12}uv + a^1_{22}v^2 \equiv \lambda(u,v)u, \\ a^2_{11}u^2 + a^2_{12}uv + a^2_{22}v^2 \equiv \lambda(u,v)v. \end{cases}$$

In particular, setting $u = 0$ or $v = 0$ we get $a^1_{22} = a^2_{11} = 0$. Therefore

$$a^1_{11}u + a^1_{12}v = \lambda(u,v) = a^2_{12}u + a^2_{22}v,$$

and so $a^1_{11} = a^2_{12}$ and $a^1_{12} = a^2_{22}$. In other words, our map is of the form

$$g_{\alpha,\beta}(z,w) = (z + \alpha z^2 + \beta zw, w + \alpha zw + \beta w^2),$$

with $(\alpha,\beta) \neq (0,0)$. Now, taking $\chi(z,w) = (az, bw)$ it is easy to check that $\chi^{-1} \circ g_{\alpha,\beta} \circ \chi = g_{a\alpha, b\beta}$, and thus our map is conjugated to $g_{1,0}$, $g_{0,1}$ or $g_{1,1}$. But taking $\chi(z,w) = (w,z)$ we get $\chi^{-1} \circ g_{0,1} \circ \chi = g_{1,0}$, and taking $\chi(z,w) = (z - w, w)$ we get $\chi^{-1} \circ g_{1,1} \circ \chi = g_{1,0}$, and we are done. □

The dynamics of the map (∞) is very easy to describe. Every line through the origin is f-invariant. The line $\{z = 0\}$ is the fixed point set $\mathrm{Fix}(f)$ of f. If for $\lambda \neq 0$ we parametrize the line $\{w = \lambda z\}$ by $\zeta \mapsto (\zeta, \lambda\zeta)$, then the action of f on this line is given by the standard quadratic function $\zeta \mapsto \zeta + \zeta^2$. In particular, every such line contains a parabolic curve. All directions are non-degenerate characteristic directions but $[0:1]$, which is tangent to the fixed point set. Furthermore, the origin is a singular non-reduced point of $\mathrm{Fix}(f)$, and f is tangential to $\mathrm{Fix}(f)$.

Questions: Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be a dicritical germ tangent to the identity, of order 2. **(a)** Is a pointed neighbourhood of the origin foliated by disjoint invariant curves where f is the identity or has a parabolic fixed point at the origin? **(b)** Does the germ of $\mathrm{Fix}(f)$ at the origin contain at most one irreducible component? **(c)** When is such an f topologically conjugated to our model map (∞)? The obstruction here is that the fixed point set of such an f might consists of the origin only: for instance, this happens for

$$(z,w) \mapsto (z + z^2 - zw^2, w + zw + w^k),$$

which then cannot be topologically conjugated to (∞). Notice that the line $\{z = 0\}$ is still invariant, but the action there is $\zeta \mapsto \zeta + \zeta^k$. Is this the general case, that is the action on the invariant curves is the standard quadratic map except at most on one curve, where the action is $\zeta \mapsto \zeta + \zeta^k$ or the identity? If this is so, are two maps with the same k topologically conjugated?

3. One characteristic direction

Let us now study the maps with exactly one characteristic direction.

Proposition 3.1: *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be a quadratic map tangent to the identity. Assume that f has exactly one characteristic direction. Then f is (linearly) conjugate to one of the following three maps:*

$$f(z,w) = (z, w + z^2), \tag{1_{00}}$$

$$f(z,w) = (z + z^2, w + z^2 + zw), \tag{1_{10}}$$

$$f(z,w) = (z + zw, w + z^2 + w^2). \tag{1_{11}}$$

Proof: Up to a linear change of coordinates we can assume that $[0:1]$ is a characteristic direction, that is $a^1_{22} = 0$. We should exclude the existence of characteristic directions of the form $[1:v]$. Recalling (1.3), this means that the system

$$\begin{cases} a^1_{11} + a^1_{12}v = \lambda, \\ a^2_{11} + a^2_{12}v + a^2_{22}v^2 = (a^1_{11} + a^1_{12}v)v, \end{cases}$$

should not have solutions, that is

$$a^2_{22} = a^1_{12}, \qquad a^2_{12} = a^1_{11}, \qquad a^2_{11} \neq 0.$$

In other words, our map is of the form

$$g_{\alpha,\beta,\gamma}(z,w) = (z + \alpha z^2 + \beta zw, w + \gamma z^2 + \alpha zw + \beta w^2),$$

with $\gamma \neq 0$.

The linear change of variables keeping $[0:1]$ fixed are of the form $\chi(z,w) = (az, cz + dw)$ with $ad \neq 0$. Now we have

$$\chi^{-1} \circ g_{\alpha,\beta,\gamma} \circ \chi = g_{a\alpha + c\beta, d\beta, a^2\gamma/d};$$

in particular, the vanishing of β is a linear invariant. We then have three cases:

(a) $\alpha = \beta = 0$. In this case we choose $d/a^2 = \gamma$ and we get (1_{00}).
(b) $\beta = 0$, $\alpha \neq 0$. In this case we choose $a = \beta^{-1}$ and $d = \gamma/\beta^2$, and we get (1_{10}).
(c) $\beta \neq 0$. In this case we choose $d = \beta^{-1}$, $a = (\gamma\beta)^{-1/2}$ and $c = -\alpha/\beta(\gamma\beta)^{1/2}$ to get (1_{11}). □

Let us describe the dynamics.

• *The map* (1_{00}). This is easy. The fixed point set is again $\mathrm{Fix}(f) = \{z = 0\}$. The lines $\{z = c\}$ are f-invariant, and when $c \neq 0$ the orbits are diverging to infinity. In particular, no orbit is converging to the origin. As characteristic direction, $[1:0]$ is degenerate, and it has index -1 (as it should). Furthermore, the origin as point of $\mathrm{Fix}(f)$ is non-singular.

Question: Is every order 2 germ tangent to the identity, with only one characteristic direction, and such that O is a smooth (in the differential sense) non-singular (in the dynamical sense) point of $\mathrm{Fix}(f)$, topologically conjugated to (1_{00})?

• *The map* (1_{10}). Again, $\mathrm{Fix}(f) = \{z = 0\}$, and $[0:1]$ is a degenerate characteristic direction with index -1; the origin is singular and non-reduced, and f is tangential to its fixed point set. The rest is more complicated — but not exceedingly so; we are able to describe the dynamics in a (almost full) neighbourhood of the origin.

Let $C \subset \mathbb{C}$ be the standard cauliflower, that is the parabolic basin for the quadratic map $g(\zeta) = \zeta + \zeta^2$. It is well known (see, e.g., [F]) that setting $\zeta_n = g^n(\zeta_0)$, then ζ_n diverges as $n \to +\infty$ if and only if $\zeta_0 \notin \overline{C}$, and that ζ_n converges to 0 if and only if $\zeta_0 \in C$ — and in this case we can be more precise: $\zeta_n \sim -1/n$, that is $n\zeta_n \to -1$ as $n \to +\infty$.

In particular, it is clear that the orbit under our f of any point $(z_0,w_0)\in(\mathbb{C}\setminus\overline{C})\times\mathbb{C}$ diverges, because the first component does. We claim that, on the other hand, if $(z_0,w_0)\in C\times\mathbb{C}$ then the orbit $(z_n,w_n)=f^n(z_0,w_0)$ converges to the origin. Indeed, we already know that $\{z_n\}$ converges to zero; more precisely, $z_n\sim -1/n$. To deal with the second coordinate, we blow-up f with center in $[1:0]$, that is we make the birational change of coordinates $(z,w)=(s,st)$; see [A1] for details on blowing-up maps. We get

$$\tilde{f}(s,t)=\left(s+s^2,t+\frac{s}{1+s}\right).$$

So if we set $(s_0,t_0)=(z_0,w_0/z_0)$ and $\tilde{f}^n(s_0,t_0)=(s_n,t_n)$ we have $s_n\sim -1/n$ and

$$t_n-t_0=\sum_{j=0}^{n-1}\frac{s_j}{1+s_j}\sim -\log n.$$

Hence $w_n\sim n^{-1}\log n$ and $f^n(z,w)\to O$, as claimed. So we have a fairly complete description of the dynamics, leaving out only what happens on the invariant set $\partial C\times\mathbb{C}$, where f acts chaotically.

Question: Is every germ tangent to the identity of order 2 with exactly one characteristic direction, and such that O is a smooth (in the differential sense) singular (in the dynamical sense) point of $\mathrm{Fix}(f)$, topologically conjugated to (1_{10})?

• *The map* (1_{11}). At present we do not know much about this case, which is the first one with $\mathrm{Fix}(f)=\{O\}$. We have an invariant curve $\{z=0\}$ where f acts as the standard quadratic map $\zeta+\zeta^2$, and thus we get a parabolic curve here. In fact, $[0:1]$ is a non-degenerate characteristic direction, of index -1.

Questions: **(a)** What happens outside the invariant curve? **(b)** Is a neighbourhood of the origin foliated by invariant curves? **(c)** Is any holomorphic germ tangent to the identity, of order 2, with the origin as isolated fixed point and only one non-degenerate characteristic direction, topologically conjugate to (1_{11})?

Remark 3.1: We explicitely remark that these three maps are not topologically conjugated, nor are topologically conjugated to (∞). Indeed, (1_{11}) is the only one with an isolated fixed point; (1_{00}) is the only one with no orbits converging to the origin; and we can tell apart (1_{10}) from (∞) because in the former case there are orbits starting arbitrarily close to the origin which are diverging to infinity, whereas in the latter case there are no such orbits.

4. Two characteristic directions

There are two maps and two one-parameter families of maps with exactly two characteristic directions:

Proposition 4.1: *Let $f\in\mathrm{End}(\mathbb{C}^2,O)$ be a quadratic map tangent to the identity. Assume that f has exactly two characteristic directions. Then f is (linearly) conjugate to one of the following maps:*

(2_{001})
$$f(z,w)=(z,w+zw),$$

(2_{011})
$$f(z,w)=(z+zw,w+zw+w^2),$$

$(2_{10\gamma})$
$$f(z,w)=(z+z^2,w+\gamma zw),\qquad \gamma\neq 1,$$

$(2_{11\gamma})$
$$f(z,w)=(z+z^2+zw,w+\gamma zw+w^2),\qquad \gamma\neq 1.$$

Proof: Up to a linear change of coordinates we can assume that $[1:0]$ and $[0:1]$ are characteristic directions, which is equivalent to having $a^2_{11}=0=a^1_{22}$. We should exclude the existence of characteristic directions of the form $[u:1]$ with $u\neq 0$. Recalling (1.3), this means that the system

$$\begin{cases} a^1_{11}u^2+a^1_{12}u=(a^2_{12}u+a^2_{22})u,\\ a^2_{12}u+a^2_{22}=\lambda,\end{cases}$$

should have $u = 0$ as unique solution, that is

$$\begin{cases} a^1_{11} = a^2_{12}, \\ a^1_{12} \neq a^2_{22}, \end{cases} \qquad \text{or} \qquad \begin{cases} a^1_{11} \neq a^2_{12}, \\ a^1_{12} = a^2_{22}, \end{cases}$$

depending on which one of the two characteristic directions has multiplicity 2. Up to swapping the coordinates we can assume that the second possibility holds, or, in other words, that our map is of the form

$$g_{\alpha,\beta,\gamma}(z,w) = (z + \alpha z^2 + \beta zw, w + \gamma zw + \beta w^2),$$

with $\alpha \neq \gamma$.

The linear changes of variables keeping $[1 : 0]$ and $[0 : 1]$ fixed are of the form $\chi(z, w) = (az, cw)$ with $ac \neq 0$. Now we have

$$\chi^{-1} \circ g_{\alpha,\beta,\gamma} \circ \chi = g_{a\alpha, c\beta, a\gamma};$$

in particular, the vanishing of α and/or β are linear invariants. We then have four cases:

(a) $\alpha = \beta = 0$. In this case we choose $a = \gamma^{-1}$ and we get (2_{001}).
(b) $\alpha = 0$, $\beta \neq 0$. In this case we choose $a = \gamma^{-1}$ and $c = \beta^{-1}$, and we get (2_{011}).
(c) $\alpha \neq 0$, $\beta = 0$. In this case we choose $a = \alpha^{-1}$ to get $(2_{10\gamma})$.
(d) α, $\beta \neq 0$. In this case we choose $a = \alpha^{-1}$ and $c = \beta^{-1}$ to get $(2_{11\gamma})$. □

Remark 4.1: If we put $\gamma = 1$ in $(2_{10\gamma})$ or in $(2_{11\gamma})$ we get a dicritical map, conjugated (or equal) to the map (∞).

Let us describe what is known about the dynamics.

• *The map* (2_{001}). We have $\mathrm{Fix}(f) = \{z = 0\} \cup \{w = 0\}$. Both characteristic directions are degenerate; $[1 : 0]$ has index 0 and multiplicity 1, while $[0 : 1]$ has index -1 and multiplicity 2. The origin is a corner of f. As a point of the fixed line $\{w = 0\}$, it is singular and not reduced; as a point of the fixed line $\{z = 0\}$ is a singular point of type $(\star_2)$. The map f is tangential to $\{z = 0\}$, and not tangential to $\{w = 0\}$.

A neighbourhood of the origin is foliated by the invariant curves $\{z = c\}$, where the action has $(c, 0)$ as attractive (repulsive, indifferent) fixed point according to whether $|c + 1| < 1$ $(> 1, = 1)$. In particular, there are no orbits converging to the origin. Finally, the structure of the fixed point set clearly shows that this map is not topologically conjugated to any of the previous ones.

Question: Is every order 2 germ tangent to the identity such that O is a non-singular (in the dynamical sense) corner of $\mathrm{Fix}(f)$, the latter consists of two irreducible components, and f is tangential to one component but not to the other, topologically conjugated to (2_{001})?

• *The map* (2_{011}). We have $\mathrm{Fix}(f) = \{w = 0\}$, while $\{z = 0\}$ is an f-invariant curve where f acts as the standard quadratic map $\zeta \mapsto \zeta + \zeta^2$. The characteristic direction $[1 : 0]$ is degenerate of index 0 and multiplicity 1, while $[0 : 1]$ is non-degenerate of index -1 and multiplicity 2. As point of $L = \{w = 0\}$, the origin is singular and non-reduced; the map f is not tangential to L.

The parabolic curve inside $\{z = 0\}$ is the one associated by Theorem 0.3 to the non-degenerate characteristic direction $[0 : 1]$. The eigenvalues of df at the fixed point $(z, 0)$ are 1 and $1 + z$; therefore when $|1 + z| < 1$ (respectively, $|1 + z| > 1$) we get an f-invariant stable (respectively, unstable) curve passing through $(z, 0)$ transversally to $\mathrm{Fix}(f)$. This does not happen for the map (1_{10}), and so this map is not topologically conjugated to any of the previous ones.

Question: Is every order 2 holomorphic germ tangent to the identity with an irreducible fixed point set, with the origin as singular point, and whose differential restricted to the fixed point set is not the identity, topologically conjugate to (2_{011})?

• *The map* $(2_{10\gamma})$. This time $\mathrm{Fix}(f) = \{z = 0\}$, while $\{w = 0\}$ is an f-invariant curve where f acts as the standard quadratic map $\zeta \mapsto \zeta + \zeta^2$. The characteristic direction $[1 : 0]$ is non-degenerate of index $1/(\gamma - 1)$ and multiplicity 1, while $[0 : 1]$ is non-degenerate of index $\gamma/(1 - \gamma)$ and multiplicity 2. As point of $L = \{z = 0\}$, the origin is singular, of type $(\star_1)$ if $\gamma \notin \mathbb{Q}^+$, of type $(\star_2)$ if $\gamma = 0$, non-reduced otherwise; the map f is tangential to L. The parabolic curve inside $\{w = 0\}$ is the one associated by Theorem 0.3 to the non-degenerate characteristic direction $[1 : 0]$.

If $\gamma=0$, a neighbourhood of the origin is foliated by the f-invariant curves $\{w=c\}$, where f acts as the standard quadratic map; so the dynamics is clear, and it is topologically different from any of the previous cases.

If $\gamma\neq 0$ the situation is much more complicated, but still under control. First of all, if $z_0\notin\overline{C}$, where $C\subset\mathbb{C}$ is again the cauliflower, then the orbit of (z_0,w_0) diverges, because the first component does. If $(z_0,w_0)\in C\times\mathbb{C}$, set $(z_n,w_n)=f^n(z_0,w_0)$. We have

$$w_n=w_0\prod_{k=0}^{n-1}(1+\gamma z_k).$$

It is well known that the latter product converges (as $n\to+\infty$) to a non-zero number if and only if the series $\gamma\sum_{k=0}^{\infty}z_k$ converges, and thus never if $\gamma\neq 0$, because we know that $z_n\sim -1/n$. On the other hand, it is also known that the product converges to zero if and only if the real part of that series diverges to $-\infty$; since $w_n\sim -1/n$, this happens if and only if $\operatorname{Re}\gamma>0$. So if $\operatorname{Re}\gamma>0$, all the orbits starting inside $C\times\mathbb{C}$ converge to the origin, whereas if $\operatorname{Re}\gamma<0$ then all the orbits starting inside $C\times\mathbb{C}$ diverge to infinity.

Actually, Rivi [R, Proposition 4.4.4] proved something more precise, and somewhat surprising. On $C\times\mathbb{C}$ it is possible to make the change of coordinates

$$\begin{cases} x=z,\\ y=z^{-\gamma}w.\end{cases}$$

In the new coordinates we have $x_n=z_n$, while

$$y_1=z_1^{-\gamma}w_1=z_0^{-\gamma}w_0\big(1-\gamma z_0+O(z_0^2)\big)(1+\gamma z_0)=y_0\big(1+O(x_0^2)\big),$$

and therefore

$$y_n=y_0\prod_{j=0}^{n-1}\big(1+O(x_j^2)\big).$$

Thus y_n/y_0 converges to a never vanishing holomorphic function $h(z_0)$, and hence $w_n\sim z_0^{-\gamma}w_0h(z_0)z_n^{\gamma}$. Since $z_n\sim -1/n$, it follows that:

- if $\operatorname{Re}\gamma>1$ then the orbit of any $(z_0,w_0)\in C\times\mathbb{C}^*$ converges to the origin tangentially to the non-degenerate characteristic direction $[1:0]$;
- if $\operatorname{Re}\gamma=1$ (but $\gamma\neq 1$) then the orbit of any $(z_0,w_0)\in C\times\mathbb{C}^*$ converges to the origin *without being tangential to any direction;*
- if $1>\operatorname{Re}\gamma>0$ then the orbit of any $(z_0,w_0)\in C\times\mathbb{C}^*$ converges to the origin tangentially to the *degenerate* characteristic direction $[0:1]$;
- if $\operatorname{Re}\gamma=0$ (but $\gamma\neq 0$) then the orbit of any $(z_0,w_0)\in C\times\mathbb{C}^*$ stays bounded and bounded away from the origin.

Thus we might have an open set of orbits attracted to a degenerate characteristic direction, as well as, even more surprisingly, an open set of orbits converging to the identity without being tangential to any direction. In particular, this seems to suggest that the sign of $\operatorname{Re}\gamma$ is important for a topological classification.

Questions: **(a)** Is every order 2 holomorphic germ tangent to the identity with an irreducible fixed point set where it is non-tangential, with the origin as singular point, and whose differential restricted to the fixed point set is not the identity, topologically conjugate to (2_{100})? **(b)** Of what kind of order 2 holomorphic germs tangent to identity is a $(2_{10\gamma})$ a model?

• *The map* $(2_{11\gamma})$. In this case, $\operatorname{Fix}(f)=\{O\}$. Both $\{z=0\}$ and $\{w=0\}$ are f-invariant curves where f acts as the standard quadratic map. Both characteristic directions are non-degenerate; $[1:0]$ has index $1/(\gamma-1)$ and multiplicity 1, while $[0:1]$ has index $\gamma/(1-\gamma)$ and multiplicity 2. Thus we always have two parabolic curves, one tangent to $[1:0]$ and another tangent to $[1:0]$. Furthermore, Theorem 0.3.(iii) imply that when $\operatorname{Re}\gamma>1$ there is an open set of orbits attracted by the origin tangentially to $[1:0]$, while when $\operatorname{Re}\gamma>|\gamma|^2$ there is an open set of orbits attracted by the origin tangentially to $[0:1]$. Again, this seems to suggest that the topological classification might depend on the value of γ.

Question: What else can we say about these maps?

Remark 4.2: The maps $(2_{11\gamma})$ and (1_{11}) are not topologically conjugated. A way of seeing this is the following: if there is an f-invariant curve passing through the origin, the action on f on this curve is the same of a one-dimensional function tangent to the identity, and hence there is a parabolic curve inside the invariant curve. In particular, the invariant curve must be tangent to a characteristic direction. But for the map (1_{11}) we have only one characteristic direction, and thus all invariant curves are tangent to the same direction, while in case $(2_{11\gamma})$ we have two transversal invariant curves.

5. Three characteristic directions

This is the generic case.

Proposition 5.1: *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be a quadratic map tangent to the identity. Assume that f has exactly three characteristic directions. Then f is (linearly) conjugate to one of the following maps:*

$$(3_{100}) \qquad f(z,w) = (z + z^2 - zw, w),$$

$$(3_{\alpha 10}) \qquad f(z,w) = (z + \alpha z^2 - \alpha zw, w - zw + w^2), \qquad \alpha \neq 0, -1,$$

$$(3_{\alpha\beta 1}) \qquad f(z,w) = \big(z + \alpha z^2 + (1-\alpha)zw, w + (1-\beta)zw + \beta w^2\big), \qquad \alpha + \beta \neq 1, \alpha, \beta \neq 0.$$

Proof: Up to a linear change of coordinates we can assume that $[1:0]$, $[0:1]$ and $[1:1]$ are the characteristic directions, which is equivalent to having $a_{11}^2 = 0 = a_{22}^1$ and $a_{11}^1 + a_{12}^1 = a_{12}^2 + a_{22}^2$. In other words, our map is of the form

$$g_{\alpha,\beta,\gamma}(z,w) = (z + \alpha z^2 + (\gamma - \alpha)zw, w + (\gamma - \beta)zw + \beta w^2);$$

we can also assume $\alpha + \beta \neq \gamma$, because otherwise f is dicritical.

The linear changes of variables keeping $[1:0]$, $[0:1]$ and $[1:1]$ fixed are of the form $\chi(z,w) = (az, aw)$ with $a \neq 0$. Now we have

$$\chi^{-1} \circ g_{\alpha,\beta,\gamma} \circ \chi = g_{a\alpha, a\beta, a\gamma}.$$

We then have three cases:

(a) $\beta = \gamma = 0$. In this case we choose $a = \alpha^{-1}$ and we get (3_{100}).

(b) $\beta \neq 0$, $\gamma = 0$. In this case we choose $a = \beta^{-1}$, and we get $(3_{\alpha 10})$. Notice that if $\alpha = 0$ swapping the variables we are back in case (a).

(c) $\gamma \neq 0$. In this case we choose $a = \gamma^{-1}$ to get $(3_{\alpha\beta 1})$. Notice that the map (3_{001}) is conjugated to (3_{100}), via the conjugation $\chi(z,w) = (z - w, z)$; the map $(3_{\alpha 01})$ is conjugated to $(3_{-\alpha 10})$, via the conjugation $\chi(z,w) = (w - z, w)$; and finally, $(3_{0\beta 1})$ is conjugated, by swapping the variables, to $(3_{\beta 01})$, and hence to $(3_{-\beta 10})$. □

Remark 5.1: The maps (3_{-110}) and $(3_{\alpha(1-\alpha)1})$ are dicritical.

Remark 5.2: There are some more identifications possible, corresponding to permuting the characteristic directions. The maps $(3_{\alpha\beta 1})$ and $(3_{\beta\alpha 1})$ are conjugated by swapping the variables (that is transposing $[1:0]$ and $[0:1]$ keeping $[1:1]$ fixed). The map $(3_{\alpha\beta 1})$ is conjugated by $\chi(z,w) = (w - z, w)$ to $(3_{-\alpha 1\beta})$, and if $\beta \neq 0$ the latter is conjugated to $(3_{-(\alpha/\beta)\beta^{-1}1})$; this time we have transposed $[0:1]$ and $[1:1]$ keeping $[1:0]$ fixed. Since the whole permutation group on three elements is generated by these two transpositions, all the other identifications can be obtained composing the two already described.

Remark 5.3: Actually, (3_{100}) and $(3_{\alpha 10})$ are conjugated to $(3_{\alpha\beta 1})$ with $\alpha\beta = 0$; therefore we essentially have a unique family of maps with exactly three characteristic directions.

Let us describe what is known about the dynamics.

• *The map* (3_{100}). This is easy. We have $\mathrm{Fix}(f) = \{z = 0\} \cup \{z = w\}$ (so the origin is a corner), while the lines $\{w = c\}$ are f-invariant. We have (of course) three characteristic directions: $[1 : 0]$ is non-degenerate of index -1, while $[0 : 1]$ and $[1 : 1]$ are both degenerate of index 0. The origin is a singular $(\star_2)$ point in both components of $\mathrm{Fix}(f)$, and f is not tangential to both components.

The parabolic curve associated to $[1 : 0]$ is the one contained in the line $\{w = 0\}$, where f acts as the standard quadratic map. More generally, f acts on the line $\{w = c\}$ as the quadratic map with two fixed points: at the origin, with derivative $1 - c$, and at c, with derivative $1 + c$. In particular, one of them is always repelling. So a neighbourhood of the origin is foliated by invariant curves intersecting both irreducible components of the fixed points set, and the dynamics is dictated by the derivative at the fixed points.

Question: Is any order 2 holomorphic germ tangent to the identity with the origin as non-singular corner of the fixed points set and not tangential to both the components of the fixed point set, topologically conjugated to (3_{100})?

• *The map* $(3_{\alpha 10})$. This time $\mathrm{Fix}(f) = \{z = w\}$, while $\{z = 0\}$ and $\{w = 0\}$ are f-invariant. With respect to $\mathrm{Fix}(f)$, the origin is a singular point, of type $(\star_1)$ if $-\alpha \notin \mathbb{Q}^+$, non-reduced otherwise. Since $\alpha \neq -1$, f is non-tangential to $\mathrm{Fix}(f)$. There are three characteristic directions: $[1 : 0]$ is non-degenerate, with index $-\alpha/(1 + \alpha)$; $[0 : 1]$ is non-degenerate, with index $-1/(1 + \alpha)$; $[1 : 1]$ is degenerate, and it has index 0.

The map f restricted to $\{z = 0\}$ is the standard quadratic map, while restricted to $\{w = 0\}$ is $\zeta + \alpha\zeta^2$; in both cases we get a parabolic curve inside the invariant curve, as expected. The parabolic curves fatten up to an open domain attracted by the origin tangentially to $[0 : 1]$ if $\mathrm{Re}\,\alpha < -1$, or to an open domain attracted by the origin tangentially to $[1 : 0]$ if $\mathrm{Re}\,\alpha < -|\alpha|^2$. The differential of f along $\mathrm{Fix}(f)$ has eigenvalues 1 and $1 + (1 + \alpha)z$, with eigenvectors respectively $[1 : 1]$ (of course) and $[-\alpha : 1]$. In particular, if $|1 + (1 + \alpha)z| < 1$ (respectively, if $|1 + (1 + \alpha)z| > 1$), we find an invariant stable (unstable) curve crossing transversally $\mathrm{Fix}(f)$ at (z, z).

Question: What else can we say? Since f is non-tangential to $\mathrm{Fix}(f)$, it might be possible that a neighbourhood of the origin is foliated by invariant curves crossing $\mathrm{Fix}(f)$ transversally and where the local dynamics is dictated by the eigenvalue at the fixed point; but the situation might be more complicated. For instance, if $\alpha = 1$ the line $\{w = -z\}$ is sent onto the fixed point set. Theorem 0.3 yields a basin of attraction for $[0 : 1]$ if $\mathrm{Re}\,\alpha < -1$, and a basin of attraction for $[1 : 0]$ if $\mathrm{Re}\,\alpha + |\alpha|^2 < 0$. So one can expect a change of behavior when α crosses $\mathrm{Re}\,\alpha = -1$ or $\mathrm{Re}\,\alpha + |\alpha|^2 = 0$. In particular, they might not be all topologically conjugate.

• *The map* $(3_{\alpha\beta 1})$. This time $\mathrm{Fix}(f) = \{O\}$, and as usual we do not know much. We have three invariant curves: $\{z = 0\}$ and $\{w = 0\}$, where the action is quadratic, though non-standard (it is $\zeta + \alpha\zeta^2$ on $\{w = 0\}$, and $\zeta + \beta\zeta^2$ on $\{z = 0\}$), and $\{z = w\}$, where the action is always given by the standard quadratic map. We have three characteristic directions, all non-degenerate: $[1 : 0]$ has index $\alpha/(1 - \alpha - \beta)$; $[1 : 1]$ has index $-1/(1 - \alpha - \beta)$; and $[0 : 1]$ has index $\beta/(1 - \alpha - \beta)$. Thus we can apply the usual arguments for the existence of parabolic curves and open domains attracted by the origin, but for the moment this is it.

Questions: **(a)** What else can we say? **(b)** Is it true that a generic order 2 holomorphic germ tangent to the identity is topologically conjugated to a $(3_{\alpha\beta 1})$? **(c)** Are they topologically conjugated to each other, or not (probably not)?

Let us sum up our preliminary findings. At this level, a first important invariant for the topological classification is the number and type of characteristic directions (which is taking into account the number of fixed lines, and of invariant lines). Furthermore, it appears that being tangential or not to the fixed lines is another element useful for telling apart different topological conjugacy classes. On the other hand, we have found continuous families of maps whose topological behavior changes with the parameters; so this raises the question of whether we have only a discrete set of possible topological conjugacy classes, or we can have a continuum of distinct topological conjugacy classes, in stark contrast with the 1-dimensional situation, where we only have one conjugacy class.

6. Comparison with other classifications

As mentioned in the introduction, in the literature two other classifications of quadratic maps tangent to the identity have already appeared. The first one, due to Ueda (as quoted in [W]) is the following:

N_0: $f(z,w) = (z,w)$;
N_4: $f(z,w) = (z+z^2, w+zw)$;
$N_{3,3}$: $f(z,w) = (z, w+z^2)$;
$N_{3,2}$: $f(z,w) = (z+z^2, w+z^2+zw)$;
$N_{3,1}$: $f(z,w) = (z+zw, w+z^2+w^2)$;
$N_{2,2}(\sigma)$: $f(z,w) = \big(z+\sigma z^2, w+(1+\sigma)zw\big)$;
$N_{2,1}(\sigma)$: $f(z,w) = \big(z+\sigma z^2+zw, w+(1+\sigma)zw+w^2\big)$;
$N_1(\sigma,\tau)$: $f(z,w) = (z+\sigma z^2+(1+\tau)zw, w+(1+\sigma)zw+\tau w^2)$.

The second one is due to Rivi ([R]), and is the following:

(1) $f(z,w) = (z,w)$;
(2) $f(z,w) = (z+w^2, w)$;
(3) $f(z,w) = (z+zw, w)$;
(4) $f(z,w) = (z+z^2+w^2, w)$;
(5) $f(z,w) = (z+zw+w^2, w+w^2)$;
(6_γ) $f(z,w) = (z+\gamma zw, w+w^2)$;
(7_γ) $f(z,w) = (z+\gamma zw+w^2, w+zw)$;
(8) $f(z,w) = (z+z^2+w^2, w+zw)$;
($9_{\gamma,\delta}$) $f(z,w) = (z+z^2+\gamma zw, w+\delta zw+w^2)$, $1-\gamma\delta\neq 0$.

We can compare the three classifications as follows:

- Map (0), that is the identity, is Ueda's map N_0 and Rivi's map (1).
- Map (∞) is Ueda's map N_4, and is conjugated to Rivi's map (6_1) by swapping the variables.
- Map (1_{00}) is Ueda's map $N_{3,3}$, and is conjugated to Rivi's map (2) by swapping the variables.
- Map (1_{10}) is Ueda's map $N_{3,2}$, and is conjugated to Rivi's map (5) by swapping the variables.
- Map (1_{11}) is Ueda's map $N_{3,1}$, and is conjugated to Rivi's map (8) by swapping the variables.
- Map (2_{001}) is Ueda's map $N_{2,2}(0)$, and is conjugated to Rivi's map (3) by swapping the variables.
- Map (2_{011}) is Ueda's map $N_{2,1}(0)$, and is conjugated to Rivi's map ($7_{\pm 2i}$) via the biholomorphism $\chi(z,w) = (z\mp iw, \pm w)$.
- Map ($2_{10\gamma}$) is conjugated to Ueda's map $N_{2,2}\big(1/(\gamma-1)\big)$ via the biholomorphism $\chi(z,w) = \big(z/(\gamma-1), w\big)$, and is conjugated to Rivi's map (6_γ) by swapping the variables.
- Map ($2_{11\gamma}$) is conjugated to Ueda's map $N_{2,1}\big(1/(\gamma-1)\big)$ via the biholomorphism $\chi(z,w) = \big(z/(\gamma-1), w\big)$, and is conjugated to Rivi's map ($9_{\gamma,1}$) by swapping the variables.
- Map (3_{100}) is conjugated to Ueda's map $N_1(-1,0)$ via the biholomorphism $\chi(z,w) = (-z,-w)$, and is conjugated to Rivi's map (4) via the biholomorphism $\chi(z,w) = (z-iw, -2iw)$.
- Map ($3_{\alpha 10}$) is conjugated to Ueda's map $N_1\big(-\alpha/(1+\alpha), -1/(1+\alpha)\big)$ via $\chi(z,w) = -(1+\alpha)^{-1}(z,w)$, and is conjugated to Rivi's map ($7_{1/(1+\alpha)\sqrt{\alpha}}$) via $\chi(z,w) = (1+\alpha)^{-1}(z-\alpha^{-1/2}w, z+\alpha^{1/2}w)$.
- Map ($3_{\alpha\beta 1}$) is conjugated to Ueda's map $N_1\big(\alpha/(1-\alpha-\beta), \beta/(1-\alpha-\beta)\big)$ via the biholomorphism $\chi(z,w) = (1-\alpha-\beta)^{-1}(z,w)$, and is conjugated to Rivi's map ($9_{(1-\alpha)/\beta,(1-\beta)/\alpha}$) via the biholomorphism $\chi(z,w) = (\alpha^{-1}z, \beta^{-1}w)$.

References

[A1] M. Abate: *Diagonalization of non-diagonalizable discrete holomorphic dynamical systems.* Amer. J. Math. **122** (2000), 757–781.

[A2] M. Abate: *The residual index and the dynamics of holomorphic maps tangent to the identity.* Duke Math. J. **107** (2001), 173–207.

[AT] M. Abate, F. Tovena: *Parabolic curves in* $\mathbb{C}^3$. Abstr. Appl. Anal. **2003** (2003), 275–294.

[ABT] M. Abate, F. Bracci, F. Tovena: *Index theorems for holomorphic self-maps.* To appear in Ann. of Math., 2003.

[BM] F.E. Brochero Martínez: *Groups of germs of analytic diffeomorphisms in* $(\mathbb{C}^2, O)$. J. Dynamic. Control Systems **9** (2003), 1–32.

[C] C. Camacho: *On the local structure of conformal mappings and holomorphic vector fields.* Astérisque **59–60** (1978), 83–94.

[É1] J. Écalle: **Les fonctions résurgentes. Tome I: Les algèbres de fonctions résurgentes.** Publ. Math. Orsay **81-05,** Université de Paris-Sud, Orsay, 1981.

[É2] J. Écalle: **Les fonctions résurgentes. Tome III: L'équation du pont et la classification analytique des objects locaux.** Publ. Math. Orsay **85-05,** Université de Paris-Sud, Orsay, 1985.

[É3] J. Écalle: *Iteration and analytic classification of local diffeomorphisms of* $\mathbb{C}^\nu$. In **Iteration theory and its functional equations,** Lect. Notes in Math. **1163,** Springer-Verlag, Berlin, 1985, pp. 41–48.

[F] P. Fatou: *Sur les équations fonctionnelles, I.* Bull. Soc. Math. France **47** (1919), 161–271.

[Ha1] M. Hakim: *Analytic transformations of* $(\mathbb{C}^p, 0)$ *tangent to the identity.* Duke Math. J. **92** (1998), 403–428.

[Ha2] M. Hakim: *Transformations tangent to the identity. Stable pieces of manifolds.* Preprint, 1997.

[I] Yu.S. Il'yashenko: *Nonlinear Stokes phenomena.* In **Nonlinear Stokes phenomena,** Adv. in Soviet Math. **14,** Am. Math. Soc., Providence, 1993, pp. 1–55.

[K] T. Kimura: *On the iteration of analytic functions.* Funk. Eqvacioj **14** (1971), 197–238.

[M] J. Milnor: **Dynamics in one complex variable.** Vieweg, Braunschweig, 2000.

[Mo] L. Molino: *Parabolic curves at singular points.* In preparation, 2003.

[R] M. Rivi: **Local behaviour of discrete dynamical systems.** Ph.D. Thesis, Università di Firenze, 1999.

[V] S.M. Voronin: *Analytic classification of germs of conformal maps* $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ *with identity linear part.* Func. Anal. Appl. **15** (1981), 1–17.

[W] B.J. Weickert: *Attracting basins for automorphisms of* $\mathbb{C}^2$. Invent. Math. **132** (1998), 581–605.