数理解析研究所講究録 1447 巻 2005 年 187-197

Julia set describes quantum tunneling in the presence of chaos

Akira Shudo

Department of Physics, Tokyo Metropolitan University,
1-1 Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan
shudo@phys.metro-u.ac.jp

1. Introduction

Tunneling phenomenon is peculiar to quantum mechanics and no counterparts exist in classical mechanics. Features of tunneling are nevertheless strongly influenced by the underlying classical dynamics (see a recent review[1, 2, 3, 4]) A promising approach to see the connection of these two opposites is to carry out the complex semiclassical analysis, which allows us to describe the tunneling phenomena in terms of complex classical trajectories.

The aim is this short report is to present that several recent results on higher-dimensional complex dynamical systems are certainly helpful to our understanding of quantum tunneling in multi-dimensions, especially in the presence of chaos. Since detailed reports will be published elsewhere[5], we here describe several crucial points in our arguments.

To be precise, we introduce a two-dimensional area preserving map;

$$F : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} H'(p) - V'(q) \\ q + H'(p) - V'(q) \end{pmatrix}. \tag{1}$$

Taking $H(p) = p^2/2$ and $V(q) = K \sin q$ gives the Chirikov-Taylor standard map, and $H(p) = p^2/2$ and $V(q) = cq - q^3/3$ the area-preserving Hénon map. The canonical form of the latter is written as,

$$f \equiv f_a : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ y^2 + a - x \end{pmatrix}. \tag{2}$$

These are related via an affine change of coordinate $(p, q) = (y - x, y - 1)$ with $a = 1 - c$. As usual, we here regard that the area-preserving map is a model for Hamiltonian flow. As shown below, it is possible to construct quantum mechanics of the area preserving map, thereby quantum tunneling can also appear as well as usual quantum tunneling in Hamiltonian systems.

We first review some well-know facts on the transition caused the classical dynamics (1). Let us consider two points (p, q) and (p', q') in phase space. Trivially, a point (p', q') is reachable from (p, q) under the classical dynamics if and only if there exist a certain number of iteration N such that two points are connected via the mapping relation, *i.e.*, $(p', q') = F^N(p, q)$. On the other hand, if the map has the mixing property, and so ergodicity of the map with respect to a certain invariant measure, the notion of transition can be generalized and defined differently. That is, we can say that the transition between (p, q) and (p', q') is allowed via the mapping rule in the sense that for arbitrary neighborhoods of each point there exist an orbit that connects these two neighborhoods. In other words, in classical mechanics, an orbit in an ergodic invariant component can itinerate arbitrarily close to any other points in the same component.

However, if an area preserving map is neither hyperbolic not in completely integrable, quasi-periodic and chaotic components coexist in phase space in general and no ergodic measure exists such that its support covers the entire phase space. The coexistence of a variety of ergodic components, which are usually intermingled in a self-similar way, is a typical situation. Such a non-hyperbolic phase space is sometimes called *mixed phase space*. Note that the transition between different invariant components is not allowed in the sense mentioned above. That is, for a given point (p, q) contained in a certain KAM curve for instance, there exists a region $\mathcal{R}$ in phase space such that a certain neighborhood of (p, q) cannot reach the region $\mathcal{R}$ under the iteration of the map. In a converse way, for a point (p', q') contained in a certain chaotic component, there exists a region $\mathcal{R}'$ encircled by KAM curves such that a neighborhood of (p', q') cannot enter into $\mathcal{R}'$. We here call that these are *classically forbidden processes*.

2. Formulation of quantum dynamics in the map and its semiclassical approximation

A standard recipe to construct quantum mechanics in the map is given by expressing the unitary operator in the discretized Feynman path integral form. The one-step unitary operator is given as,

$$\hat{U} = \exp\Big\{-\frac{i}{\hbar}H_0(p)\Big\}\Big\{-\frac{i}{\hbar}V(q)\Big\}. \tag{3}$$

In the momentum (p-) representation, for example, the n-step quantum propagator is explicitly written as,

$$K(p_n; p_0) = < p_n|U^n|p_0 > = \int \cdots \int \prod_j dq_j \prod_j dp_j \exp\Big[\frac{i}{\hbar}S(\{q_j\}, \{p_j\})\Big], \tag{4}$$

where $S(\{q_j\}, \{p_j\})$ denotes the action functional along each path,

$$S(\{q_j\}, \{p_j\}) = \sum_{j=1}^{n}\Big[H_0(p_j) + V(q_j)) + q_j(p_j - p_{j-1})\Big]. \tag{5}$$

The square modulus $| < p_n|U^n|p_0 > |^2$ provides the transition probability from an initial state $|p_0 >$ to a final state $|p_n >$.

If we take the coherent representation, the propagator is expressed as $K(q_n, p_n; q_0, p_0) =< q_n, p_n|\hat{U}^n|q_0, p_0 >$, where $|q, p >= |a >$ denotes the coherent state with $a = (q + ip)/\sqrt{2}$. The position and momentum representations of the coherent state are

$$< q'|q, p > = (\pi\hbar)^{-1/4}\exp\Big[\frac{i}{2\hbar}(2q' - q)p - \frac{1}{2\hbar}(q' - q)^2\Big], \tag{6}$$

$$< p'|q, p > = (\pi\hbar)^{-1/4}\exp\Big[\frac{-i}{2\hbar}(2p' - p)q - \frac{1}{2\hbar}(p' - p)^2\Big]. \tag{7}$$

Now our interest is to clarify how quantum dynamics reflects the underlying classical dynamics. The most well established way to connect them is to perform the semiclassical approximation to the quantum propagator. In the small $\hbar$ limit, the multiple integral $< p_n|\hat{U}^n|p_0 >$ or $< q_n, p_n|\hat{U}^n|q_0, p_0 >$ can be evaluated by the method of stationary phase, which is formally equivalent to the saddle point approximation. Here we only show the final expression for the *semiclassical propagator*. In p-representation, we have

$$K^{sc}(p_n; p_0) = \sum_\gamma A_\gamma(p_0, q_0)\exp\Big\{\frac{i}{\hbar}S_\gamma(p_0, q_0)\Big\}, \tag{8}$$

where the summation is taken over all classical paths γ satisfying given initial and final momenta, $p_0 = \alpha$ and $p_n = \beta$. $A_\gamma(p_0, q_0) = [2\pi\hbar(\partial p_n/\partial q_0)_{p_0}]^{-\frac{1}{2}}$ stands for the amplitude factor associated with the stability of each orbit γ, and $S_\gamma(p_0, q_0)$ is the corresponding classical action.

An important remark is that if we take the p-representation, p_0 and p_n should be real-valued since they both are observables. This implies that the canonical conjugate variable q_0 does not have any constraint and may take not only real values but also complex ones. This ensures purely quantum effects within the semiclassical framework. Therefore, by extending the initial angle as $q_0 = \xi + i\eta \quad (\xi, \eta \in \mathbf{R})$, we have a representation for semiclassically contributing complex paths on the initial Lagrangian manifold as

$$\mathcal{M}_n^{\alpha,\beta} \equiv \{(p_0, q_0 = \xi + i\eta) \in \mathbf{C}^2 \mid p_0 = \alpha,\ p_n = \beta,\ \ \alpha, \beta \in \mathbf{R}\}. \tag{9}$$

If there exist no classical paths on the real phase space connecting between the two states specified by $p_0 = \alpha$ and $p_n = \beta$, we should say that this process is classically forbidden, and bridged only by complex classical paths.

As for the coherent state representation, we have a similar semiclassical expression,

$$K^{sc}(q_n, p_n; q_0, p_0) = \sum_\gamma A_\gamma(p_0, q_0) \exp\left\{\frac{i}{\hbar} S_\gamma(p_0, q_0)\right\}, \tag{10}$$

where the summation is taken over all classical paths satisfying given initial and final coherent states, *i.e.*, $q_0 + ip_0 = q_\alpha + ip_\alpha$, $q_n - ip_n = q_\beta + ip_\beta$. Note that the variables $q_\alpha, p_\alpha, q_\beta, p_\beta$ take real values whereas q_0, p_0, q_0, p_0 can take complex ones[7, 8]. Introducing the variables $Q = q + ip$ and $P = q - ip$ where $q, p \in \mathbf{C}$, semiclassically contributing complex paths are given as

$$\mathcal{M}_n^{\alpha,\beta} \equiv \{(Q_0, P_0) \in \mathbf{C}^2 \mid Q_0 = \alpha,\ P_n = \beta,\ \ \alpha = q_\alpha + ip_\alpha,\ \beta = q_\beta - ip_\beta,\ \}. \tag{11}$$

In either case, the manifold representing an initial and final state is one-dimensional complex manifold, and thus the space of the search parameter forms one-dimensional complex plane. This fact does not depend on which representation we choose. This is interpreted as a manifestation of uncertainty principle of quantum mechanics.

3. Quantum tunneling and the Julia set

We here discuss several aspects of quantum tunneling based on semiclassical expressions of quantum propagator. As explained in the previous section, the classical counterpart of an initial or final quantum state is a one-dimensional complex Lagrangian manifold(= two-dimensional real manifold). So, our task is to see how such a one-dimensional complex manifold evolves under the iteration of the mapping (1). We hereafter limit ourselves to the Hénon map since recent progresses on the study of the complex Hénon map is so fruitful and these well fit to what we just want know.

1. Let us first consider the situation where an initial and final state and the time step n are given, that is $p_0 = \alpha$ and $p_n = \beta$. We here adopt the propagator in the p-representation. The conditions for initial and final states give an algebraic equation.

$$(\beta, q_n) = F^n(\alpha, q_0). \tag{12}$$

We regards this as a 2^n-th degree algebraic equation for q_0. As an immediate consequence of the fundamental theorem of algebra, it has 2^n solutions in general. In our context this means that we always have 2^n complex classical paths connecting initial and final states.

Obviously, the complex orbits, some of which may describe genuine tunneling transition, proliferate exponentially as a function of the time step n. This is one way of characterization of quantum tunneling in the presence of chaos. However, it is rather formal. Exponential increase of complex orbits comes merely from the degree of the map, and the nature of the non-wondering set of the map itself does not matter in this description. Even in case of the elementary map[9], which is a class of polynomial diffeomorphism not generating chaos, we have the same conclusion, meaning that an argument of this level does not make any contrast between chaotic and non-chaotic systems

2. The second one more directly concerns characters of the dynamics and especially employs mathematical results on the complex Hénon map. Let us consider the situation where an initial manifold in $\mathbf{R}^2$ is confined in a certain KAM region. KAM tori themselves or stochastic regions sandwiched by KAM tori confine the orbits in them, the orbits contained in such a region cannot escape to outside chaotic or different KAM regions. That is, the (real) classical dynamics is confined within a certain subregion in phase space, and the transition to other regions is forbidden.

However, assume that there exists a saddle point Q on $\mathbf{R}^2$, which lies outside the region bounds the initial manifold. If the whole initial manifold of the semiclassical propagator intersects with a stable manifold of the saddle Q, both are extended in $\mathbf{C}^2$ space, then there at least exist a point on the initial manifold which approaches the saddle Q as $n \to \infty$. That means, even if the transition to the outside regions is forbidden in $\mathbf{R}^2$, there is a complex orbit which goes out from that region and approaches some point located outside. We may expect further; the existence of an intersection point between the initial complexified manifold and the stable manifold implies uncountably many intersection points in general. Similarly, if there exist such a saddle Q, we can expect that there are infinitely many similar saddles in $\mathbf{R}^2$. In this way, we recognize that there are uncountablely many complex orbits that can proceed to outer regions. Notice that this argument has not taken into account the final state p_n explicitly, and so not a precise specification of tunneling orbits yet. However, extensive numerical results tell us that those type of complex orbits are indeed responsible for quantum tunneling transitions[4].

A mathematical formulation for an intuitive argument mentioned above is possible especially using the recent results on the Hénon map. For the semiclassical propagator in the p representation, we consider the following hypersurface induced by the set $\mathcal{M}_n^{\alpha,\beta}$:

$$\mathcal{M}_n^{*,\beta} \equiv \{(p,q) \in \mathbf{C}^2 \mid p_n = \beta\}, \tag{13}$$

and define the compact set,

$$\mathcal{C}^\beta \equiv \bigcap_{m=0}^{\infty} \overline{\bigcup_{n>m} \mathcal{M}_n^{*,\beta}}, \tag{14}$$

and

$$\mathcal{C} \equiv \bigcup_{\beta \in \mathbf{R}} \mathcal{C}^\beta, \tag{15}$$

where the limit is taken in the Hausdorff topology. Then we have the following claim.

Proposition (Ishii) *$K^+ \supset \mathcal{C}^\beta \supset J^+$ for every $\beta \in \mathbf{R}$. In particular, $K^+ \supset \mathcal{C} \supset J^+$.*

Here, $K^\pm = \{\ (x,y) \mid \{f^{\pm n}(x,y)\}_{n>0}$ is bounded $\}$ and $J^\pm = \partial K^\pm$ are the filled-in Julia set and the Julia set in the forward(resp. backward) direction, respectively. In this case, the set $\mathcal{M}_n^{*,\beta}$ does not specify the initial state p_0. But, the claim represents an expected aspect of complex orbits which contribute to the semiclassical propagator. We remark that the convergent theorem of currents established recently play a crucial role[10, 11].

In the speculation 1 in which the final state is not specified, we have assumed that an initial manifold intersects with stable manifolds of the saddles. Here, we consider a special situation where the initial manifold is put exactly on a certain KAM curve. KAM curves are expressed parametrically as[12, 13]

$$\mathcal{C}_\omega : \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2\pi\omega + u(\varphi,\omega) - u(\varphi - 2\pi\omega,\omega) \\ \varphi + u(\varphi,\omega) \end{pmatrix}, \tag{16}$$

where $u(\varphi,\omega)$ is determined by the following functional equation:

$$u(\varphi + 2\pi\omega,\omega) - 2u(\varphi,\omega) + u(\varphi - 2\pi\omega,\omega) = V'(\varphi + u(\varphi,\omega)). \tag{17}$$

The dynamics on the curve $\mathcal{C}_\omega$ is given in the φ-variable as a constant rotation $\varphi_{n+1} = \varphi_n + 2\pi\omega$.

Now suppose that the initial and final states are given by KAM curves, each of which is specified by the rotation number ω and ω' respectively. This leads an expression for the propagator as $< \mathcal{C}_{\omega'}|U^n|\mathcal{C}_\omega >$. Clearly, real classical orbits connecting two states exist if and only if $\omega = \omega'$ since $\mathcal{C}_\omega$(resp. $\mathcal{C}_{\omega'}$) is an invariant curve. However, in a similar way as the semiclassical treatment in the p-representation, we can extend our initial value plane to the complex plane. Recall that, for a given rotation number ω, the existence of an analytic KAM curve is equivalent to the existence of a positive radius of convergency of Lindstedt series,

$$u(\varphi,\omega) \equiv \sum_{k=1}^{\infty} K^k \sum_{\nu \le k} \mathrm{e}^{i\nu\varphi} u_\nu^{(k)}(\omega), \tag{18}$$

where K stands for the perturbation strength of the map under consideration. Therefore, extending the angle variable φ to the complex plane as $\varphi = \varphi' + i\varphi''$ gives our complexified initial manifold of semiclassically contributing orbits. The initial value plane thus complexified is nothing but an analytical extension for the KAM torus with a given rotation number ω, or a complexified KAM torus. KAM curves with different rotation numbers give different invariant sets, there are no classical orbits connecting such different KAM curves as long as the Fourier expansion (18) provides a complex analytic function.

However, as studied in Ref. [14], one can make analytical continuation of the complex KAM curves at most to certain domains in φ-plane and there possibly exist natural boundaries. (The radius of convergence as a function of ω is called the critical function[12].) The existence of the natural boundary implies that KAM curves cannot be globally invariant in complex plane and any initial states cannot be bounded even within KAM curves.

3. The third way of characterizing the quantum tunneling in chaotic maps is as follows. To this end, we pay our attention to a remarkable result on the complex equilibrium measure derived in the argument developed by Bedford and Smillie[10]. To be precise, we first present several basic theorems in the argument.

Theorem1(Bedford-Smillie,Sibony-Fornaess)
For a complex one-dimensional locally closed sub-manifold M in either $J^\pm$ or an algebraic variety, there is a constant $\gamma > 0$ so that

$$\lim_{n\to+\infty} \frac{1}{2^n}[f^{\mp n}M] = \gamma \cdot dd^c G^\pm(x,y) \tag{19}$$

in the sense of current, where $[M]$ *is the current of integration of* M, i.e. $[M](\phi) \equiv \int_M \phi|_M$. *In this statement,* $G^{\pm}(x, y)$ *represents the Green function given by*

$$G^{\pm}(x,y) \equiv \lim_{n\to\pm\infty} \frac{1}{2^n} \log^{+} \|f^n(x,y)\|, \tag{20}$$

where dd^c *is the complex Laplacian,*

$$dd^c u \equiv 2i \sum_{j,k} \frac{\partial u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k \tag{21}$$

The statement asserts that arbitrary algebraic curves in $\mathbf{C}^2$, for example our initial manifold given as $p_0 = \alpha \in \mathbf{R}$, converge to the support of $dd^c G^{\pm}(x, y)$. It is particularly important to note that it holds irrespective of the nonlinear parameter a in the Hénon map, meaning that convergence occurs even in mixed phase space. There is no such a unique set to which arbitrary manifolds converge in the real phase space. Thus, this convergent property is particularly intrinsic in the complex dynamics.

The Green function $G^{\pm}(x, y)$ is related to the Julia set as

$$\operatorname{supp} \mu^{\pm} = J^{\pm}, \tag{22}$$

where $\mu^{\pm}$ is induced by the Green function as

$$\mu^{\pm} \equiv \frac{1}{2\pi} dd^c G^{\pm}(x, y). \tag{23}$$

The relation between $J^{\pm}$ and the support of $\mu^{\pm}$ was also proved as[10],

$$\operatorname{supp} \mu^{\pm} = J^{\pm}. \tag{24}$$

Introducing $\mu \equiv \mu^{+} \wedge \mu^{-}$, it was also shown $J^* \equiv \operatorname{supp} \mu \subset J$[10]. In particular, if the map f is hyperbolic, then $J^* = J$ holds[10]. The complex equilibrium measure μ thus defined becomes a unique maximal entropy probability measure[10, 15, 16]. Furthermore we have,

Theorem2(Bedford-Smillie) μ *is mixing and the hyperbolic measure.*

Ergodicity immediately follows from the mixing property. Here the hyperbolic measure means that characteristic exponent λ_1 and λ_2 with respect to μ satisfies $\lambda_1 > 0 > \lambda_2$. The result again makes a very sharp contrast to the real-domain dynamics in mixed phase space.

With these in mind, we consider the case with mixed phase space. In the 2-dimensional real phase space, KAM curves occupy a positive area, which is a consequence of the KAM theorem. Although there is no rigorous result as to the area of chaotic domains, it is believed that chaotic regions have a positive Lebesgue measure as well.

Now we ask how invariant sets coexist in complex phase space. The filled-in Julia set $K \equiv K^{+} \cap K^{-}$ is the set of non-escaping points both in the forward and backward directions. Since $J^{\pm}$ are defined as boundaries of $K^{\pm}$, if $K^{\pm}$ have no interior points, then $K^{\pm} = J^{\pm}$ and $J = J^{+} \cap J^{-} = K^{+} \cap K^{-} = K$ follow.

Recall that the orbits on KAM curves in $\mathbf{R}^2$ are bounded both in the forward and backward directions, so they are obviously contained in the filled-in Julia set K. Therefore, if $K^{\pm}$ has no interior points, then KAM curves are necessarily contained in J. Furthermore, if $J^* = J$ holds even in non-hyperbolic parameter regimes, we conclude that KAM curves are contained in J^*. This might sound a bit puzzling because at least in $\mathbf{R}^2$ a major role of KAM curves is to bound the orbits in a certain subspace in phase space while the mixing property mentioned above implies itineracy of (complex) orbits in an entire phase space domain.

An interesting consequence of this working hypothesis is that all KAM curves are bridged via the Julia set J. More precisely stated, due to ergodicity on the measure μ, there necessarily exist an orbit which is placed arbitrarily close to a certain KAM curve and reach some other KAM curve within any desired precision. It is needless to say such an orbit moves in complex space because the KAM curves always serve as barriers in $\mathbf{R}^2$.

Such a situation is quite suggestive to our tunneling problem because ergodicity on μ ensures the transition over the dynamical barriers in the real space. Taking the coherent representation, we can formulate it more explicitly. Suppose an initial wavepacket, whose center is specified by the center of a minimum wavepacket as(q_α, p_α), is located on a KAM torus, and evaluate the propagator $< q_n, p_n|\hat{U}^n|q_0, p_0 >$ where the final state is given as $q_n - ip_n = q_\beta - ip_\beta$. The corresponding initial manifold in the semiclassical propagator is given by the set $\mathcal{M}_n^{\alpha,\beta}$ (see the definition (11)).

Due to the mixing property, that is our basic ansatz in the argument, for any neighborhood U_α, U_β of initial and final points, (q_α, p_α) and (q_β, p_β), there exist a time step N such that $f^N(U_\alpha) \cap U_\beta \neq \emptyset$. Since the neighborhood U_α should be taken as an open set in $\mathbf{C}^2$, the orbit connecting between U_α and U_β may not be contained in the initial manifold $\mathcal{M}_n^{\alpha,\beta}$. However, we can find another initial state (q'_α, p'_α) that is taken arbitrarily close to the original point (q_α, p_α) which contains a desired orbit. In other words, although one cannot say that a set $\mathcal{M}_n^{\alpha,\beta}$ always contains a connecting orbit, there is a wavepacket arbitrarily close to the original one whose initial plane $\mathcal{M}_n^{\alpha',\beta'}$ contains such a connecting orbit. The tunneling transition, reflecting the mixing property of the complex measure μ, takes place in this way.

4. Some numerical verifications

One of our crucial working hypothesis is that $K^\pm$ has no interior points. At present the best known result on this issue is; in case $|\delta| = 1$, $vol(K) = vol(K^+) = vol(K^-) < \infty$, where δ denotes the Jacobian of the Hénon map. (In case of $|\delta| < 1$, it was shown that $vol(K^-) = 0$, $vol(K^+) = 0$ or ∞, also if $|\delta| > 1$, then $vol(K^+) = 0$, $vol(K^-) = 0$ or ∞[19].)

Below, we present some pieces of numerical evidence implying $vol(K^+) = 0$ in the area-preserving case. First we enumerate how the number of non-escaping orbits decreases as a function of the time step n. We prepare an ensemble of initial points which is located around an elliptic fixed point on the real plane, and measure the number of orbits which remain in a fixed ball in $\mathbf{C}^2$. As shown in Fig. 1, the number of non-escaping orbits remains constant during some initial time steps and then decreases algebraically. That is, even if the orbits appear to be trapped around an elliptic fixed point for a certain time interval, they finally escape to infinity. But their motion is quite sticky like the motion around KAM curves in the real phase space. Such an exceedingly slow escaping behavior would be due to the fact that typical complex trajectories initially located in the vicinity of an elliptic point are trapped around complexified KAM curves and takes a very long time to escape from it [5].

One more numerical experiment is to plot the number of iterations during which the orbits stay in a finite ball before they escape to infinity. The initial points are put on a real 1-dimensional closed circle which is again close to an elliptic fixed point on the real plane. A series of plots in Fig. 2 displays that in every scale the orbits bounded in a finite region do not have positive measure on the initial circle. Magnification of a small interval produces similar spiky peaks, which suggest that the bounded orbits are distributed in a self-similar way. These numerical results also imply that $vol(K^+) = 0$.

The Siegel disk and Hermann ring may form the interior domain of the filled-in Julia set

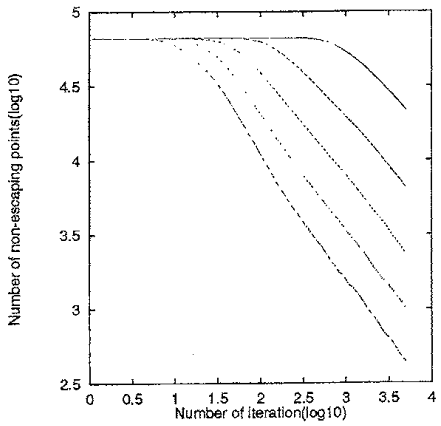

Figure 1: The number of non-escaping orbits of the Hénon map as a function of the iteration step. Initial points are placed on the boxes whose center is at a real fixed point of the map (2) with $a = 0.1$. The initial box size is given as 0.1, 0.2, 0.3, 0.4, 0.5 from the top to the bottom lines respectively.

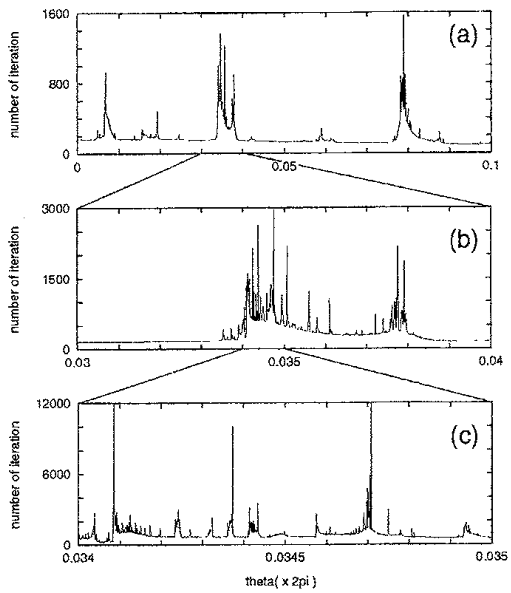

Figure 2: The number of iterations during which the orbits stay in a finite region. Initial points is placed on a circle in the complex plane given as $x = (r\cos\theta - x_{fix}, 0.1)$, $y = (r\sin\theta - y_{fix}, 0.1)$ $(\leq \theta < 2\pi)$. where (x_{fix}, y_{fix}) denotes a fixed point of the map (2) with $a = 0.1$, and the radius r is set to 0.4. The figure (b) is a magnification of a part of (a), and also (c) is a magnification of a part of (b).

in case of 1-dimensional complex maps. However, in case of the 2-dimensional area-preserving map, a necessary condition to realize Siegel disks or Hermann rings, namely, a condition for linearization around a fixed point is not satisfied: in order to make a linearization around a fixed point, the eigenvalues λ_1, λ_2 of linearized matrix around a given fixed point should satisfy so-called non-resonant condition:

$$\prod_{i=1}^{2} \lambda_i^{k_i} - \lambda_j \neq 0 \tag{25}$$

for any $j = 1, 2$ with $|\sum_{i=1}^{2} k_i| \geq 2$. but for an elliptic fixed point of the 2-dimensional area-preserving map we always have a pair of eigenvalues, λ and λ^{-1}. This evidently breaks the non-resonant condition.

The Birkhoff normal form is known as another type of normalization around an equilibrium point. However, there is a rigorous proof showing that the possibility of Birkhoff type normalization, in other words, the convergency of the normal form is a necessary and sufficient condition of complete integrability of the system[17]. Since we are concerned with the mixed system, normalization of such a type cannot obviously be realized.

5. Concluding remarks

In the present note, what we wanted to emphasize is that recent developments of complex dynamical systems in several dimensions certainly contribute to our understanding of quantum tunneling phenomena in chaotic systems. The semiclassical method is used as a bridge between quantum mechanics and the corresponding classical mechanics. As demonstrated in Ref. [4, 6], the semiclassical approximation works quite well even in chaotic maps, so we can interpret various characters in quantum tunneling phenomena in terms of classical mechanics. Since quantum tunneling is a classically forbidden process and is not described by the real classical dynamics, the use of complex classical dynamics is essential, and thereby the results on higher dimensional complex dynamics will make us go beyond speculations derived from numerical studies. .

Indeed, the second and third arguments presented in section 3 fully employ several mathematical results developed in the last decade. In particular, the convergent theorem of currents is a key ingredient to prove the statement given as the title of the present report[5, 18]. Furthermore concerning the third point, our working hypotheses, that is $J^* = J$ and the fact that $K^{\pm}$ does not have interior points, lead an interesting situation: KAM curves are contained in the Julia set and as a result of the mixing property of μ any KAM curve can be accessed from other KAM curves in complex domain. This is in a very sharp contrast to the real dynamics in which KAM curves play the role of dynamical barriers in phase space. In this sense, we may say that the mixing property of μ, exactly represents penetration due to quantum tunneling effects.

In this respect, of much interest is to make clear the relation between the Julia set and the natural boundary of KAM curves, the existence of which has been suggested numerically and analyzed extensively. The best choice of the initial manifold in the semiclassical propagator, in order to confine initial states into themselves as much as possible, would be complexified KAM curves. The presence of natural boundaries implies that such a confinement is impossible. To the author's knowledge, the role of the Julia set and the link between natural boundaries and the Julia set have not been clarified yet even in the numerical sense.

Finally, we mention some important ingredients in applying complex semiclassical analysis, we have completely skipped in this report. As stated in section 2, an basic idea of the semiclassical approximation, which corresponds to the derivation from eq. (8) to (10), is to apply the saddle point method to multiple integrals. The saddle point condition just gives the classical mapping rule (1), and thus the saddles contributing in the sum are complex classical orbits. However, due to the *Stokes phenomenon*, not all of the complex classical orbits necessarily contribute to the final semiclassical propagator. The Stokes phenomenon is discontinuous change of asymptotic solutions, and it occurs not only the saddle point method but the differential equation with a large parameter in general. Therefore, coping with the Stokes phenomenon in multi-dimensions would be a crucial step in carrying out the complex semiclassical approach. Fortunately, recent developments in mathematics, so-called *exact WKB analysis*, enable us to treat asymptotic expansions on the analytical basis via Borel-Laplace transform[20], and rather provide a recipe to extend to multi-dimensional problems or higher order differential equations[21]. A novel aspect in treating Stokes phenomenon in multiple integral or more generally higher-order differential equations is that Stokes curves can cross each other[22, 21, 23].

We will report elsewhere that some heuristic arguments, combined with a computer-assisted proof, work in the treatment of Stokes geometry for the quantized Hénon map[24]. A main idea is to impose a self-consistent condition to a given Stokes graphs by introducing new Stokes lines together with new turning points. This task is essentially equivalent to determine the Riemann sheet structure of the Borel transform(or *adjacency* in another context[25]).

The present note is written on the basis of the collaboration with Y. Ishii and K.S. Ikeda.

References

[1] S. C. Creagh, in *Tunneling in complex systems* ed. by S. Tomsovic (World Scientific, Singapore, 1998) 35.

[2] O. Bohigas, S. Tomsovic and D. Ullmo Phys. Rep. **223**(1993)43; S. Tomsovic and D. Ullmo Phys. Rev E **50**(1994)145.

[3] S.D. Frischat and E. Doron, Phys. Rev. E**57**(1999)1421.

[4] A. Shudo and K. S. Ikeda, Phys. Rev. Lett. **74** (1995) 682; Physica D**115** (1998) 234; Prog. Theor. Phys. Suppl. **139**(2000) 246.

[5] A. Shudo, Y. Ishii and K.S. Ikeda, to be submitted.

[6] T. Onishi, A. Shudo, K. S. Ikeda and K. Takahashi, Phys. Rev. E **64** (2001) 025201(R).

[7] J.R. Klauder, In *Path Integrals,* edited by G.J. Papadopoulos and J.T. Devreese, NATO Advanced Summer Institute (Plenum, New York, 1978), p. 5; J.R. Klauder, in *Random Media,* edited by G. Papanicolauou (Springer-Verlag, New York, 1987) p. 163.

[8] S. Adachi, Ann. Phys. **195** (1989) 45.

[9] S. Friedland and J. Milnor, Ergod Th. and Dynam. Sys. **9** (1989) 67.

[10] E. Bedford, and J. Smillie, Invent. Math. **103** (1991) 69 ; J. Amer. Math. Soc. **4** (1991)657; Math. Ann. **294** (1992) 395.

[11] J.E. Fornæss and N. Sibony, Duke Math. J. **65** (1992) 345.

[12] S. Marmi, J. Phys. A **23** (1990) 3447.

[13] A. Berretti, and L. Chierchaia, Nonlinearity, **3** (1990) 39; A. Berretti and S. Marmi, Phys. Rev. Lett. **68** (1992) 1443; A. Berretti, A. Celletti, L. Chierchaia and C. Falcolini, J. Stat. Phys. **66** (1992) 1613.

[14] J.M. Greene and I.C. Percival, Physica D **3** (1981) 540; I.C. Percival, Physica D **6** (1982) 67.

[15] J. Smillie, Ergod. Th. and Dynam. Sys. **10** (1990) 823.

[16] E. Bedford, M. Lyubich and J. Smillie, Invent. Math. **112** (1993) 77.

[17] H. Ito, Comment. Math. Helv. **64** (1989) 412; Math. Ann. **292** (1992) 411.

[18] A. Shudo, Y. Ishii and K.S. Ikeda, J. Phys. A **35** (2002) L225.

[19] S. Morosawa, Y. Nishimura, M. Taniguchi and T. Ueda, *Holomorphic dynamics* (Cambridge Univ. Press, 1999).

[20] A. Voros, Ann. Inst. H. Poincare, A **39** (1983) 211.

[21] T. Aoki, T. Kawai and Y. Takei, in *Méthodes résurgentes, Analyse algébrique des perturbations singuliéres*, L. Boutet de Monvel ed. 69 (1994); Asian J. Math. **2** (1998) 625; J. Math. Phys. **42** (2001) 3691.

[22] H.L. Berk, W.M. Nevins and K.V. Roberts, J. Math. Phys., **23**(1982) 988.

[23] C.J. Howls, Proc. R. Soc. Lond. A, **453** (1997) 2271.

[24] A. Shudo and K.S. Ikeda, *Stokes geometry for quantized Hénon map*, preprint.

[25] M.V. Berry and C.J. Howls, Proc. L. Soc. London, **434** (1991) 657.