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On the complex basin of real attractors

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For a unimodal real-analytic dynamics, we define the Hausdorff and hyperbolic dimensions of the local basin of attraction from the complex plane to a real attractor. Then we discuss these notions for two classes of maps (attractors): Misiurewicz's maps, and infinitely-renormalizable maps with bounded combinatorics.

1 Definitions

Let f be a real-analytic map of an interval I into itself, such that $f : I \to I$ is unimodal. By the definition, f extends to an analytic map of a complex neighborhood V of I onto another neighborhood W of I, such that f' vanishes in V at a unique (critical) point $c \in I$. Denote by $\ell = \ell(f) \in 2\mathbb{N}$ the order of f at c.

Let A be a (metric) attractor of $f : I \to I$ (in the sense of [8]), i.e. a closed forward invariant subset of I, such that its *real* basin $B_{att}(A, I) = \{x \in I | \omega(x) \subset A\}$ has a positive (one-dimensional) Lebesgue measure, and for any other closed invariant proper subset A' of A, the measure of $B_{att}(A, I) \setminus B_{att}(A', I)$ is positive. Here $\omega(x)$ is the set of limit points of the forward orbit $f^n(x), n > 0$.

Given a neighborhood U of A *in the plane*, introduce two subsets of U:

$B_{att}(A, U)$ is the basin of attraction of A in U: this is the set of all points $z \in U$, such that all the iterates $f^n(z)$ of z are well-defined, contained in U, and $\omega(z) \subset A$.

$B_{hyp}(A, U)$ is the union of closed subsets X of U, such that $f(X) \subset X$ and f is expanding on X (by the latter we mean that there are $n \geq 1$ and $a > 1$, such that $|(f^n)'(x)| \geq a$ for every $x \in X$).

Note that if A contains no an expanding subset, the sets $B_{att}(A, U)$ and $B_{hyp}(A, U)$ are disjoint.

Definition 1.1 *Denote by $D_{att}(A)$ the infinum of the Hausdorff dimensions of the sets $B_{att}(A, U)$. We call $D_{att}(A)$ the dimension of the (complex) basin of the attractor A.*

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Similarly, denote by $D_{hyp}(A)$ the infinum of the Hausdorff dimensions of the sets $B_{hyp}(A,U)$, and call $D_{hyp}(A)$ the local hyperbolic dimension of the (complex) basin of A.

Note that $D_{hyp}(A)$ is defined by analogy to the hyperbolic dimension of the Julia set [10].

Since A is a metric attractor on $\mathbb{R}$, $D_{att}(A) \geq 1$.

Below $HD(F)$ denotes the Hausdorff dimension of a set F in $\mathbf{R}^n$.

Comment 1 *Assume that f extends to a polynomial-like map from V onto W [1]. Then $B_{att}(A,V)$ and $B_{hyp}(A,V)$ are dense subset of its Julia set. Furthermore, for every neighborhood U of A, $B_{att}(A,V) = \cup_{n\geq 0} f^{-n}(B_{att}(A,U))$. Therefore, $HD(B_{att}(A,U)) = HD(B_{att}(A,V))$, i.e. $HD(B_{att}(A,U))$ is independent on the neighborhood U. Thus*

$$D_{att}(A) = HD(B_{att}(A,V)),$$

that is $D_{att}(A)$ is just the Hausdorff dimension of the (global) basin of attraction of A.

Let us call a function $f \mapsto \beta(f)$ an *invariant* if $\beta(f) = \beta(g)$ for any two topologically conjugate f and g with the same critical order $\ell(f) = \ell(g)$.

2 Misiurewicz's maps

Assume that $f : I \to I$ has no attracting or parabolic periodic orbits. Assume also that f is Misiurewicz: the critical point c of f is not recurrent. Then f has a metric attractor A which is the union of finitely many disjoint intervals. Here we prove:

In this setting, the dimension of the basin of A and the local hyperbolic dimension of the basin of A are equal to 1:

$$D_{att}(A) = D_{hyp}(A) = 1.$$

Example. Let $f(z) = 2z^\ell - 1$, $\ell \in 2\mathbb{N}$. Then $A = [-1,1]$, and the above statement means that the Hausdorff dimension of the basin of attraction in the plane to the attractor A is equal to the local hyperbolic dimension is equal to 1. Note that the hyperbolic dimension of the Julia set in this (Collet-Eckmann) case is equal to the Hausdorff dimension of the Julia set which is bigger than one (for $\ell > 2$).

Let us prove that $D_{att}(A) = D_{hyp}(A) = 1$ for Misiurewicz's maps. We start by a remark that it is enough to consider points (either of the basin of attraction or of a hyperbolic set) whose forward orbits never hit the interval I. Fix such a point z_0.

First we prove that $D_{att}(A) = 1$.

Let $\omega(z_0) \subset A$.

(a). Show that the critical point c belongs to $\omega(z_0)$. Assume the contrary. Then $\omega(z_0)$ is an expanding (for f) Cantor subset of I. Then it is easy to see (using for example Proposition 10.1 (Construction of Cantor repeller) of [5]) that there is $\delta > 0$, so that if a point z is off $\omega(z_0)$, then some iterate of z is outside of δ-neighborhood of $\omega(z_0)$. It follows that some iterate of z_0 must hit $\omega(z_0) \subset I$, a contradiction.

(b). Construction below is very similar (though much simpler) to one from Theorems A' and B'-B" of [5]. Let $J = (\hat{u}, u)$ be a small enough "symmetric" (i.e. $f(\hat{u}) = f(u)$) interval around c with the "nice" endpoints, i.e. $f^n(u) \notin J$ for all $n \geq 0$. Consider the real first entry map R_J to J: for every $x \in I$, such that there is $n \geq 1$, so that $f^n(x) \in J$, define $R_J(x) = f^{n(x)}(x)$, with the minimal $n(x)$ as above. Then the domain of definition of R_J consists of countably many disjoint open intervals Δ_i, so that $\cup_i \Delta_i$ is dense in I and does not contain c. Note that

(b1) each branch $R_J : \Delta_i \to J$ extends to a diffeomorphism onto a fixed (i.e. independent of J) neighborhood of c.

By choosing J more carefully, one can further assume that

(b2) $dist(\Delta_i, \partial J)/|\Delta_i|$ tends to infinity as $|J| \to 0$ uniformly on i.

(For example, take a sequence J_n of critical pieces of appropriate real Yoccoz's partition; then end points of J_n are preimages of an invariant and expanding under f closed set, and therefore one can pass from a small neighborhood of an end point of J_n to a fixed big scale with bounded distortion.)

(b1)-(b2) allow us to construct a complex extension of the real map R_J as follows (cf. Theorem B" of [5]). Denote by $D(K)$ the round disk based on a real interval K as diameter. Then each inverse branch $R_J^{-1} : J \to \Delta_i$ extends to a well-defined univalent map $\hat{R}_J^{-1} : D(J) \to V_i$, where all V_i are pairwise disjoint, disjoint with the boundary of $D(J)$ and, moreover, if Δ_i is contained in J, then the modulus of the annulus $D(J) \setminus V_i$ tends to infinity uniformly on i as $|J| \to 0$. We complete the map $\hat{R}_J : \cup_i V_i \to D(J)$ by some complex components as follows. Denote by $J/2$ the symmetric interval around c of the lenght $|J|/2$. Then for any component $\Delta = \Delta_i$ of the first entry map which is contained inside $|f(J/2)|$-neighborhood of the critical value $f(c)$, consider ℓ components of $f^{-1}(V_i)$ and denote them by $W_{\Delta,k}$, $k = 1, 2, ..., \ell$. Define $\hat{R}_J | W_{\Delta,k} = \hat{R}_J \circ f$. Observe that for those component Δ_i which intersect $|f(J)|$-neighborhood of the critical value $f(c)$, $|\Delta_i|/|f(J)|$ tends to zero uniformly on such Δ_i as $|J| \to 0$. Indeed, this follows from the fact that $\omega(f(c))$ is expanding for f, so that, on the one hand, one can pass from any neighborhood of $f(c)$ to a fixed scale with bounded distortion, on the other hand, any Δ_i is an isomorphic preimage of small interval J. Hence, all $W_{\Delta,k}$ as above are contained in $D(J)$ (which is round disk). Let us consider the set of all $W_{\Delta,k}$ together with V_i and re-denote all them by W_j. Thus we end up with the "complex first entry map to $D(J)$":

$$\hat{R}_J : \cup_j W_j \to D(J).$$

It follows from (b1)-(b2),

(b3) $\alpha(J) \to \infty$ as $|J| \to 0$, where $\alpha(J) := \inf_i dist(c, W_i)/diam(W_i)$.

As it follows from the above discussion, (b4) each branch $\hat{R}_J : W_j \to D(J)$ is asymptotically linear: the distortion of $\hat{R}_J : W_j \to D(J)$ tends to 1 uniformly on j as $|J| \to 0$.

(c). Let us fix a small enough interval J and the corresponding complex first entry map $\hat{R}_J : \cup_j W_j \to D(J)$ as in (b). Then choose a neighborhood U of I which obeys the following property: if Δ_i is any component of the real first entry map R_J to J, such that Δ_i is not contained in the $|f(J/2)|$-neighborhood $f(c)$, then, for the complex extension V_i of Δ_i, any component of $f^{-1}(V_i)$, which is disjoint with the interval I, is disjoint also with U.

Claim. If $z_0 \in B_{att}(A, U)$, then the whole forward orbit $\{f^n(z_0)\}_{n>0}$ belongs to the domain of definition $\cup_j W_j$ of $\hat{R}_J$.

Indeed, consider any $n > 0$ and show that $f^n(z_0) \in \cup_j W_j$. By (a), $c \in \omega(z_0)$, hence, there is $m > n$, such that $f^m(z_0) \in D(J)$. Let us go back from $f^m(z_0)$ to $f^n(z_0)$. By the choice of U, we conclude that all $f^k(z_0), k = m-1, m-2, ..., n$ are contained in $\cup_j W_j$.

(d). By the claim, for any interval J around c as above, there is a neighborhood U of I, such that $B_{att}(A, U)$ is a subset of the set $X(R_J)$ of non-escaping points of the map $\hat{R}_J : \cup_j W_j \to D(J)$. Thus it is enough to show that the Hausdorff dimension h of $X(R_J)$ tends to 1 as $|J| \to 0$. In turn, h is equal to the Hausdorff dimension h_c of the set $X_c(R_J)$ of non-escaping points of the first return map $\hat{R}_c$ to $D(J)$, i.e. of the restriction of $\hat{R}_J$ to the components which are inside $D(J)$: $\hat{R}_c = \hat{R}| \cup_j \{W_J | W_j \subset D(J)\}$.

By (b4), h_c is equal asymptotically (as $|J| \to 0$) to the root of the equation $\psi(\theta) = 1$, where

$$\psi(\theta) = \sum_{W_j \subset D(J)} [diam(W_j)/|J|]^\theta.$$

On the other hand, each $W_j \subset D(J)$ intersects one and only one of the 2ℓ components of the preimage by f of the $|f(J)|$-neighborhood of $f(c)$ with $f(c)$ deleted, and the lenght of the intersection I_j is asymptotically equal to $diam(W_j)$. Thus we have $\sum_j |I_j| \leq 2\ell|J|$. By (b3), for every $0 < \alpha < 1$ and all $|J|$ small enough, $|I_j| \leq \alpha|J|$ for every I_j.

Therefore, for any fixed $\delta > 0$ small enough, and any $0 < \alpha < 1$ we have asymptotically $\psi(1+\delta) \leq 2\ell|J|(\alpha|J|)^\delta/|J|^{1+\delta} = 2\ell\alpha^\delta$, i.e. $\psi(1+\delta) \to 0$ as $|J| \to 0$. Together with $\psi(0) = \infty$ it implies that $h = h_c \to 1$ as $|J| \to 0$.

Thus we have proved that $D_{att}(A) = 1$.

The proof that $D_{hyp}(A) = 1$ is very similar though the step (a) should be replaced by the following.

(a'). Let us fix a small enough disk $D(J)$ centered at c. Then for every small enough neighborhood U of A we have that every expanding invariant for f closed subset $X \subset U \backslash I$ intersects $D(J)$. Indeed, otherwise there is an expanding closed real set $Y \subset I$, a sequence of (complex) neighborhoods U_n of Y, which shrink to Y, and a sequence of expanding invariant sets $X_n \subset U_n \setminus I$. But this is impossible because the forward orbit of any point close enough to Y leaves a definite neighborhood of Y, see (a).

After that the steps (b)-(d) are essentially not changed. Note that A contains a neighborhood of c. Then it follows from Step (b) that the hyperbolic dimension of A is 1, hence, $D_{hyp}(A) \geq 1$. Then Step (d) implies that $D_{hyp}(A) = 1$.

3 Infinitely-renormalizable maps

Let $f : I \to I$ be a real-analytic infinitely-renormalizable unimodal map. Then the Cantor set $A = A(f) = \omega(c)$ is a metric attractor (which is called solenoid, or Feigenbaum-type attractor). It is proved in [4], Theorem 11.1, that some real renormalization $\tilde{f} = R^{n_0} f$ of f extends to a polynomial-like map. Then $\tilde{f}$ has a unique attractor on the real line, which is $A(\tilde{f}) = \omega_{\tilde{f}}(c)$, and it is clear that $D_{att}(A(f)) = D_{att}(A(\tilde{f}))$, $D_{hyp}(A(f)) = D_{hyp}(A(\tilde{f}))$.

Let us consider the case when f has a bounded combinatorics. Note that in this case the Hausdorff dimension of the attractor $A(f)$ itself is an invariant as the convergence of renormalization implies [7]. The dimensions of the basin are also invariants:

If $f, g : I \to I$ are two real-analytic unimodal maps which are infinitely-renormalizable with bounded combinatorics and which are topologically conjugate by h, and such that $\ell(f) = \ell(g)$, then, for $A = A(f)$,

$$D_{att}(A) = D_{att}(h(A)) \quad \textit{and} \quad D_{hyp}(A) = D_{hyp}(h(A)).$$

Indeed, as we know some real renormalization $R^{n_0} f$ of f and $R^{n_0} g$ of g extend to polynomial-like maps. By the convergence of the renormalizations [11], [9], there is a sequence of quasi-conformal maps $h_k : \mathbb{C} \to \mathbb{C}$, so that, for every $n > 0$, h_k conjugates the complex dynamics of the following polynomial-like maps: renormalization $R^{kn_0} f$ of $R^{n_0} f$ and renormalization $R^{kn_0} g$ of $R^{n_0} g$, and such that the dilatation of h_k tends to 1 as $k \to \infty$. Obviously, h_k maps the basin of attraction $B(R^{kn_0} f)$ of $A(R^{kn_0} f)$ onto corresponding basin $B(R^{kn_0} g)$ for g. Hence, $HD(B(R^{kn_0} f))/HD(B(R^{kn_0} g)) \to 1$. On the other hand, for every $k > 0$, $HD(B(R^{kn_0} f)) = HD(B(R^{n_0} f)) = D_{att}(A(f))$, and the same for g.

Same argument holds for the local hyperbolic dimension.

Comment 2 *This statement and its proof hold for other combinatorics whenever it is known that the quasi-conformal distance between the corresponding renormalizations tends to zero (i.e. for Fibonacci one).*

Let us look closer at Feigenbaum's maps f [2]. By the above, for every even order ℓ of f we have the following real numbers, which depend merely on ℓ: the Hausdorff dimension d_ℓ of Feigenbaum's attractor $A(f)$ itself; the Hausdorff dimension $D_{att,\ell} = D_{att}(A(f))$, and the local hyperbolic dimension $D_{hyp,\ell} = D_{hyp}(A(f))$ of the $A(f)$-basin in the plane. Consider the dependence of these invariants on ℓ. Apriori, $0 < d_\ell < 1$, $1 \le D_{att,\ell}, D_{hyp,\ell} \le 2$. As it is proved in [3], d_ℓ tends to a limit d_∞ as $\ell \to \infty$, where $d_\infty \in (2/3, 1)$, i.e. the limit is less than the maximal possible (which is 1); on the other hand, Theorem 1 in [6] says that $D_{att,\ell}$ and $D_{hyp,\ell}$ tend to 2 as $\ell \to \infty$, i.e. the limit is the maximal possible dimension. Note that results of [6] should hold for any bounded combinatorics.

One should compare it to Misiurewicz's maps where we prove that the dimension and the local hyperbolic dimension of the basin are equal to 1, i.e. minimal possible for every ℓ.

Questions:

1. Is it true that always $D_{att}(A) = D_{hyp}(A)$?

2. For which maps with (the unique) attractor A are the dimensions $D_{att}(A)$ and $D_{hyp}(A)$ invariants?

As it follows from Sections 2- 3, the latter is true for Misiurewicz's maps, and for maps which are infinitely-renormalizable with bounded combinatorics (see also Comment 2).

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