**Title**
Center conditions for polynomial differential equations: discussion of some problems (Complex Dynamics)

**Author(s)**
Movasati, Hossein

**Citation**
数理解析研究所講究録 数理解析研究所講究録

**Issue Date**
2005-08

**URL**
http://hdl.handle.net/2433/47656

**Type**
Departmental Bulletin Paper

**Textversion**
publisher

Kyoto University
Center conditions for polynomial differential equations: discussion of some problems

HOSSEIN MOVASATI
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5, 37073 Göttingen
Email: movasati@uni-math.gwdg.de

Abstract
Classifications of irreducible components of the set of polynomial differential equations with a fixed degree and with at least one center singularity lead to some other new problems on Picard-Lefschetz theory and Brieskorn modules of polynomials. In this article we explain these problems and their connections to such classifications.

0 Introduction

The set of polynomial 1-forms $\omega = P(x,y)dy - Q(x,y)dx$, $\deg P, \deg Q \leq d$, $d \geq 2$ is a vector space of finite dimension and we denote by $\mathcal{F}(d)$ its projectivization. Its subset $\mathcal{F}(d)$ containing all $\omega$'s with $P$ and $Q$ relatively prime and $\deg(\omega) := \max\{\deg P, \deg Q\} = d$ is Zariski open in $\overline{\mathcal{F}(d)}$. We denote the elements of $\mathcal{F}(d)$ by $\mathcal{F}(\omega)$ or $\mathcal{F}$ if there is no confusion about the underlying 1-form $\omega$ in the text. Any $\mathcal{F}(\omega)$ induces a holomorphic foliation $\mathcal{F}$ in $\mathbb{C}^2$ i.e., the restrictions of $\omega$ to the leaves of $\mathcal{F}$ are identically zero. Therefore, we name an element of $\mathcal{F}(d)$ a (holomorphic) foliation of degree $d$.

The points in $\text{sing}(\mathcal{F}(\omega)) = \{P = 0, Q = 0\}$ are called the singularities of $\mathcal{F}(\omega)$. A singularity $p \in \mathbb{C}^2$ of $\mathcal{F}(\omega)$ is called reduced if $(P_x Q_y - P_y Q_x)(p) \neq 0$. A reduced singularity $p$ is called a center singularity or center for simplicity if there is a holomorphic coordinates system $(\tilde{x}, \tilde{y})$ around $p$ with $\tilde{x}(p) = 0, \tilde{y}(p) = 0$ such that in this coordinates system $\omega \wedge d(\tilde{x}^2 + \tilde{y}^2) = 0$. One can call $f := \tilde{x}^2 + \tilde{y}^2$ a local first integral around $p$. The leaves of $\mathcal{F}$ around the center $p$ are given by $\tilde{x}^2 + \tilde{y}^2 = c$. Therefore, the leaf associated to the constant $c$ contains the one dimensional cycle $\{(\tilde{x} \sqrt{c}, \tilde{y} \sqrt{c}) \in \mathbb{R}^2, \tilde{x}^2 + \tilde{y}^2 = 1\}$ which is called the vanishing cycle. We consider the subset of $\mathcal{F}(d)$ containing $\mathcal{F}(\omega)$'s with at least one center and we denote its closure in $\overline{\mathcal{F}(d)}$ by $\mathcal{M}(d)$. It turns out that $\mathcal{M}(d)$ is an algebraic subset of $\mathcal{F}(d)$ (see for instance [Mo1]). Now the problem of identifying the irreducible components of $\mathcal{M}(d)$ arises. This problem is also known by the name "Center conditions" in the context of real polynomial differential equations. Let us introduce some of irreducible components of $\mathcal{M}(d)$.

For $n \in \mathbb{N} \cup \{0\}$, let $\mathcal{P}_n$ denote the set of polynomials of degree at most $n$ in $x$ and $y$ variables. Let also $d_i \in \mathbb{N}$, $i = 1, 2, \ldots, s$ with $\sum_{i=1}^s d_i = d - 1$ and $\mathcal{L}(d_1, \ldots, d_s)$ be the set of logarithmic foliations

$$\mathcal{F}(f_1 \cdots f_s \sum_{i=1}^s \lambda_i \frac{df_i}{f_i}), \ f_i \in \mathcal{P}_{d_i}, \ \lambda_i \in \mathbb{C}$$

For practical purposes, one assumes that $\deg f_i = d_i, \lambda_i \in \mathbb{C}^*, \ 1 \leq i \leq s$ and that $f_i$'s intersect each other transversally, and one obtains an element in $\mathcal{F}(d)$. Such a foliation

1Keywords: Holomorphic foliations, holonomy.
has the logarithmic first integral $f_1^{\lambda_1} \cdots f_s^{\lambda_s}$. Since $\mathcal{L}(d_1, \ldots, d_s)$ is parameterized by $\lambda_i$ and $f_i$’s it is irreducible.

**Theorem 1.** ([Mo2]) The set $\mathcal{L}(d_1, \ldots, d_s)$ is an irreducible component of $\mathcal{M}(d)$, where $d = \sum_{i=1}^s d_i - 1$.

In the case $s = 1$ we can assume that $\lambda_1 = 1$ and so $\mathcal{L}(d+1)$ is the space of foliations of the type $\mathcal{F}(df)$, where $f$ is a polynomial of degree $d+1$. This case is proved by Ilyashenko in [II].

In general the aim is to find $d_i \in \mathbb{N} \cup \{0\}, i = 1, 2, \ldots, k$ and parameterize an irreducible component $X = X(d_1, d_2, \ldots, d_k)$ of $\mathcal{M}(d)$ by $\mathcal{P}_{d_1} \times \mathcal{P}_{d_2} \times \cdots \times \mathcal{P}_{d_k}$. In the above example $k = 2s$ and $d_{s+1} = \cdots d_{2s} = 0$. Once we have done this, we can reformulate the fact that $X$ is an irreducible component of $\mathcal{M}(d)$ in a meaningful way as follows:

**Theorem 2.** There exists an open dense subset $U$ of $X$ with the following property: for all $\mathcal{F} \in U$ parameterized with $f_i \in \mathcal{P}_{d_i}$, $i = 1, 2, \ldots, k$ and a center $p \in \mathbb{C}^2$ of $\mathcal{F}$ let $\mathcal{F}_\epsilon$ be a holomorphic deformation of $\mathcal{F}$ in $\mathcal{F}(d)$ such that its unique singularity $p_\epsilon$ near $p$ is still a center. Then there exist polynomials $f_{i\epsilon} \in \mathcal{P}_{d_i}$ such that $\mathcal{F}_\epsilon$ is parameterized by $f_{i\epsilon}$’s. Here $f_{i\epsilon}$’s are holomorphic in $\epsilon$ and $f_{i0} = f_i$.

The above theorem also says that the persistence of one center implies the persistence of all other type of singularities.

## 1 Usual method

To prove theorems like Theorem 2 usually one has to take $U$ the complement of $X \cap \text{sing}(\mathcal{M}(d))$ in $X$. But this is not an explicite description of $U$. In practice one defines $U$ by conditions like: $f_i$, $i = 1, 2, \ldots, k$ is of degree $d_i$, $f_i$’s have no common factors, $\{f_i \neq 0\}$’s intersect each other transversally and so on. To prove Theorem 2, after finding such an open set $U$, it is enough to prove that for at least one $\mathcal{F} \in U$

\begin{equation}
T_\mathcal{F}X = T_\mathcal{F}\mathcal{M}(d)
\end{equation}

where $T_\mathcal{F}$ means the tangent bundle at $\mathcal{F}$. Note that for a foliation $\mathcal{F} \in X$ the equality (1) does not imply that $\mathcal{F} \in U$. There may be an irreducible component of $\mathcal{M}(d)$ of dimension lower than the dimension of $X$ such that it passes through $\mathcal{F}$ and its tangent space at $\mathcal{F}$ is a subset of $T_\mathcal{F}X$. For this reason after proving (1) for $\mathcal{F}$ with some generic conditions on $f_i$’s, we may not be sure that $U$ defined by such generic conditions on $f_i$’s is $X - (X \cap \text{sing}(\mathcal{M}(d)))$. However, in the bellow $U$ can mean $X - (X \cap \text{sing}(\mathcal{M}(d)))$ or some open dense subset of $X$.

An element $\mathcal{F}$ of the irreducible component $X$ may have more than one center. The deformation of $\mathcal{F}$ within $X$ may destroy some centers but it preserves at least one center. Therefore, we have the notion of stable and unstable center for elements of $X$. A stable center of $\mathcal{F}$ is a center which persists after any deformation of $\mathcal{F}$ within $X$. An unstable center is a center which is not stable. It is natural to ask

**P 1.** Are all the centers of a foliation $\mathcal{F} \in U$ stable?

The answer is positive for $X = \mathcal{L}(d_1, d_2, \ldots, d_s)$ in Theorem 1. Every element $\mathcal{F} \in U$ has $d^2 - \sum_{i<j} d_id_j$ stable center. Here $U$ means just an open dense subset of $X$. 

The inclusion $\subset$ in the equality (1) is trivial. To prove the other side $\supset$, we fix a stable center singularity $p$ of $\mathcal{F}$ and make a deformation $\mathcal{F}_\varepsilon(\omega + \omega_1 + \cdots)$ of $\mathcal{F} = \mathcal{F}(\omega)$. Here $\omega_1$ represents an element $[\omega_1]$ of $T_X \mathcal{M}(d)$. Let $f$ be a local first integral in a neighborhood $U'$ of $p$, $s$ a holomorphic function in $U'$ such that $\omega = sdf$, $\delta$ a vanishing cycle in a leaf of $\mathcal{F}$ in $U'$ and $\Sigma \simeq (\mathbb{C}, 0)$ a transverse section to $\mathcal{F}$ in a point $p \in \delta$. We assume that the transverse section $\Sigma$ is parameterized by $t = f |_\Sigma$. The holonomy of $\mathcal{F}$ along $\delta$ is identity. Let $h_\varepsilon(t)$ be the holonomy of $\mathcal{F}_\varepsilon$ along the path $\delta$. It is a holomorphic function in $\varepsilon$ and $t$ and by hypothesis $h_0(t) = t$. We write the Taylor expansion of $h_\varepsilon(t)$ in $\varepsilon$

$$h_\varepsilon(t) = t = M_1(t)\varepsilon + M_2(t)\varepsilon^2 + \cdots + M_i(t)\varepsilon^i + \cdots, \text{ if } M_i(t) = \frac{\partial^i h_\varepsilon}{\partial \varepsilon^i} |_{\varepsilon=0}$$

The function $M_i$ is called the $i$-th Melnikov function of the deformation $\mathcal{F}_\varepsilon$ along the path $\delta$. It is well-known that the first Melnikov function is given by

$$M_1(t) = -\int_{\delta_1} \frac{\omega_1}{s}$$

where $\delta_1$ is the lifting up of $\delta$ in the leaf through $t \in \Sigma$, and the multiplicity of $M_1$ at $t = 0$ is the number of limit cycles (more precisely the number of fixed points of the holonomy $h_\varepsilon$) which appears around $\delta$ after the deformation (see for instance [Mo1]). This fact shows the importance of these functions in the local study of Hilbert 16-th problem.

Now, if in the deformation $\mathcal{F}_\varepsilon$ the deformed singularity $p_\varepsilon$ near $p$ is center then $h_\varepsilon = \text{id}$ and in particular

$$\int_{\delta_t} \frac{\omega_1}{s} = 0, \forall t \in \Sigma$$

Let $T^*_X X$ be the set of $[\omega_1] \in T^*_X \mathcal{F}(d)$ with the above property. It is easy to check that the above definition does not depend on the choice of $f$ (see [Mo1]). We have seen that $T_X \mathcal{M}(d) \subset T^*_X X$. The following question arises:

**P 2.** Is $T_X \mathcal{M}(d) = T^*_X X$?

If the answer is positive then it means that form the vanishing of integrals (2) one must be able to prove that $\omega_1 \in T^*_X X$. Otherwise, calculating more Melnikov functions to get more and more information on $\omega_1$ is necessary. The proof of Theorem 1 with $s = 1$ shows that the answer of P2 is positive in this case. However, the answer of P2 for $X = \mathcal{L}(d_1, d_2, \ldots, d_s)$ is not known.

2 Some singularities of $\mathcal{M}(d)$

The method explained in the previous section has two difficulties: First, identifying $U := X \cap \text{sing} \mathcal{M}(d)$ and second to know the dynamics and topology of the original foliation $\mathcal{F}$. A way to avoid these difficulties is to look for foliations $\mathcal{F}(df)$, where $f$ is a degree $d + 1$ polynomial in $\mathbb{C}^2$. We already know that such foliations lie in the irreducible component $\mathcal{L}(d+1)$. But if we take $f$ a non-generic polynomial then $\mathcal{F}(df)$ may lie in other irreducible components of $\mathcal{M}(d)$ and even worse, $\mathcal{F}(df)$ may not be a smooth point of such irreducible components.

**P 3.** Do all irreducible components of $\mathcal{M}(d)$ intersect $\mathcal{L}(d+1)$?
If the answer of the above question is positive then the classification of irreducible components of $\mathcal{M}(d)$ leads to the classification of polynomials of degree $d + 1$ in $\mathbb{C}^2$ according to their Picard-Lefschetz theory and Brieskorn modules. If not, we may be interested to find an irreducible component $X$ which does not intersect $\mathcal{L}(d + 1)$. In any case, the method which we are going to explain below is useful for those $X$ which intersect $\mathcal{L}(d + 1)$.

The foliation $\mathcal{F} = \mathcal{F}(df)$ has a first integral $f$ and so it has no dynamics. The function $f$ induces a $(C^\infty)$ locally trivial fibration on $\mathbb{C} - C$, where $C$ is a finite subset of $\mathbb{C}$. The points of $C$ are called critical values of $f$ and the associated fibers are called the critical fibers. We have Picard-Lefschetz theory of $f$ and the action of monodromy

$$\pi_1(\mathbb{C} - C, b) \times H_1(f^{-1}(b), \mathbb{Q}) \to H_1(f^{-1}(b), \mathbb{Q})$$

where $b \in \mathbb{C} - C$ is a regular fiber. Let $\delta' \in H_1(f^{-1}(b), \mathbb{Q})$ be the monodromy of $\delta$ (the vanishing cycle around a center singularity of $\mathcal{F}(df)$) along an arbitrary path in $\mathbb{C} - C$ with the end point $b$. From analytic continuation of the integral (2) one concludes that $\int_{\pi_1(\mathbb{C} - C), \delta} \omega = 0$.

**P 4. Determine the subset** $\pi_1(\mathbb{C} - C), \delta \subset H_1(f^{-1}(b), \mathbb{Q})$.

In the case of a generic polynomial $f$, Ilyashenko has proved that in P4 the equality happens. To prove Theorem 1, I have used a polynomial $f$ which is a product of $d + 1$ lines in general position and I have proved that $\pi_1(\mathbb{C} - C), \delta$ together with the cycles at infinity generate $H_1(f^{-1}(b), \mathbb{Q})$. Cycles at infinity are cycles around the points of compactification of $f^{-1}(b)$.

Parallel to the above topological theory, we have another algebraic theory associated to each polynomial. The Brieskorn module $H = \partial_{\mathbb{C}[t]} \nabla^{i} df$, where $\partial_{t}^{i}, i = 0, 1, 2$ is the set of polynomial differential $i$-forms in $\mathbb{C}^2$, is a $\mathbb{C}[t]$-module in a natural way and we have the action of Gauss-Manin connection

$$\nabla : HC \to HC$$

where $HC$ is the localization of $H$ over the multiplicative subgroup of $\mathbb{C}[t]$ generated by $t - c$, $c \in C$ (see [Mo2]).

**P 5. Find the torsions of $H$ and classify the kernel of the maps** $\nabla^i = \nabla \circ \nabla \circ \cdots \circ \nabla$

$i$-times.

When $f$ is the product of lines in general position then $H$ has not torsions and the classification of the kernel of $\nabla^i$ is done in [Mo2] using a theorem of Cerveau-Matei.

Solutions to the both problems P4 and P5 are closely related to the position of $\mathcal{F}(df)$ in $\mathcal{M}(d)$. Using solutions to P4 and P5 one calculates the Melnikov functions $M_{i}$'s by means of integrals of 1-forms (the data of the deformation) over vanishing cycles and one calculates the tangent cone $TC_{\mathcal{F}} \mathcal{M}(d)$ of $\mathcal{F} = \mathcal{F}(df)$ in $\mathcal{M}(d)$ and compare it with the tangent cone of suspicious irreducible components of $\mathcal{M}(d)$. For instance, to prove Theorem 1, we have taken $f$ the product of $d + 1$ lines in general position and we have proved that

$$\cup_{i=1}^{4} d_i = d - 1 TC_{\mathcal{F}} \mathcal{L}(d_1, d_2, \ldots, d_s) = TC_{\mathcal{F}} \mathcal{M}(d)$$

All the varieties $\mathcal{L}(d_1, \ldots, d_s), \sum_{i=1}^{s} d_i = d - 1$ pass through $\mathcal{F} = \mathcal{F}(df)$.
P 6. Are \( \mathcal{L}(d_1, \ldots, d_s) \)'s all irreducible components of \( \mathcal{M}(d) \) through \( \mathcal{F}(df) \)?

Note that the equality (3) does not give an answer to this problem. There may be an irreducible component of \( \mathcal{M}(d) \) through \( \mathcal{F}(df) \) and different form \( \mathcal{L}(d_1, d_2, \ldots, d_s) \)'s such that its tangent cone at \( \mathcal{F}(df) \) is a subset of (3). In this case the definition of other notions of tangent cone based on higher order 1-forms in the deformation of \( \mathcal{F}(df) \) seems to be necessary.

The first case in which one may be interested to use the method of this section can be:

P 7. Let \( l_i = 0, \ i = 0, 1, \ldots, d \) be lines in the real plane and \( m_i, \ i = 0, 1, \ldots, d \) be integer numbers. Put \( f = l_0^{m_0} \cdots l_d^{m_d} \). Find all irreducible components of \( \mathcal{M}(d) \) through \( \mathcal{F}(df) \).

In this problem the line \( l_i \) has multiplicity \( m_i \) and it would be interesting to see how the classification of irreducible components through \( \mathcal{F}(df) \) depends on the different arrangements of the lines \( l_i \) in the real plane and the associated multiplicities. In particular, we may allow several lines to pass through a point or to be parallel. When there are lines with negative multiplicities then we have a third kind of singularities \( \{l_i = 0\} \cap \{l_j = 0\} \) called dicritical singularities, where \( l_i \) (resp. \( l_j \)) has positive (resp. negative) multiplicity. They are infinitesimal points of \( f \) and are characterized by this property that there are infinitely many leaves of the foliation passing through the singularity. Also in this case there are saddle critical points of \( f \) which are not due to the intersection points of the lines with positive (resp. negative) multiplicity. The reader may analyze the situation by the example \( f = \frac{14k}{15} \).

3 Looking for irreducible components of \( \mathcal{M}(d) \)

To apply the methods of previous sections one must find some irreducible subsets of \( \mathcal{M}(d) \) and then one conjectures that they must be irreducible components of \( \mathcal{M}(d) \). The objective of this section is to do this.

Classification of codimension one foliations on complex manifolds of higher dimension is a subject related to center conditions. We state the problem in the case of \( \mathbb{C}^n, \ n > 2 \) which is compatible with this text. However, the literature on this subject is mainly for projective spaces of dimension greater than two (see [CL]).

The set of polynomial 1-forms \( \omega = \sum_{i=1}^{n} P_i(x)dx_i, \deg P_i \leq d \) is a vector space of finite dimension and we denote by \( \overline{\mathcal{F}(n, d)} \) its projectivization. Its subset \( \mathcal{F}(n, d) \) containing all \( \omega \)'s with \( P_i \)'s relatively prime and \( \deg(\omega) := \max\{\deg P_i, i = 1, 2, \ldots, n\} = d \) is Zariski open in \( \overline{\mathcal{F}(n, d)} \). An element \( [\omega] \in \overline{\mathcal{F}(n, d)} \) induces an holomorphic foliation \( \mathcal{F} = \mathcal{F}(\omega) \) in \( \mathbb{C}^n \) if and only if \( \omega \) satisfies the integrability condition

(4) \[ \omega \wedge d\omega = 0 \]

This is an algebraic equation on the coefficients of \( \omega \). Therefore, the elements of \( \mathcal{F}(n, d) \) which induce a holomorphic foliation in \( \mathbb{C}^n \) form an algebraic subset, namely \( \mathcal{M}(n, d) \), of \( \mathcal{F}(n, d) \). Now we have the problem of identifying the irreducible components of \( \mathcal{M}(n, d) \). We define \( \mathcal{F}(2, d) := \mathcal{F}(d) \) and \( \mathcal{M}(2, d) := \mathcal{M}(d) \).

Let us be given a polynomial map \( F : \mathbb{C}^2 \to \mathbb{C}^n, \ n \geq 2 \) and a codimension one foliation \( \mathcal{F} = \mathcal{F}(\omega) \) in \( \mathbb{C}^n \). In the case \( n > 2 \), let us suppose that \( F \) is regular in a point \( p \in \mathbb{C}^2 \). This implies that \( F \) around \( p \) is a smooth embedding. We assume that \( F(\mathbb{C}^2, p) \) has a tangency with the leaf of \( \mathcal{F} \) through \( F(p) \). In the case \( n = 2 \), we assume that \( F \) is singular
at $p$. In both cases, after choosing a generic $F$ and $\mathcal{F}$, the pullback of $\mathcal{F}$ by $F$ has a center singularity at $p$.

**P 8.** Fix an irreducible component $X$ of $\mathcal{F}(n, d)$. Is

$$\{ F^*\mathcal{F}, F \in X, \deg f_i \leq d_i, i = 1, 2, \ldots, n \}$$

where $F = (f_1, f_2, \ldots, f_n)$, an irreducible component of $\mathcal{M}(d'')$ for some $d'' \in \mathbb{N}$?

For instance in Theorem 1, the elements of $\mathcal{L}(d_1, d_2, \ldots, d_s)$ are pull backs of holomorphic foliations $\mathcal{F}(x_1x_2 \cdots x_s \sum_{i=1}^{s} \lambda_i \frac{dx}{x_i})$, $\lambda_i \in \mathbb{C}$ in $\mathbb{C}^s$ by the polynomial maps $F = (f_1, f_2, \ldots, f_s)$, $\deg f_i \leq d_i$.

Another way to find irreducible subsets of $\mathcal{M}(d)$ is by looking for foliations of lower degree. Take a polynomial of degree $d$ in $\mathbb{C}^2$ with the generic conditions considered by Ilyashenko, i.e. $f$ has non degenerated singularities with distinct images. Now $\mathcal{F}(df)$ has degree $d - 1$ which is less than the degree of a generic foliation in $\mathcal{F}(d)$.

**P 9.** Classify all irreducible components of $\mathcal{M}(d)$ through $\mathcal{F}(df)$.

All $\mathcal{L}(d_1, \ldots, d_s)$'s pass through $\mathcal{F}(df)$. There are other candidates as follows:

1. $A_i = \{ \mathcal{F}(\frac{df}{p} + d(\frac{q}{p})) | \deg(p) = 1, \deg(q) = d \} i = 0, 1, 2, \ldots, d$;

2. $B_1 = \{ \mathcal{F}(\frac{dq}{p} + d(p)) | \deg(p) = 1, \deg(q) = d \}$;

An element of $A_i$ (resp. $B_1$) has a first integral of the type $p \exp^{q/p}$ (resp. $q \exp^p$). These candidates are supported by Dulac's classification (see [Du] and [CL] p.601) in the case $d = 2$.

We can look at our problem in a more general context. Let $M$ be a projective complex manifold of dimension two. We consider the space $\mathcal{F}(L)$ of holomorphic foliations in $M$ with the normal line bundle $L$ (see for instance [Mo1]). Let also $\mathcal{M}(L)$ be its subset containing holomorphic foliation with at least one center singularity. Again $\mathcal{M}(L)$ is an algebraic subset of $\mathcal{F}(L)$ and one can ask for the classification of irreducible components of $\mathcal{M}(L)$. For $M = \mathbb{C}P(2)$ some irreducible components of $\mathcal{M}(L)$ are identified in [Mo1].

**P 10.** Prove a theorem similar to Theorem 1 for an arbitrary projective manifold of dimension two.

In this generality one must be careful about trivial centers which we explain now. Let $\mathcal{F}$ be a holomorphic foliation in $\mathbb{C}^2$ and 0 a regular point of $\mathcal{F}$. We make a blow up (see [CaSa]) at 0 and we obtain a divisor $\mathbb{C}P(1)$ which contains exactly one singularity of the blow up foliation and this singularity is a center.

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