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Dynamics of polynomial semigroups with bounded postcritical set in the plane

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Abstract

We investigate dynamics of polynomial semigroups of which post-critical sets in the plane are bounded. The Julia set may not be connected in general. We show that for such a polynomial semigroup, there exists an intrinsic total order in the space of all connected components of the Julia set and each connected component of the Fatou set is either simply or doubly connected.

We classify the class of polynomial semigroups with bounded post-critical set in the plane.

Furthermore, we investigate wordwise dynamics of such semigroups. Using uniform fiberwise quasiconformal surgery on a fiber bundle, we show that if the Julia set of such a semigroup is disconnected, then there exists a family of quasicircles with uniform distortion which is parametrized by the Cantor set.

1 Introduction and the main results

A **rational semigroup** is a semigroup generated by non-constant rational maps on $\overline{\mathbb{C}}$ with the semigroup operation being the composition of maps([HM1]). A **polynomial semigroup** is a semigroup generated by non-constant polynomial maps. Research on the dynamics of rational semigroups was initiated by A. Hinkkanen and G.J. Martin ([HM1]), who were interested in the role of the dynamics of polynomial semigroups while studying various one-complex-dimensional moduli spaces for discrete groups, and F. Ren's group([ZR], [GR]).

Definition 1.1. Let G be a rational semigroup. We set

$$F(G) = \{z \in \overline{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\},\ J(G) = \overline{\mathbb{C}} \setminus F(G).$$

$F(G)$ is called the **Fatou set** for G and $J(G)$ is called the **Julia set** for G. The backward orbit $G^{-1}(z)$ of z and **the set of exceptional points** $E(G)$ are defined by: $G^{-1}(z) = \cup_{g\in G} g^{-1}(z)$ and $E(G) = \{z \in \overline{\mathbb{C}} \mid \sharp G^{-1}(z) < \infty\}$. For any subset A of $\overline{\mathbb{C}}$, we set $G^{-1}(A) = \cup_{g\in G} g^{-1}(A)$. We denote by $\langle h_1, h_2, \dots \rangle$ the rational semigroup generated by the family $\{h_i\}$. The Julia set of the semigroup generated by a single map g is denoted by $J(g)$. For any polynomial g, we set $K(g) := \{z \in \mathbb{C} \mid \cup_{n\in\mathbb{N}} g^n(z) :$ bounded in $\mathbb{C}\}$.

For a polynomial semigroup G, we set

$$\hat{K}(G) := \{z \in \mathbb{C} \mid \bigcup_{g\in G} g(z) : \text{ bounded in } \mathbb{C}\}.$$

Furthermore, we set Rat $:= \{h : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \mid h :$ holomorphic $\}$, with uniform convergence topology on $\overline{\mathbb{C}}$.

Definition 1.2. Let G be a rational semigroup. We set

$$P(G) = \overline{\bigcup_{g\in G} \{\text{all critical values of } g\}}.$$

This is called the **postcritical set** for G.

Question 1. Let G be a polynomial semigroup such that each element $g \in G$ is of degree at least two. If $P(G) \setminus \{\infty\}$ is bounded in $\mathbb{C}$, then $J(G)$ is connected?

The answer is **NO.**

Example 1.3 ([SY]). Let $G = \langle z^3, \frac{z^2}{4} \rangle$. Then $P(G) \setminus \{\infty\} = \{0\}$ (which is bounded in $\mathbb{C}$) and $J(G)$ is disconnected($J(G)$ is a Cantor family of round circles). Furthermore, small pertubation H of G still satisfies that $P(G) \setminus \{\infty\}$ is bounded in $\mathbb{C}$ and that $J(H)$ is disconnected. ($J(H)$ is a Cantor family of quasi-circles with uniform dilatation.)

Question 2. What happens if $P(G) \setminus \{\infty\}$ is bounded in $\mathbb{C}$ and $J(G)$ is disconnected?

1.1 Connected components of Julia sets

We present some results on connected components of the Julia set of a polynomial semigroup with bounded postcritical set in the plane. Furthermore, we classify such semigroups. The proofs are given in the section 3.1.

Theorem 1.4. *Let G be a rational semigroup generated by a family $\{h_\lambda\}_{\lambda\in\Lambda}$ such that each h_λ is not an elliptic Möbius transformation of finite order.*

Suppose that there exists a connected component A of $J(G)$ such that $\sharp A > 1$ and $\cup_{\lambda\in\Lambda} J(h_\lambda) \subset A$. Then, $J(G)$ is connected.

Definition 1.5. Let $\mathcal{G}$ be the set of all polynomial semigroups G with the following properties:

- each element of G is of degree at least two, and
- $P(G) \setminus \{\infty\}$ is bounded in $\mathbb{C}$.

Furthermore, we set $\mathcal{G}_c = \{G \in \mathcal{G} \mid J(G) : \text{connected}\}$ and $\mathcal{G}_d = \{G \in \mathcal{G} \mid J(G) : \text{disconnected}\}$.

Notation: For a polynomial semigroup G, we denote by $\mathcal{J}$ the set of all connected components of $J(G)$ which are included in $\mathbb{C}$.

Definition 1.6. For any connected compact sets K_1 and K_2 in $\mathbb{C}$, "$K_1 \leq K_2$" means that $K_1 = K_2$ or K_1 is included in a bounded component of $\mathbb{C} \setminus K_2$. "$K_1 < K_2$" means $K_1 \leq K_2$ and $K_1 \neq K_2$. Note that "$\leq$" is a partial order in the space of all non-empty compact connected set in $\mathbb{C}$.

Theorem 1.7. *Let $G \in \mathcal{G}$ (possibly infinitely generated). Then*

1. *$(\mathcal{J}, \leq)$ is totally ordered.*
2. *Each connected component of $F(G)$ is either simply or doubly connected.*
3. *For any $g \in G$ and any connected component J of $J(G)$, we have that $g^{-1}(J)$ is connected. Let $g^*(J)$ be the connected component of $J(G)$ containing $g^{-1}(J)$. If $J \in \mathcal{J}$, then $g^*(J) \in \mathcal{J}$. If $J_1, J_2 \in \mathcal{J}$ and $J_1 \leq J_2$, then $g^{-1}(J_1) \leq g^{-1}(J_2)$ and $g^*(J_1) \leq g^*(J_2)$.*

Proposition 1.8. *Let $G \in \mathcal{G}$. If U is a connected component of $F(G)$ such that $U \cap \hat{K}(G) \neq \emptyset$, then $U \subset$int $\hat{K}(G)$ and U is simply connected. Furthermore, we have $\hat{K}(G) \cap F(G) =$int $\hat{K}(G)$.*

Notation: We classify the class $\mathcal{G}_d$ of polynomial semigroups as follows:

- type(I) $\sharp(P(G) \setminus \{\infty\}) \geq 2$
  - (a): $\infty \in F(G)$
  - (b): $\infty \in J(G)$
- type (II) $\sharp(P(G) \setminus \{\infty\}) = 1$, let $\{z_0\} = P(G) \setminus \{\infty\}$. (Note that in this case each element of the semigroup is of the form $a(z - z_0)^n + z_0$.)
  - (a): $\infty \in F(G)$
    * (i): $z_0 \in F(G)$
    * (ii): $z_0 \in J(G)$
  - (b): $\infty \in J(G)$

* (i): $z_0 \in F(G)$
* (ii): $z_0 \in J(G)$.

Theorem 1.9. *Let $G \in \mathcal{G}_d$. Under the above notation, we have the following.*

1. *If G is generated by a compact set of polynomials in* Rat, *then any subsemigroup H of G with $H \in \mathcal{G}_d$ is of type (I)(a) or (II)(a)(i).*

2. *If $\infty \in F(G)$, the connected component U_∞ of $F(G)$ containing ∞ is simply connected. Furthermore, the element $J_{\max} \in \mathcal{J}$ containing ∂U_∞ is the unique one satisfying that $J \leq J_{\max}$ for each $J \in \mathcal{J}$. Furthermore, $\infty \in F(G)$ if and only if there exists a unique maximal element in $(\mathcal{J}, \leq)$.*

3. *Whatever the type of G is, there exists a unique element $J_{\min} \in \mathcal{J}$ such that $J_{\min} \leq J$ for each element $J \in \mathcal{J}$. Furthermore, let D be the unbounded component of $\mathbb{C} \setminus J_{\min}$. Then $(P(G) \setminus \{\infty\}) \cap D = \emptyset$ and $\partial \hat{K}(G) \subset J_{\min}$.*

4. *If G is generated by a family $\{h_\lambda\}_{\lambda \in \Lambda}$, then there exist two elements λ_1 and λ_2 of Λ satisfying:*

   (a) there exist two elements J_1 and J_2 of $\mathcal{J}$ such that $J_1 \neq J_2$ and $J(h_{\lambda_i}) \subset J_i$ for each $i = 1, 2$,

   (b) $J(h_{\lambda_1}) \cap J_{\min} = \emptyset$,

   (c) for each $n \in \mathbb{N}$, we have $h_{\lambda_1}^{-n}(J(h_{\lambda_2})) \cap J(h_{\lambda_2}) = \emptyset$ and $h_{\lambda_2}^{-n}(J(h_{\lambda_1})) \cap J(h_{\lambda_1}) = \emptyset$, and

   (d) h_{λ_1} has an attracting fixed point z_1 in $\mathbb{C}$, int $K(h_{\lambda_1})$ consists of only one immediated attracting basin for z_1, and $K(h_{\lambda_2}) \subset$int $K(h_{\lambda_1})$. Furthermore, $z_1 \in$ int $K(h_{\lambda_2})$.

   Moreover, for each $g \in G$ with $J(g) \cap J_{\min} = \emptyset$, we have that g has an attracting fixed point z_g in $\mathbb{C}$, int $K(g)$ consists of only one immediate attracting basin for z_g, and $J_{\min} \subset$int $K(g)$.

5. *int $\hat{K}(G)(= \hat{K}(G) \cap F(G)) \neq \emptyset$ if and only if G is of type (I) or (II)(a)(i)or (II)(b)(i). If int $\hat{K}(G) \neq \emptyset$ then*

   (a) $\mathbb{C} \setminus J_{\min}$ is disconnected, $\sharp J \geq 2$ for each $J \in \mathcal{J}$, and

   (b) for each $g \in G$ with $J(g) \cap J_{\min} = \emptyset$, we have $J_{\min} < g^*(J_{\min})$, $g(\hat{K}(G)) \subset$ int $\hat{K}(G)$, and the unique attracting fixed point z_g of g in $\mathbb{C}$ belongs to int $\hat{K}(G)$.

6. *If G is of type (I)(b) or (II)(b), then the connected component of $J(G)$ containing ∞ is equal to $\{\infty\}$. Similarly, if G is of type (II)(a)(ii) or (II)(b)(ii), then $J_{\min} = \{z_0\}$.*

7. *$J(G)$ is uniformly perfect if and only if G is of type (I)(a) or (II)(a)(i). Here a compact set K in $\overline{\mathbb{C}}$ is said to be uniformly perfect if $\sharp K \geq 2$ and there exists a constant $C > 0$ such that each annulus A which separates K satisfies that mod $(A) \leq C$.*

8. *Suppose that G is of type (I). Let $g \in G$ and let $z_1 \in J(G) \cap \mathbb{C}$. If $g(z_1) = z_1$ and $g'(z_1) = 0$, then $z_1 \in$ int $J_{\min}$ and $J(g) \subset J_{\min}$. (**Remark:** For a rational semigroup G, if $J(G)$ is connected then each superattracting fixed point in $J(G)$ of some element $g \in G$ belongs to int $J(G)$.)*

1.2 Fiberwise Julia sets

We present some results on fiberwise dynamics of a finitely generated polynomial such that the postcritical set in the plane is bounded and the Julia set is disconnected. In particular, using the uniform fiberwise quasiconformal surgery on a fiber bundle, we show the existence of a family of quasicircles parametrized by a Cantor set with uniform distortion in the Julia set of such a semigroup. The proofs are given in section 3.2.

Definition 1.10. 1. ([S1],[S3]) Let X be a compact metric space, $g : X \to X$ be a continuous map and $f : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a continuous map. Then we say that f is a rational skew product (or fibered rational map on trivial bundle $X \times \overline{\mathbb{C}}$) over $g : X \to X$, if $\pi \circ f = \pi \circ g$ where $\pi : X \times \overline{\mathbb{C}} \to X$ denotes the projection, and for each $x \in X$, the restriction $f_x := f|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \pi^{-1}(g(x))$ of f is a non constant rational map, under the identification $\pi^{-1}(x') \cong \overline{\mathbb{C}}$ for each $x' \in X$. Let $d(x) = \deg(f_x)$, for each $x \in X$. Let $q_x^{(n)}$ be a rational map defined by: $q_x^{(n)}(y) = \pi_{\overline{\mathbb{C}}}(f_x^n((x, y)))$, for each $n \in \mathbb{N}$ and $x \in X$. Let $\pi : X \times \overline{\mathbb{C}} \to X$ and $\pi_{\overline{\mathbb{C}}} : X \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be the projections.

2. Let $G = \langle h_1, \cdots, h_m \rangle$ be a finitely generatd rational semigroup. For a fixed generator system $\{h_1, \cdots, h_m\}$, we set $\Sigma_m = \{1, \cdots, m\}^{\mathbb{N}}$, $\sigma : \Sigma_m \to \Sigma_m$, $\sigma(x_1, x_2, \cdots) := (x_2, x_3, \cdots)$. Moreover, we define a map $f : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ by: $(x, y), \mapsto (\sigma(x),\ h_{x_1}(y))$, where $x = (x_1, x_2, \cdots)$. This is called **the skew product map associated with the generator system** $\{h_1, \cdots, h_m\}$.

3. Let $f : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product over $g : X \to X$. We set
$$P(f) := \overline{\bigcup_{n \geq 0, x \in X} f^n(\text{critical points of } f_x)},$$

where the closure is taken in $X \times \overline{\mathbb{C}}$. This is called the fiber-postcritical set for f. We say that f is hyperbolic (along fibers) if $P(f) \subset F(f)$.

Definition 1.11 ([S1],[S3]). Let $f : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product over $g : X \to X$. Then,

1. $f_x^n := f^n|_{\pi^{-1}x} : \pi^{-1}x \to \pi^{-1}\sigma^n(x) \subset X \times \overline{\mathbb{C}}$.
2. we denote by $F_x(f)$ the set of points $y \in \pi^{-1}x$ which has a neighborhood U in $\pi^{-1}x$ such that $\{f_x^n : U \to X \times \overline{\mathbb{C}}\}_{n \in \mathbb{N}}$ is normal.
3. $J_x(f) := \pi^{-1}x \setminus F_x(f)$.
4. $\tilde{J}(f) := \overline{\bigcup_{x \in \Sigma_m} J_x(f)}$ in $X \times \overline{\mathbb{C}}$.
5. $\hat{J}_x(f) := \pi^{-1}x \cap \tilde{J}(f)$.
6. $\tilde{F}(f) := (\Sigma_m \times \overline{\mathbb{C}}) \setminus \tilde{J}(f)$.

Definition 1.12. Let $G = \langle h_1, \dots, h_m \rangle$ be a finitely generated polynomial semigroup. Fix the generator system $\{h_1, \dots, h_m\}$. Suppose $G \in \mathcal{G}_d$. Then we set

$$B_{\min} := \{1 \le j \le m \mid J(h_j) \subset J_{\min}\},$$

where $J_{\min}$ denotes the unique minimal element in $(\mathcal{J}, \le)$ in Theorem 1.9.3. Furthermore, let $H_{\min}$ be the subsemigroup of G which is generated by $\{h_j \mid j \in B_{\min}\}$.

Proposition 1.13. *Let $G = \langle h_1, \cdots, h_m \rangle$ be a finitely generated polynomial semigroup in $\mathcal{G}_d$. Then, we have $B_{\min} \neq \emptyset$ and $\{1, \dots, m\} \setminus B_{\min} \neq \emptyset$, under the above notation.*

Theorem 1.14. *Let $G = \langle h_1, \cdots, h_m \rangle$ be a finitely generated polynomial semigroup. Suppose $G \in \mathcal{G}_d$. Let $f : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the skew product map associated with the generator system $\{h_1, \dots, h_m\}$. Then under the above notation and notation in section 1, we have that*

1. *for each $x \in \Sigma_m \setminus \bigcup_{n \ge 0} \sigma^{-n} B_{\min}^{\mathbb{N}}$,*
   - *(a) there exists only one bounded component U_x of $\pi^{-1}(x) \setminus (J_x(f) \cup \pi_{\overline{\mathbb{C}}}^{-1}\{\infty\})$. Furthermore, the boundary ∂U_x in $\pi^{-1}x$ is equal to $J_x(f)$.*
   - *(b) each limit function of $\{f_x^n\}_n$ in U_x is constant. For each $y \in U_x$ there exists a number $n \in \mathbb{N}$ such that $\pi_{\overline{\mathbb{C}}}(f_x^n(y)) \in$int $\hat{K}(G)$.*
   - *(c) $\hat{J}_x(f) = J_x(f)$.*
   - *(d) 2-dimensional Lebesgue measure of $\hat{J}_x(f) = J_x(f)$ is equal to zero.*

2. *If $H_{\min}$ is semi-hyperbolic then G is also. If this is so, then for each $x \in \Sigma_m \setminus \bigcup_{n\geq 0} \sigma^{-n} B_{\min}^{\mathbb{N}}$, we have that $\hat{J}_x(f) = J_x(f)$ is a Jordan curve. Here, a rational semigroup H is said to be semi-hyperbolic if for each $z \in J(H)$ there exists a neighborhood U of z in $\overline{\mathbb{C}}$ and a number $N \in \mathbb{N}$ such that for each $g \in G$, $\deg(g : V \to U) \leq N$ for each connected component V of $g^{-1}(U)$. ([S1],[S4].)*

3. *For each $s \in \mathbb{N}$, we set*

$$W_s := \{x \in \Sigma_m \mid \forall l \in \mathbb{N},\ 1 \leq \exists j \leq s \text{ with } x_{l+j} \in \{1, \dots, m\} \setminus B_{\min}\}.$$

Let $\overline{f} := f|_{W_s \times \overline{\mathbb{C}}} : W_s \times \overline{\mathbb{C}} \to W_s \times \overline{\mathbb{C}}$. Then $\overline{f}$ is a hyperbolic skew product and there exists a constant $K_s \geq 1$ such that for each $x \in W_s$, $\hat{J}_x(f) = J_x(f) = J_x(\overline{f})$ is a K_s-quasicircle.

Theorem 1.15. (Existence of a Cantor family of quasicircles.) *Let $G \in \mathcal{G}_d$(possibly infinitely generated) and let V be an open set with $V \cap J(G) \neq \emptyset$. Then there exist elements g_1 and g_2 in G such that all of the following hold.*

1. *$H = \langle g_1, g_2 \rangle$ satisfies $J(H) \cap V \neq \emptyset$.*

2. *There exists an open set U in $\mathbb{C}$ satifying that $g_1^{-1}(\overline{U}) \cup g_2^{-1}(\overline{U}) \subset U$ and $g_1^{-1}(\overline{U})$ and $g_2^{-1}(\overline{U})$ are disjoint.*

3. *$H = \langle g_1, g_2 \rangle$ is a hyperbolic polynomial semigroup(i.e. $P(H) \subset F(H)$).*

4. *Let $f : \Sigma_2 \times \overline{\mathbb{C}} \to \Sigma_2 \times \overline{\mathbb{C}}$ be the skew product map associated with the generator system $\{g_1, g_2\}$. Then,*

   - *$J(H) = \bigcup_{x \in \Sigma_2} \pi_{\overline{\mathbb{C}}}(J_x(f))$ (disjoint union),*
   - *each connected component J of $J(H)$ is equal to $\pi_{\overline{\mathbb{C}}}(J_x(f))$ for some $x \in \Sigma_2$ and*
   - *there exists a constant $K \geq 1$ independent of J such that each connected component J of $J(H)$ is a K-quasicircle.*

5. *$x \mapsto J_x(f)$ is continuous and injective.*

2 Tools

To show the main results, we need some tools in this section.

2.1 Fundamental properties of rational semigroups

Lemma 2.1 ([HM1],[GR],[S1]). *Let G be a rational semigroup.*

1. *For each $f \in G$, we have $f(F(G)) \subset F(G)$ and $f^{-1}(J(G)) \subset J(G)$. Note that we do not have that the equality holds in general.*
2. *If $G = \langle h_1, \cdots, h_m \rangle$, then $J(G) = h_1^{-1}(J(G)) \cup \cdots \cup h_m^{-1}(J(G))$*
3. *If $\sharp(J(G)) \geq 3$, then $J(G)$ is a perfect set.*
4. *If $\sharp(J(G)) \geq 3$, then $\sharp E(G) \leq 2$.*
5. *If a point z is not in $E(G)$, then $J(G) \subset \overline{G^{-1}(z)}$. In particular if a point z belongs to $J(G) \setminus E(G)$, then $\overline{G^{-1}(z)} = J(G)$.*
6. *If $\sharp(J(G)) \geq 3$, then $J(G)$ is the smallest closed backward invariant set containing at least three points. Here we say that a set A is backward invariant under G if for each $g \in G$, $g^{-1}(A) \subset A$.*

Theorem 2.2 ([HM1],[GR]). *Let G be a rational semigroup. If $\sharp(J(G)) \geq 3$, then $J(G) = \{z \in \overline{\mathbb{C}} \mid \exists g \in G,\ g(z) = z,\ |g'(z)| > 1\}$. In particular, $J(G) = \overline{\bigcup_{g \in G} J(g)}$.*

2.2 Fundamental properties of fibered rational maps

Lemma 2.3. *Let $f : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product over $g : X \to X$. Then,*

1. *$f^{-1}(\tilde{J}(f)) = \tilde{J}(f) = f(\tilde{J}(f))$. $f^{-1}J_{g(x)}(f) = J_x(f)$. $\hat{J}_x(f) \supset J_x(f)$, note that* **equality does not hold in general.** *If $g : X \to X$ is a surjective and open map, then $f^{-1}\hat{J}_{g(x)}(f) = \hat{J}_x(f)$.*
2. *(**[J], [S1]**) If $\deg(f_x) \geq 2$ for each $x \in X$, then $J_x(f)$ is a non-empty perfect set. Furthermore, $x \mapsto J_x(f)$ is lower semicontinuous; i.e. for any point $(x, y) \in X \times \overline{\mathbb{C}}$ with $(x, y) \in J_x(f)$ and any sequence (x^n) in X with $x^n \to x$, there exists a sequence (x^n, y^n) in $X \times \overline{\mathbb{C}}$ with $(x^n, y^n) \in J_{x^n}(f)$ such that $(x^n, y^n) \to (x, y)$. But $x \mapsto J_x(f)$ is* **NOT** *continuous with respect to the Hausdorff topology in general.*
3. *If $\deg(f_x) \geq 2$ and f_x is a polynomial for each $x \in X$, then for each $x \in X$, we have $\infty \in \pi_{\overline{\mathbb{C}}}(F_x(f))$ and $J_x(f) = \partial K_x(f)$ (in $\pi^{-1}(x)$), where $K_x(f) := \{z \in \pi^{-1}x \mid \{\pi_{\overline{\mathbb{C}}}(f_x^n(z))\}_{n \in \mathbb{N}}$: bounded in $\mathbb{C}\}$.*

4. *If $f : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ is a skew product map associated with a generator system $\{h_1, \cdots, h_m\}$ of a rational semigroup G, then $\pi_{\overline{\mathbb{C}}}(\tilde{J}(f)) = J(G)$.*

Lemma 2.4. *Let $G = \langle h_1, \cdots, h_m \rangle$ be a finitely generated polynomial semigroup such that each h_j is of degree at least two. Let $f : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the skew product map associated with the generator system $\{h_1, \cdots, h_m\}$. Then $P(G) \setminus \{\infty\}$ is bounded in $\mathbb{C}$ if and only if $J_x(f)$ is connected for each $x \in \Sigma_m$.*

2.3 A lemma from general topology

Lemma 2.5 ([N]). *Let X be a compact metric space and let $f : X \to X$ be a continuous open map. Let A be a compact connected subset of X. Then for each connected component B of $f^{-1}(A)$, we have $f(B) = A$.*

3 Proofs of the main results

In this section, we demonstrate the main results.

3.1 Proofs of results in 1.1

Proof of Theorem 1.4: Let $\lambda \in \Lambda$ and let B be a connected component of $h_\lambda^{-1}(A)$. Then by Lemma 2.5, $h_\lambda(B) = A$. Combining this with $h_\lambda^{-1}(J(h_\lambda)) = J(h_\lambda)$, we obtain $B \cap J(h_\lambda) \neq \emptyset$. Hence $B \subset A$. This means that $h_\lambda^{-1}(A) \subset A$ for each $\lambda \in \Lambda$. Combining this with $\sharp A \geq 3$, by Lemma 2.1-6 we obtain $J(G) \subset A$. Hence $J(G) = A$ and $J(G)$ is connected. □

Lemma 3.1. *Let $G \in \mathcal{G}$ and let J be a connected component of $J(G)$, $z_0 \in J$ a point, and (g_n) a sequence of G such that $d(z_0, J(g_n)) \to 0$ as $n \to \infty$. Then $\sup_{z \in J(g_n)} d(z, J) \to 0$ as $n \to \infty$.*

Proof. Suppose there exist a connected component J' of $J(G)$ with $J' \neq J$ and a subsequence (g_{n_j}) of (g_n) such that $\min_{z \in J(g_{n_j})} d(z, J') \to 0$ as $j \to \infty$. Since $J(g_{n_j})$ is compact and connected for each j, we may assume that there exists a non-empty compact connected subset K of $\overline{\mathbb{C}}$ such that $J(g_{n_j}) \to K$ as $j \to \infty$, with respect to the Hausdorff topology. Then $K \cap J \neq \emptyset$ and $K \cap J' \neq \emptyset$. Since $K \subset J(G)$ and K is connected, it contradicts $J' \neq J$. □

Lemma 3.2. *Suppose $G \in \mathcal{G}_d$ and $\infty \in J(G)$. Then, the connected component A of $J(G)$ containing ∞ is equal to $\{\infty\}$.*

Proof. By Lemma 2.5, we obtain $g^{-1}(A) \subset A$ for each $g \in G$. Hence, if $\sharp A \geq 3$, then $J(G) \subset A$, by Lemma 2.1-6. Then $J(G) = A$ and it causes a contradiction, since $J(G)$ is disconnected. □

Proof of Theorem 1.7: First we show the statement 1. Suppose there exist elements $J_1, J_2 \in \mathcal{J}$ such that J_2 is included in the unbouded component A_1 of $\mathbb{C} \setminus J_1$ and J_1 is included in the unbounded component A_2 of $\mathbb{C} \setminus J_2$. Then we can find an $\epsilon > 0$ such that $\overline{B(J_2, \epsilon)}$ is included in the unbounded component of $\mathbb{C} \setminus \overline{B(J_1, \epsilon)}$ and $\overline{B(J_1, \epsilon)}$ is included in the unbounded component of $\mathbb{C} \setminus \overline{B(J_2, \epsilon)}$. We take a point $z_i \in J_i$, for each $i = 1, 2$. By Theorem 2.2, for each $i = 1, 2$, there exists a sequence $\{g_{i,n}\}_{n \in \mathbb{N}}$ in G such that $d(z_i, J(g_{i,n})) \to 0$ as $n \to \infty$. Then by Lemma 3.1, we obtain for each $i = 1, 2$, $\sup_{z \in J(g_{i,n})} d(z, J_i) \to 0$ as $n \to \infty$. Hence, we obtain that there exists a positive integer n_0 such that for each n with $n \geq n_0$ and each $i = 1, 2$, we have $J(g_{i,n}) \subset B(J_i, \epsilon)$. This implies $J(g_{1,n}) \subset A_{2,n}$ and $J(g_{2,n}) \subset A_{1,n}$, where $A_{i,n}$ denotes the unbounded component of $\mathbb{C} \setminus J(g_{i,n})$. Hence we obtain $K(g_{2,n}) \subset A_{1,n}$ for each $n \in \mathbb{N}$ with $n \geq n_0$. Let $n \geq n_0$ be a number and v a critical value of $g_{2,n}$ in $\mathbb{C}$. Since $P(G) \setminus \{\infty\}$ is bounded in $\mathbb{C}$, we have $v \in K(g_{2,n})$. Hence $v \in A_{1,n}$. Hence $g_{1,n}^l(v) \to \infty$. But this implies a contradiction since $P(G) \setminus \{\infty\}$ is bounded in $\mathbb{C}$. Hence we have shown the statement 1.

Next, we show the statement 2. Let F_1 be a connected component of $F(G)$. Suppose that there exist three connected components J_1, J_2 and J_3 of $J(G)$ such that they are mutually disjoint and $\partial F_1 \cap J_i \neq \emptyset$ for each $i = 1, 2, 3$. Then, by the statement 1, we may assume that we have either (1): $J_i \in \mathcal{J}$ for each $i = 1, 2, 3$ and $J_1 < J_2 < J_3$, or (2): $J_1, J_2 \in \mathcal{J}$, $J_1 < J_2$, and $\infty \in J_3$. Each of these cases implies that J_1 is included in a bounded component of $\mathbb{C} \setminus J_2$ and J_3 is included in the unbounded component of $\overline{\mathbb{C}} \setminus J_2$. But it causes a contradiction, since $\partial F_1 \cap J_i \neq \emptyset$ for each $i = 1, 2, 3$. Hence, we have shown that we have either

Case I: $\sharp\{J : \text{component of } J(G) \mid \partial F_1 \cap J \neq \emptyset\} = 1$ or

Case II: $\sharp\{J : \text{component of } J(G) \mid \partial F_1 \cap J \neq \emptyset\} = 2$.

Suppose we have Case I. Let J_1 be the connected component of $J(G)$ such that $\partial F_1 \subset J_1$. Let D_1 be the connected component of $\overline{\mathbb{C}} \setminus J_1$ containing F_1. Since $\partial F_1 \subset J_1$, we have $\partial F_1 \cap D_1 = \emptyset$. Hence, we have $F_1 = D_1$. Hence F_1 is simply connected.

Suppose we have Case II. Let J_1 and J_2 be two connected components of $J(G)$ such that $J_1 \neq J_2$ and $\partial F_1 \subset J_1 \cup J_2$. Let D be the connected component of $\overline{\mathbb{C}} \setminus (J_1 \cup J_2)$ containing F_1. Since $\partial F_1 \subset J_1 \cup J_2$, we have $\partial F_1 \cap D = \emptyset$. Hence, we have $F_1 = D$. Hence F_1 is doubly connected. Hence, we have shown the statement 2.

We now show the statement 3. Let $g \in G$ be an element and let J be a connected component of $J(G)$. Suppose $g^{-1}(J)$ is disconnected. By Lemma 2.5, there exist at most finitely many connected components of $g^{-1}(J)$. Let $\{A_j\}_{j=1}^r$ be the set of connected components of $g^{-1}(J)$. Then there exists a positive number ϵ such that denoting by B_j the connected component of $g^{-1}(B(J, \epsilon))$ containing A_j for each $j = 1, \cdots, r$, $\{B_j\}$ are mutually disjoint. By Lemma 2.5, we obtain for each connected component

B of $g^{-1}(B(J,\epsilon))$, $g(B) = B(J,\epsilon)$ and $B \cap A_j \neq \emptyset$ for some j. Hence we obtain $g^{-1}(B(J,\epsilon)) = \bigcup_{j=1}^{r} B_j$ (disjoint union) and $g(B_j) = B(J,\epsilon)$ for each j. By Theorem 2.2 and Lemma 3.1, we have that there exists a sequence (g_n) of G such that $\sup_{z\in J(g_n)} d(z, J) \to 0$ as $n \to \infty$. Let $n \in \mathbb{N}$ be a number such that $J(g_n) \subset B(J,\epsilon)$. Then it follows that $g^{-1}(J(g_n)) \cap B_j \neq \emptyset$ for each $j = 1, \cdots, r$. Moreover, we have $g^{-1}(J(g_n)) \subset g^{-1}(B(J,\epsilon)) = \bigcup_{j=1}^{r} B_j$. On the other hand, by Lemma 2.4, we have that $g^{-1}(J(g_n))$ is connected. This is a contradiction. Hence we have shown that for each $g \in G$ and each connected component J of $J(G)$, $g^{-1}(J)$ is connected. By Lemma 3.2, we obtain that if $J \in \mathcal{J}$, then $g^*(J) \in \mathcal{J}$. Let J_1 and J_2 be two elements of $\mathcal{J}$ such that $J_1 \leq J_2$. Let U_i be the unbounded component of $\mathbb{C} \setminus J_i$, for each $i = 1, 2$. Then $U_2 \subset U_1$. Let $g \in G$ be an element. Then $g^{-1}(U_2) \subset g^{-1}(U_1)$. Since $g^{-1}(U_i)$ is the unbounded connected component of $\mathbb{C} \setminus g^{-1}(J_i)$ for each $i = 1, 2$, it follows that $g^{-1}(J_1) \leq g^{-1}(J_2)$. Hence $g^*(J_1) \leq g^*(J_2)$, otherwise $g^*(J_2) \leq g^*(J_1)$ and $g^*(J_2) \neq g^*(J_1)$, and it contradicts $g^{-1}(J_1) \leq g^{-1}(J_2)$. □

Proof of Proposition 1.8: Let $g \in G$ be an element. Then we have $\hat{K}(G) \cap F(G) \subset \text{int } K(g)$. Since $h(\hat{K}(G) \cap F(G)) \subset \hat{K}(G) \cap F(G)$ for each $h \in G$, it follows that $h(U) \subset$int $K(g)$ for each $h \in G$. Hence $U \subset$int $\hat{K}(G)$. From this, it is easy to see $\hat{K}(G) \cap F(G) =$int $\hat{K}(G)$. By the maximal value principle, we see that U is simply connected. □

Proof of Theorem 1.9: First we show the statement 1. To do that, we show the following claim:

Claim 1: Let G be a rational semigroup generated by a compact set Λ in Rat. Let $z_0 \in \overline{\mathbb{C}}$ be a point such that for each $g \in \Lambda$, $g(z_0) = z_0$ and $|g'(z_0)| < 1$. Then, $z_0 \in F(G)$.

The proof of this claim is immediate. From the claim, we easily obtain the statement 1.

Next, we show the statement 2. Suppose $\infty \in F(G)$. Let F_∞ be the connected component of $F(G)$ containing ∞. Let $J \in \mathcal{J}$ be an element such that $\partial F_\infty \cap J \neq \emptyset$. Let D be the unbounded component of $\overline{\mathbb{C}} \setminus J$. Then $F_\infty \subset D$ and D is simply connected. We show $F_\infty = D$. Otherwise, there exists an element $J_1 \in \mathcal{J}$ such that $J_1 \neq J$ and $J_1 \subset D$. By Theorem 1.7-1, we have either $J_1 < J$ or $J < J_1$. Hence, it follows that $J < J_1$ and we have that J is included in a bounded component D of $\mathbb{C} \setminus J_1$. Since F_∞ is included in the unbounded component D_1 of $\overline{\mathbb{C}} \setminus J_1$, it contradicts $\partial F_\infty \cap J \neq \emptyset$. Hence, $F_\infty = D$ and F_∞ is simply connected.

Next, suppose that there exists an element $J \in \mathcal{J}$ such that $J_\alpha < J$. Then J_α is included in a bounded component of $\mathbb{C} \setminus J$. On the other hand, F_∞ is included in the unbounded component of $\overline{\mathbb{C}} \setminus J$. Since $\partial F_\infty \subset J_\alpha$, we have a contradiction. Hence, we have shown $J \leq J_\alpha$ for each $J \in \mathcal{J}$.

By Lemma 3.2 and Lemma 2.1-3, it follows that if $G \in \mathcal{G}_d$ and $\infty \in J(G)$, then there exists a sequence (J_n) of $\mathcal{J}$ such that $d(\infty, J_n) \to 0$ as $n \to \infty$. Then there exists no maximal element in $(\mathcal{J}, \leq)$. Hence, we have shown the statement 2.

Next, we show the statement 3. Since $\emptyset \neq P(G) \setminus \{\infty\} \subset \hat{K}(G)$, we have $\hat{K}(G) \neq \emptyset$. By Proposition 1.8, we have $\partial \hat{K}(G) \subset J(G)$. Let J_1 be a connected component of $J(G)$ with $J_1 \cap \partial \hat{K}(G)$. By Lemma 3.2, $J_1 \in \mathcal{J}$. Suppose that there exists an element $J \in \mathcal{J}$ such that $J < J_1$. Let $z_0 \in J$ be a point. By Theorem 2.2, there exists a sequence (g_n) in G such that $d(z_0, J(g_n)) \to 0$ as $n \to \infty$. Then by Lemma 3.1, $\sup_{z \in J(g_n)} d(z, J) \to 0$ as $n \to \infty$. Since J_1 is included in the unbounded component of $\mathbb{C} \setminus J$, it follows that for a large $n \in \mathbb{N}$, J_1 is included in the unbounded component of $\mathbb{C} \setminus J(g_n)$. But this causes a contradiction, since $J_1 \cap \hat{K}(G) \neq \emptyset$. Hence, by Theorem 1.9-1 it must hold that $J_1 \leq J$ for each $J \in \mathcal{J}$. This argument shows that if J_1 and J_2 are two connected components of $J(G)$ such that $J_i \cap \partial \hat{K}(G) \neq \emptyset$ for each $i = 1, 2$, then $J_1 = J_2$. Hence, we obtain that there exists a unique minimal element $J_{\min}$ in $(\mathcal{J}, \leq)$ and $\partial \hat{K}(G) \subset J_{\min}$.

Next, let D be the unbounded component of $\mathbb{C} \setminus J_{\min}$. Suppose $D \cap P(G) \neq \emptyset$. Let $x \in D \cap P(G)$ be a point. By Theorem 2.2 and Lemma 3.1, there exists a sequence (g_n) in G such that $\sup_{z \in J(g_n)} d(z, J_{\min}) \to 0$ as $n \to \infty$. Then, for a large $n \in \mathbb{N}$, x is in the unbounded component of $\mathbb{C} \setminus J(g_n)$. But this is a contradiction, since $g_n^l(x) \to \infty$ as $l \to \infty$, $x \in P(G) \setminus \{\infty\}$, and $P(G) \setminus \{\infty\}$ is bounded in $\mathbb{C}$. Hence, we have shown the statement 3.

Next, we show the statement 4. By Theorem 1.4, there exist $\lambda_1, \lambda_2 \in \Lambda$ and conntected components J_1, J_2 of $J(G)$ such that $J_1 \neq J_2$ and $J(h_{\lambda_i}) \subset J_i$ for each $i = 1, 2$. By Lemma 3.2, we have $J_i \in \mathcal{J}$ for each $i = 1, 2$. Then $J(h_{\lambda_1}) \cap J(h_{\lambda_2}) = \emptyset$. Since $P(G) \setminus \{\infty\}$ is bounded in $\mathbb{C}$, we may assume $J(h_{\lambda_2}) < J(h_{\lambda_1})$. Then we have $K(h_{\lambda_2}) \subset \text{int } K(h_{\lambda_1})$ and $J_2 < J_1$. By the statement 3, $J_1 \neq J_{\min}$. Hence $J(h_{\lambda_1}) \cap J_{\min} = \emptyset$. Since $P(G) \setminus \{\infty\}$ is bounded in $\mathbb{C}$, we have that $K(h_{\lambda_2})$ is connected. Let U be the connected component of int $K(h_{\lambda_1})$ containing $K(h_{\lambda_2})$. Since $P(G) \setminus \{\infty\} \subset K(h_{\lambda_2})$, it follows that there exists an attracting fixed point z_1 of h_{λ_1} in $K(h_{\lambda_2})$ and U is the immediate attracting basin for z_1 with respect to the dynamics of h_{λ_1}. Furthermore, by Lemma 2.4, $h_{\lambda_1}^{-1}(J(h_{\lambda_2}))$ is connected. Hence, $h_{\lambda_1}^{-1}(U) = U$. Hence, int $K(h_{\lambda_1}) = U$.

Suppose there exists an $n \in \mathbb{N}$ such that $h_{\lambda_1}^{-n}(J(h_{\lambda_2})) \cap J(h_{\lambda_2}) \neq \emptyset$. Then $A := \cup_{s \geq 0} h_{\lambda_1}^{-ns}(J(h_{\lambda_2}))$ is connected and its closure $\overline{A}$ contains $J(h_{\lambda_1})$. Hence $J(h_{\lambda_1})$ and $J(h_{\lambda_2})$ are included in the same connected component of $J(G)$. This is a contradiction. Hence, for each $n \in \mathbb{N}$, we have $h_{\lambda_1}^{-n}(J(h_{\lambda_2})) \cap J(h_{\lambda_2}) = \emptyset$. Similarly, for each $n \in \mathbb{N}$, we have $h_{\lambda_2}^{-n}(J(h_{\lambda_1})) \cap J(h_{\lambda_1}) = \emptyset$. Combining $h_{\lambda_1}^{-1}(J(h_{\lambda_2})) \cap J(h_{\lambda_2}) = \emptyset$ with $z_1 \in K(h_{\lambda_2})$, we obtain $z_1 \in$ int $K(h_{\lambda_2})$. Now, let $g \in G$ be an element with $J(g) \cap J_{\min} = \emptyset$. We show the

following:
Claim 2: $J_{\min} < J(g)$.

To show the claim, suppose that $J_{\min}$ is included in the unbounded component U of $\mathbb{C} \setminus J(g)$. Since $J_{\min}$ is the unique minimal element in $(\mathcal{J}, \leq)$, we have that $J(g)$ is included in the unbounded component V of $\mathbb{C} \setminus J_{\min}$. Then, there exists an $\epsilon > 0$ such that $B(J_{\min}, \epsilon) \subset U$ and $J(g)$ is included in the unbounded component V_ϵ of $\mathbb{C} \setminus B(J_{\min}, \epsilon)$. By Theorem 2.2 and Lemma 3.1, it follows that there exists an element $h \in G$ such that $J(h) \subset B(J_{\min}, \epsilon)$. Then, we obtain that $J(h) \subset U$ and $J(g)$ is included in the unbounded component of $\mathbb{C} \setminus J(h)$. Hence, we obtain $K(h) \subset U$. Since $P(G) \setminus \{\infty\} \subset K(h)$, it contradicts $P(G) \setminus \{\infty\}$ is bounded in $\mathbb{C}$. Hence, we have shown the claim.

By Claim 2, Theorem 2.2 and Lemma 3.1, there exists an element $h_1 \in G$ such that $J(h_1) < J(g)$. By an argument which we have used before, it follows that g has an attracting fixed point z_g in $\mathbb{C}$ and int $K(g)$ consists of only one immediate attracting basin for z_g. Hence, we have shown the statement 4.

Next, we show the statement 5. Suppose that G is of type (II)(a)(ii) or (II)(b)(ii). Let $\{z_0\} = P(G) \setminus \{\infty\}$. Then $z_0 \in J(G)$ and each element of G is of the form $a(z-z_0)^n + z_0$. By Lemma 2.1-3, there exists a sequence (g_n) in G such that $\max\{d(z, z_0) \mid z \in J(g_n)\} \to 0$ as $n \to \infty$. Then we see $\hat{K}(G) = \{z_0\}$. Hence $F(G) \cap \hat{K}(G) = \emptyset$. Conversely, suppose $F(G) \cap \hat{K}(G) = \emptyset$. By the statement 4, there exist two elements g_1 and g_2 of G and two elements J_1 and J_2 of $\mathcal{J}$ such that $J_1 \neq J_2$, $J(g_i) \subset J_i$ for each $i = 1, 2$, g_1 has an attracting fixed point z_0 in int $K(g_2)$ and $K(g_2) \subset$ int $K(g_1)$. Since we assume $F(G) \cap \hat{K}(G) = \emptyset$, we have $z_0 \in J(G)$. Let J be the connected component of $J(G)$ containing z_0. By Lemma 3.2, we have $J \in \mathcal{J}$. We show $J = \{z_0\}$. Suppose $\sharp J \geq 2$. Then $J(g_1) \subset \overline{\cup_{n\geq 0} g_1^{-n}(J)}$. Moreover, by Theorem 1.7-3, $g_1^{-n}J$ is connected for each $n \in \mathbb{N}$. Since $g_1^{-n}(J) \cap J \neq \emptyset$ for each $n \in \mathbb{N}$, we see that $\overline{\cup_{n\geq 0} g_1^{-n}(J)}$ is connected. Combining this with $z_0 \in$ int $K(g_2)$, $K(g_2) \subset$ int $K(g_1)$, $z_0 \in J$ and $J(g_1) \subset \overline{\cup_{n\geq 0} g_1^{-n}(J)}$, we obtain $\overline{\cup_{n\geq 0} g_1^{-n}(J)} \cap J(g_2) \neq \emptyset$. Then it follows that $J(g_1)$ and $J(g_2)$ are included in the same connected component of $J(G)$. This is a contradiction. Hence we have shown $J = \{z_0\}$. By the statement 3, we obtain $\{z_0\} = J_{\min} = P(G) \setminus \{\infty\}$. Hence, it follows that G is of type (II)(a)(ii) or (II) (b)(ii). Hence, we have shown that $F(G) \cap \hat{K}(G) = \emptyset$ if and only if G is of type (II)(a)(ii) or (II)(b)(ii).

Next, suppose int $\hat{K}(G) (= \hat{K}(G) \cap F(G)) \neq \emptyset$. Since $\partial \hat{K}(G) \subset J_{\min}$ (the statement 3) and $\hat{K}(G)$ is bounded, it follows that $\mathbb{C} \setminus J_{\min}$ is disconnected and $\sharp J_{\min} \geq 2$. Hence, $\sharp J \geq 2$ for each $J \in \mathcal{J}$. Now, let $g \in G$ be an element with $J(g) \cap J_{\min} = \emptyset$. we show $J_{\min} \neq g^*(J_{\min})$. If $J_{\min} = g^*(J_{\min})$, then $g^{-1}(J_{\min}) \subset J_{\min}$. Since $\sharp J_{\min} \geq 3$, it follows that $J(g) \subset J_{\min}$. But this is a contradiction. Hence, $J_{\min} \neq g^*(J_{\min})$. Hence, we obtain $J_{\min} <$

$g^*(J_{\min})$. Since $g(\hat{K}(G)) \subset \hat{K}(G)$, we have $g(\text{int } \hat{K}(G)) \subset\text{int } \hat{K}(G)$. Suppose $g(\partial\hat{K}(G)) \cap \partial\hat{K}(G) \neq \emptyset$. Then, since $\partial\hat{K}(G) \subset J_{\min}$ (statement 3), we obtain $g(J_{\min}) \cap J_{\min} \neq \emptyset$. This implies $g^{-1}(J_{\min}) \cap J_{\min} \neq \emptyset$. Since $g^{-1}(J_{\min})$ is connected (Theorem 1.7-3), we obtain $g^*(J_{\min}) = J_{\min}$. But this is a contradiction. Hence, it must hold $g(\partial\hat{K}(G)) \subset\text{int } \hat{K}(G)$. Hence, $g(\hat{K}(G)) \subset \text{int } \hat{K}(G)$.

By the statement 4, g has a unique attracting fixed point z_g in $\mathbb{C}$. Then, $z_g \in P(G)\setminus\{\infty\} \subset \hat{K}(G)$. Hence, $z_g = g(z_g) \in g(\hat{K}(G)) \subset \text{int } \hat{K}(G)$. Hence, we have shown the statement 5.

Next, we show the statement 6. Let G be of type (II)(a)(ii) or (II)(b)(ii) and let $\{z_0\} = P(G) \setminus \{\infty\}$. Then by the same method as in the proof of Lemma 3.2, we obtain that the connected component J of $J(G)$ with $z_0 \in J$ satisfies $J = \{z_0\}$. By the statement 3, we obtain $J = \{z_0\} = J_{\min}$. Combining this and Lemma 3.2, we obtain the statement 6.

Next, we show the statement 7. Suppose that G is of type (I)(a) or (II)(a)(i). Let A be an annulus separating $J(G)$. Then A separates $J_{\max}$ and $J_{\min}$. Let D be the unbounded component of $\mathbb{C} \setminus J_{\min}$ and let U be the connected component of $\mathbb{C} \setminus J_{\max}$containing $J_{\min}$. Then it follows that $A \subset U \cap D$. By the statement 2 and 5, we have $\sharp J_{\max} > 1$ and $\sharp J_{\min} > 1$. Hence, the doubly connected domain $U \cap D$ satisfies mod $(U \cap D) < \infty$. Hence, we obtain mod $A \leq$ mod $(U \cap D) < \infty$. Hence, $J(G)$ is uniformly perfect. Next, suppose that G is of type (I)(b) or (II)(a)(ii) or (II)(b). By Theorem 4.1 in [HM2] and the statement 6, we obtain that $J(G)$ is not uniformly perfect. Hence, we have shown the statement 7.

Next, we show the statement 8. By Theorem 4.1 in [HM2] and the statement 7, we obtain $z_1 \in\text{int } J(G)$. Furthermore, by the statement 3, we obtain $z_1 \in J_{\min}$. Hence $z_1 \in \text{int } J_{\min}$. By the statement 5b, we obtain $J(g) \subset J_{\min}$. Hence, we have shown the statement 8.

Hence, we have shown Theorem 1.9. □

3.2 Proofs of results in 1.2

Proposition 3.3. *Let $G \in \mathcal{G}$ and let $\{h_\lambda\}_{\lambda\in\Lambda}$ be a generator system of G. Let $\lambda_1, \lambda_2 \in \Lambda$ and let $J_i \in \mathcal{J}$ be an element containing $J(h_{\lambda_i})$ for each $i = 1, 2$. Suppose $J_1 \leq J_2$. Let*

$$Q = \{g \in G \mid \exists J \in \mathcal{J} \text{ with } J_1 \leq J \leq J_2,\ J(g) \subset J\}$$

and let H be the subsemigroup of G generated by Q. Then, we have $J(H) \subset ((\overline{\mathbb{C}} \setminus A_2) \cap A_1) \cup J_1$, where A_i denotes the unbounded component of $\mathbb{C} \setminus J_i$ for each $i = 1, 2$.

Proof. Let $K = J(G) \cap (((\overline{\mathbb{C}} \setminus A_2) \cap A_1) \cup J_1)$. Let $g \in Q$ and let $J \in \mathcal{J}$ be an element containing $J(g)$. Let $J_3 \in \mathcal{J}$ be an element with $J \leq J_3 \leq J_2$. We show the following:

Claim1 : $g^*(J_3) \leq J_3$. To show the claim, suppose $J_3 < g^*(J_3)$. Then by Theorem 1.7-3, we have $J \leq J_3 < g^*(J_3) \leq (g^n)^*(J_3)$ for each $n \in \mathbb{N}$. Hence, $\inf\{d(z, J) \mid z \in g^{-n}(J_3),\ n \in \mathbb{N}\} > 0$. But since $J(g) \subset J$, we obtain a contradiction. Hence the claim holds.

Similarly we obtain the following:
Claim 2: For any element $J_4 \in \mathcal{J}$ with $J_1 \leq J_4 \leq J$, we have $J_4 \leq g^*(J_4)$.

By Claim 1 and 2, we obtain that $g^{-1}(K) \subset K$ for each $g \in Q$. By Lemma 2.1-6, it follows that $J(H) \subset K$. □

Proof of Proposition 1.13: By Theorem 1.4 and Proposition 3.3. □

Lemma 3.4. *Let $f : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product over $g : X \to X$ such that f_x is a polynomial with $d(x) \geq 2$, for each $x \in X$. Let $(x, y) \in F_x(f)$ and let $q_x^{(n_j)}(y) := \pi_{\overline{\mathbb{C}}} f_x^{n_j}((x, y))$ be a sequence of rational maps which converges to a nonconstant map around y. Then there exist a domain V in $\mathbb{C}$, a point x_∞ in X, and a number $k \in \mathbb{N}$ such that $\{x_\infty\} \times \partial V \subset \tilde{J}(f) \cap P(f)$ and $q_x^{(n_j)}(y) \in V$ for each j with $j \geq k$.*

Proof. By the proof of Lemma 2.13 in [S1]. □

Proof of Theorem 1.14-1: Let (x, y_0) be a point in a bounded component of $F_x(f)$. By Lemma 3.4 and $h_j(\hat{K}(G)) \subset \mathrm{int}\ \hat{K}(G)$ for each $j \in \{1, \cdots, m\} \setminus B_{\min}$ (Theorem 1.9-5b), we obtain that there exists no non-constant limit function of the sequence $(q_x^{(n)})_{n \in \mathbb{N}}$ $(q_x^{(n)}(y) := \pi_{\overline{\mathbb{C}}} f_x^n((x, y)))$ around y_0. Since $P(G) \setminus \{\infty\} \cap J(G) \subset J_{\min}$ (Theorem 1.9-3), we obtain that the statement 1b is true. From the statement 1b, we obtain the statement 1c. By the lower semi-continuity of $x \mapsto J_x(f)$ (Lemma 2.3-2), we obtain the statement 1a. By Theorem 1.7, we obtain $h_j^{-1}(J(G)) \cap J_{\min} = \emptyset$ for each $j \in \{1, \cdots, m\} \setminus B_{\min}$. Combining this with Theorem 1.9-3 and the Koebe distortion theorem, we obtain the statement 1d. □

Proof of Theorem 1.14-2: We can easily obtain the following claim:
Claim1: $\hat{K}(H_{\min}) = \hat{K}(G)$.

By Claim 1 and $h_j(\hat{K}(G)) \subset \mathrm{int}\ \hat{K}(G)$ for each $j \in \{1, \cdots, m\} \setminus B_{\min}$ (Theorem 1.9-5b), we can easily obtain that if $H_{\min}$ is semi-hyperbolic, then G is semi-hyperbolic. If G is semi-hyperbolic, then by [S4], we obtain that for each $x \in X$, the unbounded component $A_x(f)$ of $F_x(f)$ $(A_x(f) = \overline{\mathbb{C}} \setminus K_x(f))$ is a John domain. Since $A_x(f)$ is simply connected for each $x \in X$ (Lemma 2.4), it follows that $J_x(f) = \partial A_x(f)$ (Lemma 2.3-3) is locally connected ([NV]). Hence, combining this with the statement 1a and $J_x(f) = \partial A_x(f)$ (Lemma 2.3-3), we obtain Theorem 1.14-2. □

Theorem 3.5. *(**Uniform fiberwise quasiconformal surgery**) Let $f : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product over $g : X \to X$ such that f_x is a polynomial with $d(x) \geq 2$, for each $x \in X$. Suppose that f is hyperbolic and* *there exists only one bounded component of $F_x(f)$, for each $x \in X$. Then, there exists a constant K such that for each $x \in X$, $J_x(f)$ is a K-quasicircle.*

Proof. Step 1: By [S1], the map $x \mapsto J_x(f)$ is continuous with respect to the Hausdorff topology. Hence, there exists a positive constant C_1 such that for each $x \in X$, $d(J_x(f),\ \pi^{-1}(x) \cap P(f)) > C_1$. Since X is compact, it follows that for each $x \in X$, there exists an analytic Jordan curve γ_x in $K_x(f) \cap F_x(f)$ such that:

1. $\pi^{-1}(x) \cap P(f)$ is included in a bounded component V_x of $\pi^{-1}(x) \setminus \gamma_x$.
2. $\inf_{z \in \gamma_x} d(z,\ J_x(f) \cup (\pi^{-1}(x) \cap P(f))) \geq C_2$, where C_2 is a positive constant independent of $x \in X$.
3. There exist finitely many Jordan curves $\tau_1, \cdots, \tau_k$ in $\mathbb{C}$ such that for each $x \in X$, there exists a j with $\pi_{\overline{\mathbb{C}}}(\gamma_x) = \tau_j$.

Step 2: By Theorem 2.14 (5) in [S1], there exists an $n \in \mathbb{N}$ such that for each $x \in X$, $W_x := (f_x^n)^{-1}(V_{\sigma^n(x)}) \supset V_x$ and mod $(W_x \setminus \overline{V_x}) \geq C_3$, where C_3 is a positive constant independent of $x \in X$. Since $J_x(f^n) = J_x(f)$, we may assume $n = 1$.
Step 3: For each $x \in X$, let $\varphi_x : \pi^{-1}(x) \setminus \overline{V_x} \to \pi^{-1}(x) \setminus \overline{D(0, \frac{1}{2})}$ be the Riemann map such that $\varphi_x((x, \infty)) = (x, \infty)$, under the identification $\pi^{-1} \cong \overline{\mathbb{C}}$. φ_x can be extended analytically to $\partial V_x = \gamma_x$. We define a quasi-regular map $h_x; \pi^{-1}(x) \to \pi^{-1}(\sigma(x))$ as follows:

$$h_x(z) := \begin{cases} \varphi_{\sigma(x)} f_x \varphi_x^{-1}(z), & z \in \varphi_x(\pi^{-1}(x) \setminus W_x) \\ z^{d(x)}, & z \in \overline{D(0, \frac{1}{2})} \\ \tilde{h}_x(z), & z \in \varphi_x(W_x) \setminus \overline{D(0, \frac{1}{2})}, \end{cases}$$

where $\tilde{h}_x : \varphi_x(W_x) \setminus \overline{D(0, \frac{1}{2})} \to \overline{D(0, \frac{1}{2})} \setminus D(0, (\frac{1}{2})^{d(x)})$ is a regular covering and a K-quasiregular map, where K is a constant independent of $x \in X$.
Step 4: For each $x \in X$, we define a Beltrami differential $\mu_x(z)\frac{d\bar{z}}{dz}$ on $\pi^{-1}(x)$ as follows:

$$\begin{cases} (h_{\sigma^n(x)} \cdots h_x)^*\left(\frac{\partial_{\bar{z}} \tilde{h}_{\sigma^n(x)}}{\partial_z \tilde{h}_{\sigma^n(x)}} \frac{d\bar{z}}{dz}\right), & z \in (h_{\sigma^n(x)} \cdots h_x)^{-1}(\varphi_{\sigma^n(x)}(W_{\sigma^n(x)}) \setminus \overline{D(0, \frac{1}{2})}) \\ \frac{\partial_{\bar{z}} \tilde{h}_{\sigma^n(x)}}{\partial_z \tilde{h}_{\sigma^n(x)}} \frac{d\bar{z}}{dz}, & z \in \varphi_x(W_x) \setminus \overline{D(0, \frac{1}{2})} \\ 0, & \text{otherwise.} \end{cases}$$

Then, there exists a constant k with $0 < k < 1$ such that for each $x \in X$, $\|\mu_x\|_\infty \leq k$. By the construction, we have $g_x^* \mu_{\sigma(x)} = \mu_x$, for each $x \in X$. Let $\psi_x : \pi^{-1}(x) \to \pi^{-1}(x)$ be a quasiconformal map such that $\partial_{\bar{z}} \psi_x = \mu_x \partial_z \psi_x$, $\psi_x(0) = 0$, $\psi_x(1) = 1$, and $\psi_x(\infty) = \infty$, under the identification $\pi^{-1}(x) \cong \overline{\mathbb{C}}$. Let $\hat{h}_x := \psi_{\sigma(x)} h_x \psi_x^{-1} : \pi^{-1}(x) \to \pi^{-1}(\sigma(x))$. Then, $\hat{h}_x$ is holomorphic on $\pi^{-1}(x)$. By the construction, we see that $\hat{h}_x(z) = c(x)z^{d(x)}$, where $c(x) = \psi_{\sigma(x)} h_x \psi_x^{-1}(1) = \psi_{\sigma(x)} g_x(1)$. Furthermore, by the construction again, we see that there exists a positive constant C_4 such that for each $x \in X$, $\frac{1}{C_4} \leq |g_x(1)| \leq C_4$. Hence, there exists a positive constant C_5 such that for each $x \in X$, $\frac{1}{C_5} \leq |c(x)| \leq C_5$. Let $\tilde{J}_x$ be the set of non-normality of the sequence $(\hat{h}_{\sigma^n(x)} \cdots \hat{h}_x)_n$ in π_x. Then, by $\hat{h}_x(z) = c(x)z^{d(x)}$ and $\frac{1}{C_5} \leq |c(x)| \leq C_5$ for each $x \in X$, we obtain that $\tilde{J}_x$ is a round circle. Under the identification of $\pi^{-1}(x) \cong \overline{\mathbb{C}}$, we have that the family $\{\psi_x\}_{x \in X}$ is normal in $\overline{\mathbb{C}}$. Hence, $J_x(f) = \varphi_x^{-1} \psi_x^{-1} \tilde{J}_x$ and it follows that there exists a constant K such that for each $x \in X$, $J_x(f)$ is a K-quasicircle. □

Proof of Theorem 1.14-3: Since $P(G) \setminus \{\infty\} \cap J(G) \subset J_{\min}$ (Theorem 1.9-3, it is easy to see $(\pi_{\overline{\mathbb{C}}} \hat{J}_x(f)) \cap P(G) = \emptyset$ for each $x \in X$. Hence, $\overline{f}$ is a hyperbolic skew product. Combining this with Theorem 1.14-1a and Theorem 3.5, we obtain that there exists a constant K_s such that for each $x \in W_s$, $J_x(\overline{f})$ is a K_s-quasicircle. □

Proof of Theorem 1.15: Since $J(G) = \overline{\cup_{g \in G} J(g)}$ (Theorem 2.2), there exists an element $h_1 \in G$ with $J(h_1) \cap V \neq \emptyset$. By Theorem 1.4, there exists an element $h_2 \in G$ such that the Julia set of $G_1 = \langle h_1, h_2 \rangle$ is disconnected. By Theorem 1.14-3, we can find two elements g_1 and g_2 in G_1 satisfying all of the conditions in the statement in Theorem 1.15. □

References

[GR] Z.Gong and F.Ren, A random dynamical system formed by infinitely many functions, Journal of Fudan University, **35**, 1996, 387-392.

[HM1] A.Hinkkanen and G.J.Martin, *The Dynamics of Semigroups of Rational Functions I*, Proc.London Math.Soc. (3)**73**(1996), 358-384.

[HM2] A.Hinkkanen, G.J.Martin, *Julia Sets of Rational Semigroups* , Math.Z. **222**, 1996, no.2, 161-169.

[J] M.Jonsson, *Ergodic properties of fibered rational maps* , Ark.Mat., 38 (2000), pp 281-317.

[N] S.B.Nadler, *Continuum Theory: An introduction*, Marcel Dekker, 1992.

[NV] R.Näkki and J.Väisälä, *John Discs,* Expo.math.9(1991), 3-43.

[S1] H.Sumi, *Dynamics of sub-hyperbolic and semi-hyperbolic rational semigroups and skew products*, Ergod. Th. and Dynam. Sys.(2001), **21**, 563-603.

[S2] H.Sumi, *A correction to the proof of a lemma in 'Dynamics of sub-hyperbolic and semi-hyperbolic rational semigroups and skew products'*, Ergod.Th.and Dynam.Sys.(2001), **21**, 1275-1276.

[S3] H.Sumi, *Skew product maps related to finitely generated rational semigroups,* Nonlinearity, **13**, (2000), 995-1019.

[S4] H.Sumi, *Semi-hyperbolic fibered rational maps and rational semigroups,* http://www.math.titech.ac.jp/~sumi/welcome-e.html, preprint 2001.

[St] R.Stankewitz, *Uniformly perfect sets, rational semigroups, Kleinian groups and IFS's* , Proc.Amer.Math.Soc. **128**, (2000), No.9, 2569-2575.

[SY] Y.Sun and C-C.Yang. *On the connectivity of the Julia set of a finitely generated rational semigroup*, Proc. Amer.Math.Soc., Vol.130, No.1, 49-52, 2001.

[TP] Tan Lei and K.Pilgrim, *Rational maps with disconnected Julia set,* Asterisque 261, (2000).

[ZR] W.Zhou, F.Ren, The Julia sets of the random iteration of rational functions, Chinese Bulletin, **37**(12), 1992, 969-971.