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Center conditions for polynomial differential equations: discussion of some problems [1]

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Abstract

Classifications of irreducible components of the set of polynomial differential equations with a fixed degree and with at least one center singularity lead to some other new problems on Picard-Lefschetz theory and Brieskorn modules of polynomials. In this article we explain these problems and their connections to such classifications.

0 Introduction

The set of polynomial 1-forms $\omega = P(x,y)dy - Q(x,y)dx, \deg P, \deg Q \leq d,\ d \geq 2$ is a vector space of finite dimension and we denote by $\overline{\mathcal{F}(d)}$ its projectivization. Its subset $\mathcal{F}(d)$ containing all ω's with P and Q relatively prime and $\deg(\omega) := \max\{\deg P, \deg Q\} = d$ is Zariski open in $\overline{\mathcal{F}(d)}$. We denote the elements of $\overline{\mathcal{F}(d)}$ by $\mathcal{F}(\omega)$ or $\mathcal{F}$ if there is no confusion about the underlying 1-form ω in the text. Any $\mathcal{F}(\omega)$ induces a holomorphic foliation $\mathcal{F}$ in $\mathbb{C}^2$ i.e., the restrictions of ω to the leaves of $\mathcal{F}$ are identically zero. Therefore, we name an element of $\mathcal{F}(d)$ a (holomorphic) foliation of degree d.

The points in $sing(\mathcal{F}(\omega)) = \{P = 0, Q = 0\}$ are called the singularities of $\mathcal{F}(\omega)$. A singularity $p \in \mathbb{C}^2$ of $\mathcal{F}(\omega)$ is called reduced if $(P_xQ_y - P_yQ_x)(p) \neq 0$. A reduced singularity p is called a center singularity or center for simplicity if there is a holomorphic coordinates system $(\tilde{x}, \tilde{y})$ around p with $\tilde{x}(p) = 0, \tilde{y}(p) = 0$ such that in this coordinates system $\omega \wedge d(\tilde{x}^2 + \tilde{y}^2) = 0$. One can call $f := \tilde{x}^2 + \tilde{y}^2$ a local first integral around p. The leaves of $\mathcal{F}$ around the center p are given by $\tilde{x}^2 + \tilde{y}^2 = c$. Therefore, the leaf associated to the constant c contains the one dimensional cycle $\{(\tilde{x}\sqrt{c}, \tilde{y}\sqrt{c}) \mid (\tilde{x}, \tilde{y}) \in \mathbb{R}^2, \tilde{x}^2 + \tilde{y}^2 = 1\}$ which is called the vanishing cycle. We consider the subset of $\mathcal{F}(d)$ containing $\mathcal{F}(\omega)$'s with at least one center and we denote its closure in $\overline{\mathcal{F}(d)}$ by $\mathcal{M}(d)$. It turns out that $\mathcal{M}(d)$ is an algebraic subset of $\mathcal{F}(d)$ (see for instance [Mo1]). Now the problem of identifying the irreducible components of $\mathcal{M}(d)$ arises. This problem is also known by the name "Center conditions" in the context of real polynomial differential equations. Let us introduce some of irreducible components of $\mathcal{M}(d)$.

For $n \in \mathbb{N} \cup \{0\}$, let $\mathcal{P}_n$ denote the set of polynomials of degree at most n in x and y variables. Let also $d_i \in \mathbb{N},\ i = 1, 2, \ldots, s$ with $\sum_{i=1}^{s} d_i = d - 1$ and $\mathcal{L}(d_1, \ldots, d_s)$ be the set of logarithmic foliations

$$\mathcal{F}(f_1 \cdots f_s \sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i}),\ f_i \in \mathcal{P}_{d_i},\ \lambda_i \in \mathbb{C}$$

For practical purposes, one assumes that $\deg f_i = d_i, \lambda_i \in \mathbb{C}^*,\ 1 \leq i \leq s$ and that f_i's intersect each other transversally, and one obtains an element in $\mathcal{F}(d)$. Such a foliation

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has the logarithmic first integral $f_1^{\lambda_1}\cdots f_s^{\lambda_s}$. Since $\mathcal{L}(d_1,\ldots,d_s)$ is parameterized by λ_i and f_i's it is irreducible.

Theorem 1. *([Mo2]) The set $\mathcal{L}(d_1,\ldots,d_s)$ is an irreducible component of $\mathcal{M}(d)$, where $d=\sum_{i=1}^{s} d_i - 1$.*

In the case $s=1$ we can assume that $\lambda_1=1$ and so $\mathcal{L}(d+1)$ is the space of foliations of the type $\mathcal{F}(df)$, where f is a polynomial of degree $d+1$. This case is proved by Ilyashenko in [Il].

In general the aim is to find $d_i\in\mathbb{N}\cup\{0\}, i=1,2,\ldots,k$ and parameterize an irreducible component $X=X(d_1,d_2,\ldots,d_k)$ of $\mathcal{M}(d)$ by $\mathcal{P}_{d_1}\times\mathcal{P}_{d_2}\times\cdots\times\mathcal{P}_{d_k}$. In the above example $k=2s$ and $d_{s+1}=\cdots d_{2s}=0$. Once we have done this, we can reformulate the fact that X is an irreducible component of $\mathcal{M}(d)$ in a meaningful way as follows:

Theorem 2. *There exists an open dense subset U of X with the following property: for all $\mathcal{F}\in U$ parameterized with $f_i\in\mathcal{P}_{d_i}$, $i=1,2,\ldots,k$ and a center $p\in\mathbb{C}^2$ of $\mathcal{F}$ let $\mathcal{F}_\epsilon$ be a holomorphic deformation of $\mathcal{F}$ in $\mathcal{F}(d)$ such that its unique singularity p_ϵ near p is still a center. Then there exist polynomials $f_{i\epsilon}\in\mathcal{P}_{d_i}$ such that $\mathcal{F}_\epsilon$ is parameterized by $f_{i\epsilon}$'s. Here $f_{i\epsilon}$'s are holomorphic in ϵ and $f_{i0}=f_i$.*

The above theorem also says that the persistence of one center implies the persistence of all other type of singularities.

1 Usual method

To prove theorems like Theorem 2 usually one has to take U the complement of $X\cap sing(\mathcal{M}(d))$ in X. But this is not an explicite description of U. In practice one defines U by conditions like: f_i, $i=1,2,\ldots,k$ is of degree d_i, f_i's have no common factors, $\{f_i=0\}$'s intersect each other transversally and so on. To prove Theorem 2, after finding such an open set U, it is enough to prove that for at least one $\mathcal{F}\in U$

$$T_{\mathcal{F}}X=T_{\mathcal{F}}\mathcal{M}(d) \tag{1}$$

where $T_{\mathcal{F}}$ means the tangent bundle at $\mathcal{F}$. Note that for a foliation $\mathcal{F}\in X$ the equality (1) does not imply that $\mathcal{F}\in U$. There may be an irreducible component of $\mathcal{M}(d)$ of dimension lower than the dimension of X such that it passes through $\mathcal{F}$ and its tangent space at $\mathcal{F}$ is a subset of $T_{\mathcal{F}}X$. For this reason after proving (1) for $\mathcal{F}$ with some generic conditions on f_i's, we may not be sure that U defined by such generic conditions on f_i's is $X-(X\cap sing(\mathcal{M}(d)))$. However, in the bellow U can mean $X-(X\cap sing(\mathcal{M}(d)))$ or some open dense subset of X.

An element $\mathcal{F}$ of the irreducible component X may have more than one center. The deformation of $\mathcal{F}$ within X may destroy some centers but it preserves at least one center. Therefore, we have the notion of stable and unstable center for elements of X. A stable center of $\mathcal{F}$ is a center which persists after any deformation of $\mathcal{F}$ within X. An unstable center is a center which is not stable. It is natural to ask

P 1. *Are all the centers of a foliation $\mathcal{F}\in U$ stable?*

The answer is positive for $X=\mathcal{L}(d_1,d_2,\ldots,d_s)$ in Theorem 1. Every element $\mathcal{F}\in U$ has $d^2-\sum_{i<j}d_id_j$ stable center. Here U means just an open dense subset of X.

The inclusion $\subset$ in the equality (1) is trivial. To prove the other side $\supset$, we fix a stable center singularity p of $\mathcal{F}$ and make a deformation $\mathcal{F}_\epsilon(\omega+\epsilon\omega_1+\cdots)$ of $\mathcal{F}=\mathcal{F}(\omega)$. Here ω_1 represents an element $[\omega_1]$ of $T_{\mathcal{F}}\mathcal{M}(d)$. Let f be a local first integral in a neighborhood U' of p, s a holomorphic function in U' such that $\omega = s.df$, δ a vanishing cycle in a leaf of $\mathcal{F}$ in U' and $\Sigma \simeq (\mathbb{C},0)$ a transverse section to $\mathcal{F}$ in a point $p\in\delta$. We assume that the transverse section Σ is parameterized by $t = f\mid_\Sigma$. The holonomy of $\mathcal{F}$ along δ is identity. Let $h_\epsilon(t)$ be the holonomy of $\mathcal{F}_\epsilon$ along the path δ. It is a holomorphic function in ϵ and t and by hypothesis $h_0(t)=t$. We write the Taylor expansion of $h_\epsilon(t)$ in ϵ

$$h_\epsilon(t)-t = M_1(t)\epsilon + M_2(t)\epsilon^2+\cdots+M_i(t)\epsilon^i+\cdots,\ i!.M_i(t)=\frac{\partial^i h_\epsilon}{\partial\epsilon^i}\mid_{\epsilon=0}$$

The function M_i is called the i-th Melnikov function of the deformation $\mathcal{F}_\epsilon$ along the path δ. It is well-known that the first Melnikov function is given by

$$M_1(t) = -\int_{\delta_t}\frac{\omega_1}{s}$$

where δ_t is the lifting up of δ in the leaf through $t\in\Sigma$, and the multiplicity of M_1 at $t=0$ is the number of limit cycles (more precisely the number of fixed points of the holonomy h_ϵ) which appears around δ after the deformation (see for instance [Mo1]). This fact shows the importance of these functions in the local study of Hilbert 16-th problem.

Now, if in the deformation $\mathcal{F}_\epsilon$ the deformed singularity p_ϵ near p is center then $h_\epsilon = id$ and in particular

(2) $$\int_{\delta_t}\frac{\omega_1}{s}=0,\ \forall t\in\Sigma$$

Let $T^*_{\mathcal{F}}X$ be the set of $[\omega_1]\in T_{\mathcal{F}}\mathcal{F}(d)$ with the above property. It is easy to check that the above definition does not depends on the choice of f (see [Mo1]). We have seen that $T_{\mathcal{F}}\mathcal{M}(d)\subset T^*_{\mathcal{F}}X$. The following question arises:

P 2. *Is $T_{\mathcal{F}}\mathcal{M}(d) = T^*_{\mathcal{F}}X$?*

If the answer is positive then it means that form the vanishing of integrals (2) one must be able to prove that $\omega_1\in T_{\mathcal{F}}X$. Otherwise, calculating more Melnikov functions to get more and more information on ω_1 is necessary. The proof of Theorem 1 with $s=1$ shows that the answer of P2 is positive in this case. However, the answer of P2 for $X=\mathcal{L}(d_1,d_2,\ldots,d_s)$ is not known.

2 Some singularities of $\mathcal{M}(d)$

The method explained in the previous section has two difficulties: First, identifying $U := X\cap sing(\mathcal{M}(d))$ and second to know the dynamics and topology of the original foliation $\mathcal{F}$. A way to avoid these difficulties is to look for foliations $\mathcal{F}(df)$, where f is a degree $d+1$ polynomial in $\mathbb{C}^2$. We already know that such foliations lie in the irreducible component $\mathcal{L}(d+1)$. But if we take f a non-generic polynomial then $\mathcal{F}(df)$ may lie in other irreducible components of $\mathcal{M}(d)$ and even worse, $\mathcal{F}(df)$ may not be a smooth point of such irreducible components.

P 3. *Do all irreducible components of $\mathcal{M}(d)$ intersect $\mathcal{L}(d+1)$?*

If the answer of the above question is positive then the classification of irreducible components of $\mathcal{M}(d)$ leads to the classification of polynomials of degree $d+1$ in $\mathbb{C}^2$ according to their Picard-Lefschetz theory and Brieskorn modules. If not, we may be interested to find an irreducible component X which does not intersect $\mathcal{L}(d+1)$. In any case, the method which we are going to explain bellow is useful for those X which intersect $\mathcal{L}(d+1)$.

The foliation $\mathcal{F} = \mathcal{F}(df)$ has a first integral f and so it has no dynamics. The function f induces a (C^∞) locally trivial fibration on $\mathbb{C} - C$, where C is a finite subset of $\mathbb{C}$. The points of C are called critical values of f and the associated fibers are called the critical fibers. We have Picard-Lefschetz theory of f and the action of monodromy

$$\pi_1(\mathbb{C} - C, b) \times H_1(f^{-1}(b), \mathbb{Q}) \to H_1(f^{-1}(b), \mathbb{Q})$$

where $b \in \mathbb{C} - C$ is a regular fiber. Let $\delta' \in H_1(f^{-1}(b), \mathbb{Q})$ be the monodromy of δ (the vanishing cycle around a center singularity of $\mathcal{F}(df)$) along an arbitrary path in $\mathbb{C} - C$ with the end point b. From analytic continuation of the integral (2) one concludes that $\int_{\pi_1(\mathbb{C}-C).\delta} \omega = 0$.

P 4. *Determine the subset* $\pi_1(\mathbb{C} - C).\delta \subset H_1(f^{-1}(b), \mathbb{Q})$.

In the case of a generic polynomial f, Ilyashenko has proved that in P4 the equality happens. To prove Theorem 1, I have used a polynomial f which is a product of $d+1$ lines in general position and I have proved that $\pi_1(\mathbb{C} - C).\delta$ together with the cycles at infinity generate $H_1(f^{-1}(b), \mathbb{Q})$. Cycles at infinity are cycles around the points of compactification of $f^{-1}(b)$.

Parallel to the above topological theory theory, we have another algebraic theory associated to each polynomial. The Brieskorn module $H = \frac{\Omega^1}{d\Omega^0 + \Omega^0 df}$, where $\Omega^i, i = 0, 1, 2$ is the set of polynomial differential i-forms in $\mathbb{C}^2$, is a $\mathbb{C}[t]$-module in a natural way and we have the action of Gauss-Manin connection

$$\nabla : H_C \to H_C$$

where H_C is the localization of H over the multiplicative subgroup of $\mathbb{C}[t]$ generated by $t - c,\ c \in C$ (see [Mo2]).

P 5. *Find the torsions of H and classify the kernel of the maps $\nabla^i = \nabla \circ \nabla \circ \cdots \circ \nabla$ i-times.*

When f is the product of lines in general position then H has not torsions and the classification of the kernel of ∇^i is done in [Mo2] using a theorem of Cerveau-Mattei.

Solutions to the both problems P4 and P5 are closely related to the position of $\mathcal{F}(df)$ in $\mathcal{M}(d)$. Using solutions to P4 and P5 one calculates the Melnikov functions M_i's by means of integrals of 1-forms (the data of the deformation) over vanishing cycles and one calculates the tangent cone $TC_{\mathcal{F}}\mathcal{M}(d)$ of $\mathcal{F} = \mathcal{F}(df)$ in $\mathcal{M}(d)$ and compare it with the tangent cone of suspicious irreducible components of $\mathcal{M}(d)$. For instance, to prove Theorem 1, we have taken f the product of $d+1$ lines in general position and we have proved that

(3) $$\cup_{\sum_{i=1}^s d_i = d-1} TC_{\mathcal{F}}\mathcal{L}(d_1, d_2, \ldots, d_s) = TC_{\mathcal{F}}\mathcal{M}(d)$$

All the varieties $\mathcal{L}(d_1, \ldots, d_s)$, $\sum_{i=1}^s d_i = d - 1$ pass through $\mathcal{F} = \mathcal{F}(df)$.

P 6. Are $\mathcal{L}(d_1,\dots,d_s)$'s all irreducible components of $\mathcal{M}(d)$ through $\mathcal{F}(df)$?

Note that the equality (3) does not give an answer to this problem. There may be an irreducible component of $\mathcal{M}(d)$ through $\mathcal{F}(df)$ and different form $\mathcal{L}(d_1,d_2,\dots,d_s)$'s such that its tangent cone at $\mathcal{F}(df)$ is a subset of (3). In this case the definition of other notions of tangent cone based on higher order 1-forms in the deformation of $\mathcal{F}(df)$ seems to be necessary.

The first case in which one may be interested to use the method of this section can be:

P 7. Let $l_i = 0$, $i = 0,1,\dots,d$ be lines in the real plane and m_i, $i = 0,1,\dots,d$ be integer numbers. Put $f = l_0^{m_0}\cdots l_d^{m_d}$. Find all irreducible components of $\mathcal{M}(d)$ through $\mathcal{F}(df)$.

In this problem the line l_i has multiplicity m_i and it would be interesting to see how the classification of irreducible components through $\mathcal{F}(df)$ depends on the different arrangements of the lines l_i in the real plane and the associated multiplicities. In particular, we may allow several lines to pass through a point or to be parallel. When there are lines with negative multiplicities then we have a third kind of singularities $\{l_i = 0\}\cap\{l_j = 0\}$ called dicritical singularities, where l_i (resp. l_j) has positive (resp. negative) multiplicity. They are indeterminacy points of f and are characterized by this property that there are infinitely many leaves of the foliation passing through the singularity. Also in this case there are saddle critical points of f which are not due to the intersection points of the lines with positive (resp. negative) multiplicity. The reader may analyze the situation by the example $f = \frac{l_0 l_1}{l_2 l_3}$.

3 Looking for irreducible components of $\mathcal{M}(d)$

To apply the methods of previous sections one must find some irreducible subsets of $\mathcal{M}(d)$ and then one conjectures that they must be irreducible components of $\mathcal{M}(d)$. The objective of this section is to do this.

Classification of codimension one foliations on complex manifolds of higher dimension is a subject related to center conditions. We state the problem in the case of $\mathbb{C}^n$, $n > 2$ which is compatible with this text. However, the literature on this subject is mainly for projective spaces of dimension greater than two (see [CL]).

The set of polynomial 1-forms $\omega = \sum_{i=1}^n P_i(x)dx_i, \deg P_i \leq d$ is a vector space of finite dimension and we denote by $\overline{\mathcal{F}(n,d)}$ its projectivization. Its subset $\mathcal{F}(n,d)$ containing all ω's with $P_i's$ relatively prime and $\deg(\omega) := \max\{\deg P_i, i = 1,2,\dots,n\} = d$ is Zariski open in $\overline{\mathcal{F}(n,d)}$. An element $[\omega] \in \overline{\mathcal{F}(n,d)}$ induces a holomorphic foliation $\mathcal{F} = \mathcal{F}(\omega)$ in $\mathbb{C}^n$ if and only if ω satisfies the integrability condition

$$\omega \wedge d\omega = 0 \tag{4}$$

This is an algebraic equation on the coefficients of ω. Therefore, the elements of $\mathcal{F}(n,d)$ which induce a holomorphic foliation in $\mathbb{C}^n$ form an algebraic subset, namely $\mathcal{M}(n,d)$, of $\mathcal{F}(n,d)$. Now we have the problem of identifying the irreducible components of $\mathcal{M}(n,d)$. We define $\mathcal{F}(2,d) := \mathcal{F}(d)$ and $\mathcal{M}(2,d) := \mathcal{M}(d)$.

Let us be given a polynomial map $F : \mathbb{C}^2 \to \mathbb{C}^n$, $n \geq 2$ and a codimension one foliation $\mathcal{F} = \mathcal{F}(\omega)$ in $\mathbb{C}^n$. In the case $n > 2$, let us suppose that F is regular in a point $p \in \mathbb{C}^2$. This implies that F around p is a smooth embedding. We assume that $F(\mathbb{C}^2, p)$ has a tangency with the leaf of $\mathcal{F}$ through $F(p)$. In the case $n = 2$, we assume that F is singular at p. In both cases, after choosing a generic F and $\mathcal{F}$, the pullback of $\mathcal{F}$ by F has a center singularity at p.

P 8. *Fix an irreducible component X of $\mathcal{F}(n,d)$. Is*

$$\{F^*\mathcal{F}, \mathcal{F} \in X, \ \deg f_i \le d_i, \ i = 1, 2, \ldots, n\}$$

where $F = (f_1, f_2, \ldots, f_n)$, an irreducible component of $\mathcal{M}(d'')$ for some $d'' \in \mathbb{N}$?

For instance in Theorem 1, the elements of $\mathcal{L}(d_1, d_2, \ldots, d_s)$ are pull backs of holomorphic foliations $\mathcal{F}(x_1 x_2 \cdots x_s \sum_{i=1}^{s} \lambda_i \frac{dx_i}{x_i})$, $\lambda_i \in \mathbb{C}^*$ in $\mathbb{C}^s$ by the polynomial maps $F = (f_1, f_2, \ldots, f_s)$, $\deg f_i \le d_i$.

Another way to find irreducible subsets of $\mathcal{M}(d)$ is by looking for foliations of lower degree. Take a polynomial of degree d in $\mathbb{C}^2$ with the generic conditions considered by Ilyashenko, i.e. f has non degenerated singularities with distinct images. Now $\mathcal{F}(df)$ has degree $d-1$ which is less than the degree of a generic foliation in $\mathcal{F}(d)$.

P 9. Classify all irreducible components of $\mathcal{M}(d)$ through $\mathcal{F}(df)$.

All $\mathcal{L}(d_1, \ldots, d_s)$'s pass through $\mathcal{F}(df)$. There are other candidates as follows:

1. $A_i = \{\mathcal{F}(\frac{dp}{p} + d(\frac{q}{p^i})) \mid deg(p) = 1, deg(q) = d\}$ $i = 0, 1, 2, \ldots, d$;

2. $B_1 = \{\mathcal{F}(\frac{dq}{q} + d(p)) \mid deg(p) = 1, deg(q) = d\}$;

An element of A_i (resp. B_1) has a first integral of the type pe^{q/p^i} (resp. qe^p). These candidates are supported by Dulac's classification (see [Du] and [CL] p.601) in the case $d = 2$.

We can look at our problem in a more general context. Let M be a projective complex manifold of dimension two. We consider the space $\mathcal{F}(L)$ of holomorphic foliations in M with the normal line bundle L (see for instance [Mo1]). Let also $\mathcal{M}(L)$ be its subset containing holomorphic foliation with at least one center singularity. Again $\mathcal{M}(L)$ is an algebraic subset of $\mathcal{F}(L)$ and one can ask for the classification of irreducible components of $\mathcal{M}(L)$. For $M = \mathbb{C}P(2)$ some irreducible components of $\mathcal{M}(L)$ are identified in [Mo1].

P 10. Prove a theorem similar to Theorem 1 for an arbitrary projective manifold of dimension two.

In this generality one must be careful about trivial centers which we explain now. Let $\mathcal{F}$ be a holomorphic foliation in $\mathbb{C}^2$ and 0 a regular point of $\mathcal{F}$. We make a blow up (see [CaSa]) at 0 and we obtain a divisor $\mathbb{C}P(1)$ which contains exactly one singularity of the blow up foliation and this singularity is a center.

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