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The Foliated Geodesic Flow on Riccati Equations *

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Abstract

We introduce the geodesic flow on the leaves of a holomorphic foliation with leaves of dimension 1 and hyperbolic, corresponding to the unique complete metric of curvature -1 compatible with its conformal structure. We do these for the foliations associated to Riccati equations, which are the projectivisation of the solutions of a linear ordinary differential equations over a finite Riemann surface of hyperbolic type S, and may be described by a representation $\rho : \pi_1(S) \to GL(n, \mathbb{C})$. We give conditions under which the foliated geodesic flow has a generic repellor-attractor statistical dynamics. That is, there are measures μ^+ and μ^- such that for almost any initial condition with respect to the Lebesgue measure class the statistical average of the foliated geodesic flow converges for negative time to μ^- and for positive time to μ^+ (i.e. μ_+ is the unique SRB-measure and its basin has total Lebesgue measure). These measures are ergodic with respect to the foliated geodesic flow.

Introduction

The objective of this work is to propose a method for understanding the statistical properties of the leaves of a holomorphic foliation, and which may be carried out for a simple class of holomorphic foliations: those obtained from the solutions of Riccati Equations. The method consists in using the canonical metric of curvature -1 that the leaves have as Riemann surfaces, the Poincaré metric, and then to flow along foliated geodesics. One is interested in understanding the statistics of this foliated geodesic flow. In particular, in determining if the foliated geodesic flow has an SRB-measure (for Sinaï , Ruelle and Bowen [11], [10], [3]), or physical measure, which means that a set of geodesics of positive Lebesgue measure have a convergent time statistics, which is shared by all the geodesics in this set, called the basin of attraction of the SRB-measure. The SRB-measure is the spatial measure describing this common time statistics of a significant set of geodesics.

The Riccati equations are projectivisations of linear ordinary differential equations over a finite hyperbolic Riemann surface S (i.e. compact minus a finite number of points and with universal cover the upper half plane). Locally they have the form

$$\frac{dw}{dz} = A(z)w \quad , \quad w \in \mathbb{C}^n \ , \ z \in \mathbb{C} \quad , \quad A : \mathbb{C} \to Mat_{n,n}(\mathbb{C})$$

with A holomorphic. These equations may be equivalently defined by giving the monodromy representations

$$\tilde{\rho} : \pi_1(S, z_0) \to GL(n, \mathbb{C}) \qquad , \qquad \rho : \pi_1(S, z_0) \to PGL(n, \mathbb{C}) \tag{1}$$

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and suspending them, to obtain flat $\mathbb{C}^n$ and $\mathbb{C}P^{n-1}$ bundles over S

$$E_{\tilde{\rho}} \to S, \qquad , \qquad M_\rho \to S. \tag{2}$$

The graphs of the local flat sections of these bundles are the 'solutions' to the linear differential equation defined by the monodromy (1) and define holomorphic foliations $\mathcal{F}_{\tilde{\rho}}$ and $\mathcal{F}_\rho$ of $E_{\tilde{\rho}}$ and M_ρ whose leaves $\mathcal{L}$ project as a covering to the base surface S.

Introduce to the finite hyperbolic Riemann surface S the Poincaré metric, to the unit tangent bundle $q : T^1S \to S$ the geodesic flow $\varphi : T^1S \times \mathbb{R} \to T^1S$ and the Liouville measure $dLiouv$ (hyperbolic area element in S and Haar measure on T^1_pS, normalised to volume 1). We may introduce on the leaves $\mathcal{L}$ of the foliations $\mathcal{F}_{\tilde{\rho}}$ and $\mathcal{F}_\rho$ the Poincaré metric, which is the pull back of the Poincaré metric of S by the covering map $q : \mathcal{L} \to S$. The unit tangent bundle $T^1_{\mathcal{F}_{\tilde{\rho}}}$ to the foliation $\mathcal{F}_{\tilde{\rho}}$ in $E_{\tilde{\rho}}$ is canonically isomorphic to the vector bundle $q^*E_{\tilde{\rho}}$ over T^1S, that we denote by E. In the same way the unit tangent bundle $T^1_{\mathcal{F}_\rho}$ of the foliation $\mathcal{F}_\rho$ is canonically identified to the projectivisation $Proj(E)$ of the vector bundle E over T^1S. Introduce on E and on $Proj(E)$ the foliated geodesic flows $\tilde{\Phi}$ and Φ (see (2.2)), obtained by flowing along the foliated geodesics. Introduce also on E a continuous Hermitian inner product $|\,.\,|_v$.

Given a vector $v \in T^1S$ we have the geodesic

$$\mathbb{R} \to T^1S \qquad , \qquad t \to \varphi(v,t)$$

determined by the initial condition v and given $w_0 \in E_v$ we also have the foliated geodesic

$$\mathbb{R} \to E \qquad , \qquad t \to \tilde{\Phi}(w_0,t)$$

which is the solution to the linear differential equation defined by (1) along the foliated geodesic determined by v and w_0. The function

$$t \to |\tilde{\Phi}(w_0,t)|_{\varphi(v,t)}$$

describes the type of growth of the solution of (1) along the geodesic γ_v with initial condition $w_0 \in E_v$ and the function

$$t \to \frac{|\tilde{\Phi}(w_1,t)|_{\varphi(v,t)}}{|\tilde{\Phi}(w_2,t)|_{\varphi(v,t)}}$$

describes the relative growth of the solution of (1) along the geodesic γ_v with initial condition $w_1 \in E_v$ with respect to the growth of the solution of (1) along γ_v with the initial condition $w_2 \in E_v$.

We say that the Riccati equation has a **section of largest expansion** σ^+ if for Liouville almost any point v on T^1S we may measurably define a splitting $E_v = F_v \oplus G_v$ by linear spaces, which is invariant by the foliated geodesic flow $\tilde{\Phi}$ with F_v of dimension 1 and with the property that the map $t \to \tilde{\Phi}(w_1,t)$ with initial condition $w_1 \in F_v$ grows more rapidly than the maps $t \to \tilde{\Phi}(w_2,t)$ for any $w_2 \in G_v$. That is, for almost any $v \in T^1S$, for any compact set $K \subset T^1S$ and for any sequence $(t_n)_{n\in\mathbb{N}}$ of times such that $\varphi(v,t_n) \in K$ and $\lim_{n\to\infty} t_n = +\infty$, one has:

$$\lim_{n\to\infty} \frac{|\tilde{\Phi}(w_1,t_n)|_{\varphi(v,t_n)}}{|\tilde{\Phi}(w_2,t_n)|_{\varphi(v,t_n)}} = \infty, \quad \text{for all non-zero} \quad w_1 \in F_v, \quad \text{and} \quad w_2 \in G_v.$$

So the section of largest expansion is defined as $\sigma^+ := Proj(F) : T^1S \to Proj(E)$. Similarly, we may define a **section σ^- of largest contraction** (see (3.1)).

An elementary argument of Linear Algebra suggests that a section $\sigma^+ = Proj(F)$ of largest expansion is attracting all the points in $Proj(E) - Proj(G)$ as they flow according to the action of the foliated geodesic flow Φ. In fact, we prove:

Theorem 1. *Let S be a finite hyperbolic Riemann surface and $\tilde{\rho} : \pi_1(S, z_0) \to GL(n, \mathbb{C})$ a representation having a section σ^+ of largest expansion, then $\mu^+ = \sigma_*^+(dLiouv)$ is a Φ−invariant ergodic measure on $T^1\mathcal{F}_\rho$ which is an SRB-measure for the foliated geodesic flow Φ of the Riccati equation, whose basin has total Lebesgue measure in $T^1\mathcal{F}_\rho$. Similarly, if σ^- is the section of largest contraction, then $\mu^- = \sigma_*^-(dLiouv)$ is a Φ−invariant ergodic measure which is an SRB-measure whose basin has total Lebesgue measure in $T^1\mathcal{F}_\rho$, for negative times.*

In the case that both $\sigma^\pm$ exist, the foliated geodesic flow has a very simple 'north to south pole dynamics': almost everybody is being born in μ^- and is dying on μ^+. If the sections $\sigma^\pm$ are continuous disjoint sections defined on all T^1S then it is easy to imagine this north to south pole dynamics (see section 7 for an example). If $\sigma^\pm$ are only measurable, then they describe more subtle phenomena.

The Lyapunov exponents measure the exponential rate of growth (for the metric $|\ .\ |_v$ in the vectorial fibers) of the solutions of the linear equation along the geodesics (definition 4.2):

$$\lim_{t \to \pm\infty} \frac{1}{t} \log |\tilde{\Phi}(w_0, t)|_{\varphi(v,t)}.$$

Let S be a finite hyperbolic Riemann surface, $\tilde{\rho}$: $\pi_1(S, z_0) \to GL(n, \mathbb{C})$ a representation and E the previously constructed bundle. The association of initial conditions to final conditions for the linear equation in E over the geodesic flow of S, after a measurable trivialisation of the bundle, gives rise to a measurable multiplicative cocycle over the geodesic flow on T^1S

$$\tilde{A} : T^1S \times \mathbb{R} \longrightarrow GL(n, \mathbb{C})$$

(see (2.4)). The integrability condition

$$\int_{T^1S} log^+ \|\tilde{A}_{\pm 1}\| dLiouv < +\infty, \tag{3}$$

where $\|\ \|$ is the operator norm and $\tilde{A}_t := \tilde{A}(\cdot, t)$, asserts that the amount of expansion of $\tilde{A}_\pm 1$ is Liouville integrable.

As a consequence of the multiplicative Ergodic Theorem of Oseledec applied to the foliated geodesic flow we obtain:

Corollary 2. *Let S be a finite hyperbolic Riemann surface, $\tilde{\rho}$: $\pi_1(S, z_0) \to GL(n, \mathbb{C})$ a representation and let $\tilde{A}$ be the measurable multiplicative cocycle over the geodesic flow on T^1S satisfying the integrability condition* (3)*, then:*

- *The Lyapunov exponents $\lambda_1 < \cdots < \lambda_k$ of $\tilde{\Phi}$ are well defined and are constant on a subset of T^1S of total Liouville measure. Denote by $F_i(v)$ the corresponding Lyapunov spaces.*
- *For every $i \in \{1, \ldots, k\}$, $\lambda_{k+1-i} = -\lambda_i$ and $dim(F_{k+1-i}) = dim(F_i)$.*
- *If $dimF_k = 1$, denote by σ^+ the section corresponding to F_k and σ^- the section correponding to F_1, then $\sigma^\pm$ are sections of largest expansion and contraction, respectively.*

From now on by the *Lyapunov exponents of the linear equation obtained from the representation* $\tilde{\rho}$ we will understand the Lyapunov exponents of the above multiplicative cocycle $\tilde{A}$ over the geodesic flow on T^1S obtained from the foliated geodesic flow on E and satisfying the integrability condition (3). The relationship between the section of largest expansion and the Lyapunov exponents is:

Theorem 3. *Let S be a finite hyperbolic Riemann surface, $\tilde{\rho}\colon \pi_1(S, z_0) \to GL(n, \mathbb{C})$ a representation satisfying the integrability condition (3), then there exists a section of largest expansion if and only if the largest Lyapunov exponent is positive and simple, if and only if the smallest Lyapunov exponent is negative and simple, and if and only if there is a section of largest contraction.*

So, a section of largest expansion is an extension for non-integrable cocycles $\tilde{A}$ of the notion of having a simple largest Lyapunov exponent. We give an example of this in section 6.

In order to apply Oseledec's Theorem, the prevailing hypothesis is the integrability condition (3). This condition is always satisfied if the base Riemann surface is compact, and more generally:

Theorem 4. *If S is a finite hyperbolic Riemann surface, $\tilde{\rho}$ a representation* (1) *then the multiplicative cocycle $\tilde{A}$ satisfies the integrability condition* (3) *if and only if the monodromy $\tilde{\rho}$ around each of the punctures of S corresponds to a matrix with all its eigenvalues of norm 1.*

This paper is organised as follows. In section 1 we recall the Riccati equations and in section 2 we set up the foliated geodesic flow on Riccati equations. In section 3 we introduce SRB-measures and prove Theorem 1. In section 4 we prove Corollary 2 and Theorem 3.

We now describe the results obtained in parallel but independent works ([1], [2]). If S is a compact hyperbolic Riemann surface, then the foliated geodesic flow is a linear or projective multiplicative cocycle over a hyperbolic dynamical system. This led us to think that it could be possible to adapt Fustenberg's theory of the existence of a positive Lyapunov exponent for random products of matrices. This has been carried out in [2]. That is, we give a condition on the representation so that if S is a compact hyperbolic Riemann surface, then there is a positive and a negative Lyapunov exponent. Morover, this condition is satisfied for an open and dense set in the space of representations for $n = 2$ or 3. Hence Theorem 1 implies that for the general representation with $n = 2$ or 3 the foliated geodesic flow of the Riccati equation has a unique SRB-measure.

Assume that S is a finite hyperbolic Riemann surface, that the image of the representation $\rho : \pi_1(S) \to PSL(2, \mathbb{C})$ does not leave invariant a probability measure on $\mathbb{C}P^1$ and that the multiplicative cocycle satisfies the integrability condition (3), then it is shown in [1] that projecting the measures $\mu^{\pm}$ to the projective bundle M_ρ over S gives rise to a measure ν which describes effectively the statistical behaviour of the leaves of the foliation $\mathcal{F}_\rho$: For any compact set $K \subset M_\rho$, for any sequence $(x_n \in K)_{n\in\mathbb{N}}$ and any sequence of real numbers $(r_n)_{n\in\mathbb{N}}$ tending to $+\infty$ the family of probability measures $\nu_{r_n}(x_n)$ obtained by normalizing the area element on the disk $D_{r_n}(x_n)$ in the leafwise Poincaré metric converges towards ν for the weak topology when n tends to $+\infty$.

1 The Riccati Equation

1.1 Linear Ordinary Differential Equations

The classical linear ordinary differential equation is

$$\frac{dw}{dz} = A(z)w \qquad , \qquad z \in \mathbb{C},\ w \in \mathbb{C}^n \tag{1.1}$$

where $A(z)$ is a matrix of rational functions (see [4]). The fundamental property of this equation is that locally in z we can find a basis of independent solutions of (1.1) which accept analytic continuation to the universal covering space of $S := \bar{\mathbb{C}} - \text{poles}(A)$ as holomorphic vector valued functions w satisfying the monodromy relation:

$$w(T_\gamma(z)) = \tilde{\rho}(\gamma)(w(z)) \qquad , \qquad \gamma \in \pi_1(S, z_0)$$

where T_γ is the covering transformation corresponding to the close loop γ and

$$\tilde{\rho} : \pi_1(S, z_0) \to GL(n, \mathbb{C}) \tag{1.2}$$

is the monodromy representation of the equation. The linear automorphism $\tilde{\rho}(\gamma) : \mathbb{C}^n \to \mathbb{C}^n$ contains the information of how the initial conditions are transformed to final conditions by solving (1.1) along the closed loop γ based at z_0.

Another classical construction of linear ordinary differential equations is the suspension ([8]). Assume given a hyperbolic Riemann surface S and a representation (1.2). We construct from this data a vector bundle $E_{\tilde{\rho}}$ over S and an equation of type (1.1). Let $\mathbb{H}^+$ be the upper half plane, considered as the universal covering space of S, with covering transformations (4) giving rise to the canonical representation $\tilde{\rho}_{can}$ of the fundamental group of S. Consider the trivial bundle $\tilde{E} := \mathbb{H}^+ \times \mathbb{C}^n$ on the upper half plane $\mathbb{H}^+$ and the $\pi_1(S, z_0)$-action on $\tilde{E}$

$$(z, w) \to (\tilde{\rho}_{can}(\gamma)z, \tilde{\rho}(\gamma)w) \qquad , \qquad \gamma \in \pi_1(S, z_0). \tag{1.3}$$

The quotient of $\tilde{E}$ by this action gives rise to a vector bundle $E_{\tilde{\rho}}$ over S. On $\tilde{E}$ we can consider the equation given by $\tilde{A} = 0$ (i.e. $\frac{dw}{dz} = 0$). Its solutions are the constant functions. Since this equation $\tilde{A}$ is invariant under the action in (1.3), it descends to a linear ordinary differential equation on $E_{\tilde{\rho}}$ which is holomorphic over S. The construction gives directly that the monodromy transformation of this equation is the given representation $\tilde{\rho}$. The graphs of the local solutions to (1.1) form a holomorphic foliation $\mathcal{F}_{\tilde{\rho}}$ in $E_{\tilde{\rho}}$.

1.2 The Riccati Equation

Riccati equations may be obtained from a linear ordinary differential equation as (1.1) or (1.2) by projectivising the linear variables of the vector bundle $E_{\tilde{\rho}}$ over the Riemann surface S. Denoting $\zeta_j := \frac{w_j}{w_1}$ with $j = 2, \ldots, n$, the Riccati equation associated to (1.1) in affine coordinates takes the form of a quadratic polynomial in $\zeta_2, \ldots, \zeta_n$ with rational coefficients in z:

$$\begin{pmatrix} \frac{d\zeta_2}{dz} \\ \cdots \\ \frac{d\zeta_n}{dz} \end{pmatrix} = \begin{pmatrix} a_{21} \\ \cdots \\ a_{n1} \end{pmatrix} + \begin{pmatrix} a_{22} - a_{11} & a_{23} & \cdots \\ a_{32} & a_{33} - a_{11} & \cdots \\ \cdots & \cdots & a_{nn} - a_{11} \end{pmatrix} \begin{pmatrix} \zeta_2 \\ \cdots \\ \zeta_n \end{pmatrix} - (a_{12}\zeta_2 + \cdots + a_{1n}\zeta_n) \begin{pmatrix} \zeta_2 \\ \vdots \\ \zeta_n \end{pmatrix} \tag{1.4}$$

where $A = (a_{ij}(z))$ is the matrix of rational functions in (1.1). Similarly, given a representation $\tilde{\rho}$ as in (1.2) we may also construct from the projectivised representation ρ in (1) its suspension $M_\rho = Proj(E_{\tilde{\rho}})$ which gives a manifold which is a $\mathbb{C}P^{n-1}$ bundle over S with a flat connection. The set of flat sections form a foliation $\mathcal{F}_\rho$ of M_ρ which is the projectivisation of the foliation $\mathcal{F}_{\tilde{\rho}}$ in $E_{\tilde{\rho}}$. The foliations so constructed, will be called **Riccati foliations**.

2 The Foliated Geodesic Flow on Linear and Riccati Equations

2.1 The Geodesic Flow on Finite Hyperbolic Riemann Surfaces

We say that S is a finite hyperbolic Riemann surface if S is conformally equivalent to $\bar{S}-\{p_1,\dots,p_r\}$, where $\bar{S}$ is a compact Riemann surface of genus g and $g>1$ or $g=1$ with $r\geq 1$ or $g=0$ with $r\geq 3$. In such a case S has as a universal covering space the Poincaré upper half plane $\mathbb{H}^+$, with its complete metric of curvature -1 given by $ds=\frac{|dz|}{y}$. We introduce on S the hyperbolic metric induced by the Poincaré metric via the universal covering map. For the measure associated to the hyperbolic metric, the surface S has finite area.

Let T^1S be the unit tangent bundle of S. The Liouville measure $dLiouv$ on T^1S is the measure obtained from the hyperbolic area element in S and Haar measure $d\theta$ on unit vectors, normalised so as to have volume 1. The geodesic flow

$$\varphi: T^1S\times\mathbb{R}\to T^1S \qquad \varphi_t:=\varphi(\ ,t) \tag{2.1}$$

is obtained by flowing along the geodesics (see [6] p. 209). The geodesic flow leaves invariant the Liouville measure.

Theorem 2.1 (Hopf-Birkhoff). ([6] p. 217, 136) *Let S be a finite hyperbolic Riemann surface, then the Liouville measure is ergodic with respect to the geodesic flow and the generic geodesic of S is statistically distributed in T^1S according to the Liouville measure: For all Liouville integrable functions h on T^1S and for almost any $v_z\in T^1S$ with respect to the Liouville measure*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t h(\varphi(v_z,t))dt=\int_{T^1S} hdLiouv$$

2.2 The Foliated Geodesic Flows

Let S be a finite hyperbolic Riemann surface, and $\tilde{\rho}$ and ρ representation as in (1) and let $\mathcal{F}_{\tilde{\rho}}$ and $\mathcal{F}_\rho$ be the foliations constructed in section 1. If $\mathcal{L}$ is a leaf of the foliation $\mathcal{F}_{\tilde{\rho}}$ or $\mathcal{F}_\rho$, then the projection map $p:\mathcal{L}\to S$ is a covering map, and hence the pull back of the Poincaré metric of S induces a metric to the leaves of $\mathcal{F}$, which coincides with the Poincaré metric of each leaf $\mathcal{L}$ of $\mathcal{F}$. This is the Poincaré metric of the foliations $\mathcal{F}_{\tilde{\rho}}$ or $\mathcal{F}_\rho$.

Let $T^1\mathcal{F}_{\tilde{\rho}}$ be the manifolds formed by those tangent vectors to $E_{\tilde{\rho}}$ and M_ρ which are tangent to $\mathcal{F}_{\tilde{\rho}}$ and $\mathcal{F}_\rho$ and are of unit length with respect to the Poincaré metrics of the foliations. The derivative of the projection map $E_{\tilde{\rho}}, M_\rho\to S$ induces the commutative diagram

$$\begin{array}{ccc} T^1\mathcal{F}_{\tilde{\rho}} & \xrightarrow{q} & E_{\tilde{\rho}} \\ \downarrow & & \downarrow \\ T^1S & \xrightarrow{q} & S \end{array} \qquad\qquad \begin{array}{ccc} T^1\mathcal{F}_{\rho} & \to & M_{\rho} \\ \downarrow & & \downarrow \\ T^1S & \to & S \end{array}$$

The foliated geodesic flows $\tilde{\Phi}$ and Φ are defined by following geodesics along the leaves and is compatible with the geodesic flow φ on S, giving rise to the commutative diagram

$$\begin{array}{ccccccc} \tilde{\Phi}: & T^1\mathcal{F}_{\tilde{\rho}} & \times & \mathbb{R} & \to & T^1\mathcal{F}_{\tilde{\rho}} \\ & \downarrow & & \downarrow & & \downarrow \\ \varphi: & T^1S & \times & \mathbb{R} & \to & T^1S \end{array} \quad , \quad \begin{array}{ccccccc} \Phi: & T^1\mathcal{F}_{\rho} & \times & \mathbb{R} & \to & T^1\mathcal{F}_{\rho} \\ & \downarrow & & \downarrow & & \downarrow \\ \varphi: & T^1S & \times & \mathbb{R} & \to & T^1S \end{array} . \tag{2.2}$$

For any $v\in T^1S$ and $t\in\mathbb{R}$, the flow $\tilde{\Phi}_t:=\tilde{\Phi}(\ ,t)$ induces a linear isomorphism

$$\tilde{A}(v,t):=\tilde{\Phi}(v,\ ,t)|_{E_{\tilde{\rho},v}}:E_{\tilde{\rho},v}\to E_{\tilde{\rho},\varphi(v,t)}. \tag{2.3}$$

between the $\mathbb{C}^n$-fibres. After a measurable trivialisation of the bundles by choosing measurably an othonormal basis of the fibers, the foliated geodesic flows may be seen as measurable multiplicative cocycles over the geodesic flow on T^1S:

$$\tilde{A} : T^1S \times \mathbb{R} \to GL(n, \mathbb{C}) \quad , \quad \tilde{A}(v, t_1 + t_2) = \tilde{A}(\varphi(v, t_1), t_2)\tilde{A}(v, t_1) \quad , \quad t_1, t_2 \in \mathbb{R}. \tag{2.4}$$

Moreover the usual operator norm in $GL(n, \mathbb{C})$ coincides with the operator norm of (2.3) as Hermitian spaces with the metrics induced from the fibre bundle metric.

3 SRB-measures for Riccati Equations

3.1 SRB-measures

Let M be a differentiable manifold. The Lebesgue measure class is the set of measures whose restriction on any chart U has a smooth strictly-positive Radom-Nikodyn derivative with respect to $dx_1 \wedge dx_2 \cdots \wedge dx_n$ where the x_i are coordinates on U. A set $E \subset M$ has *zero Lebesgue measure* if there is a measure μ in the Lebesgue class such that $\mu(E) = 0$.

Let X be a complete vector field on the manifold M, and denote by φ_t its flow. A probability measure μ on M is *invariant by* X if for any $t \in \mathbb{R}$ one has $\varphi_{t*}(\mu) = \mu$. The **basin** $B(\mu)$ of an $X-$invariant probability μ is the set of points $x \in M$ such that the positive time average along its orbit of any continuous function $h \colon M \to \mathbb{R}$ with compact support coincides with the integral of the function by μ. In formula:

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T h(\varphi_t(x))dt = \int_M h d\mu$$

.

Definition 3.1. *An $X-$invariant probability measure in M is an* **SRB-measure** *if its basin has non-zero Lebesgue measure in M.*

3.2 Key Idea to Build SRB-measures for Riccati Equations

Let S be a finite hyperbolic Riemann surface and $\tilde{\rho}$ and ρ representations as in (1) and $\mathcal{F}_{\tilde{\rho}}$ and $\mathcal{F}_\rho$ the foliations in $E_{\tilde{\rho}}$ and M_ρ constructed in section 2. Consider a continuous Hermitian metric $|\cdot|_x$ on the fiber $E_{\tilde{\rho},x}$ of $E_{\tilde{\rho}}$ and for each point $x \in S$ we endow the corresponding Fubini-Study (Riemannian) metric $|\cdot|_x$ on $M_{\rho,x} = Proj(E_{\tilde{\rho},x})$. The bundles $q^*E_{\tilde{\rho}} \simeq T^1\mathcal{F}_{\tilde{\rho}}$ and $q^*M_\rho \simeq T^1\mathcal{F}_\rho$ over T^1S are endowed in a natural way with the induced Hermitian or Fubini-Study metric, respectively.

Definition 3.2. *Under the above setting, assume that the vector bundle $E\colon = T^1\mathcal{F}_{\tilde{\rho}} \to T^1S$ admits a measurable splitting $E_v = F_v \oplus G_v$, defined for v in a subset $\mathcal{A}$ of T^1S, and verifying the following hypothesis:*

1. *$\mathcal{A}$ has total Lebesgue measure in T^1S;*
2. *$\mathcal{A}$ is invariant by the geodesic flow φ;*
3. *the splitting is invariant by the foliated geodesic flow $\tilde{\Phi}$: for every $t \in \mathbb{R}$ and every $v \in \mathcal{A}$,*

$$F_{\varphi(v,t)} = \tilde{\Phi}(F_v, t) \quad and \quad G_{\varphi(v,t)} = \tilde{\Phi}(G_v, t);$$

4. *$dim(F_v) = 1$;*

5. *for any* $v \in \mathcal{A}$, *for any compact set* $K \subset T^1S$ *and for any sequence* $(t_n)_{n\in\mathbb{N}}$ *of times such that* $\varphi(v,t_n) \in K$ *and* $\lim_{n\to\infty} t_n = +\infty$, *one has:*

$$\lim_{n\to\infty} \frac{|\tilde{\Phi}(w_1,t_n)|_{\varphi(v,t_n)}}{|\tilde{\Phi}(w_2,t_n)|_{\varphi(v,t_n)}} = \infty, \quad \textit{for all non-zero} \quad w_1 \in F_v, \quad \textit{and} \quad w_2 \in G_v.$$

Under the above hypothesis denote by $\sigma^+\colon \mathcal{A} \subset T^1S \to T^1\mathcal{F}_\rho$ *the mesurable section defined by letting* $\sigma^+(v)$ *be the point of* $Proj(E_v)$ *corresponding to the line* F_v. *A section* σ^+ *verifying the above hypothesis is called a* **section of largest expansion**.

Similarly, one defines the **section σ^- of largest contraction** by requiring

$$\lim_{n\to\infty} \frac{|\tilde{\Phi}(w_1,t_n)|_{\varphi(v,t_n)}}{|\tilde{\Phi}(w_2,t_n)|_{\varphi(v,t_n)}} = 0, \quad \text{for all non-zero} \quad w_1 \in F_v, \quad \text{and} \quad w_2 \in G_v. \tag{3.1}$$

where we are imposing the condition that the measurable sub-bundle F is 1 dimensional.

Proof of Theorem 1: σ^+ induces an isomorphism of the measure $dLiouv$ and $\mu^+ = \sigma^+_* dLiouv$, so that the invariance and the ergodicity of μ^+ follow from those of $dLiouv$ and of σ^+.

Let $h\colon T^1\mathcal{F}_\rho \to \mathbb{R}$ be a continuous function with compact support, and denote by K the projection of this compact set on T^1S. The function $h \circ \sigma^+\colon T^1S \to \mathbb{R}$ is measurable and bounded, so it belongs in $\mathcal{L}^1(dLiouv)$. As the Liouville measure is a φ ergodic probability on T^1S, there is an invariant set $Y_h \subset T^1S$ of total Lebesgue measure such that, for $v \in Y_h$, the average

$$\frac{1}{T}\int_0^T h \circ \sigma^+(\varphi(v,t))dt \;\to\; \int_{T^1S} h \circ \sigma^+ dLiouv = \int_{T^1\mathcal{F}_\rho} hd\mu^+. \tag{3.2}$$

For each $v \in Y_h$ we denote by $\mathcal{Y}_h(v)$ the set of points in the fiber $y \in Proj(E_v)$ corresponding to a line of $E_v \setminus G_v$. We denote by $\mathcal{Y}_h$ the union $\mathcal{Y}_h = \bigcup_{v\in Y_h} \mathcal{Y}_h(v) \subset M_\rho$.The set $\mathcal{Y}_h$ is invariant by Φ because Y_h is invariant by φ and the bundle G is $\tilde{\Phi}$−invariant. By Fubini's theorem, the set $\mathcal{Y}_h$ has total Lebesgue measure in M_ρ.

Claim . *For every* $w \in \mathcal{Y}_h$, *the average* $\frac{1}{T}\int_0^T h(\tilde{\Phi}(w,t))dt$ *converges to* $\int_{T^1\mathcal{F}_\rho} hd\mu^+$

Before proving the claim let us show that this concludes the proof of Theorem 1: There is a countable family h_i, $i \in \mathbb{N}$ of continuous functions with compact support which is dense (for the uniform topology) in the set of all continuous functions of $T^1\mathcal{F}$ with compact support. Look now at the set $\mathcal{Y} = \bigcap_0^\infty \mathcal{Y}_{h_j}$: It is invariant by Φ, has total Lebesgue measure, and is contained in the basin of μ^+ by the claim. This proves Theorem 1.

Now we prove the claim: Let $w \in \mathcal{Y}_h(v)$, for some $v \in Y_h$, and denote $w_0 = \sigma^+(v)$. As the section σ^+ is invariant by the foliated geodesic flow, for any t, $\Phi(w_0,t) = \sigma^+(\varphi(v,t))$; so for any $T \in \mathbb{R}$ the averages $\frac{1}{T}\int_0^T h \circ \Phi(w_0,t)dt$ and $\frac{1}{T}\int_0^T h \circ \sigma^+(\varphi(v,t))dt$ are equal and we get by (3.2)

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T h(\Phi(w_0,t))dt = \int_{T^1\mathcal{F}_\rho} hd\mu^+.$$

Consider a non-zero vector $\tilde{w}$ in the linear space E_v in the line corresponding to w. We can write in a unique way $\tilde{w} = \tilde{w}_0 + \tilde{w}_1$ where $\tilde{w}_0 \in F_v$ and $\tilde{w}_1 \in G_v$. Notice that $\tilde{w}_0 \neq 0$ projects on $w_0 \in Proj(E_v)$. By hypothesis 5 in Definition 3.2, when $t \in \mathbb{R}$ is very large, either $\varphi_t(v) \notin K$ or

$\frac{|\tilde{\Phi}(w_0,t_n)|_{\varphi(v,t_n)}}{|\tilde{\Phi}(w_1,t_n)|_{\varphi(v,t_n)}}$ is very large and so the distance (for the Fubini-Study metrics) between $\Phi(w,t)$ and $\Phi(w_0,t)$ is very small, goes to zero. Now we decompose the averages $\frac{1}{T}\int_0^T h(\varphi_t(w))dt$ in two parts, one corresponding to the times t such that $\varphi(v,t) \notin K$, and the other to the times such that $\varphi(v,t) \in K$. The first part is uniformly zero (for both w and w_0). Moreover for large t such that $\varphi(v,t) \in K$, the difference $h(\Phi(w_0,t)) - h(\Phi(w,t))$ goes to zero. So the averages of h along the orbits of w and w_0 converge to the same limit, which is $\int_{T^1\mathcal{F}_\rho} hd\mu^+$. □

Remark 3.3. 1. The existence of a section of largest expansion does not depend of the choice of the continuous Hermitian metrics on the fibers.

2. Theorem 1 does not use our specific hypothesis (2-dimensional basis, geodesic flow, holomorphic foliation). One has:

   Theorem 1′: *Let B be a manifold and φ a flow on B admiting an ergodic invariant probability λ which is absolutely continuous (with strictly positive density) with respect to Lebesgue measure. Let $\tilde{\rho}$: $\pi_1(B) \to GL(n,\mathbb{C})$ be a representation, $(E_{\tilde{\rho}}, \tilde{\mathcal{F}}_{\tilde{\rho}})$ be the vector bundle endowed with the suspension foliation, and $M_\rho = (Proj(E_{\tilde{\rho}}), \mathcal{F}_\rho)$ the suspension of the corresponding representation ρ: $\pi_1(B) \to PGL(n,\mathbb{C})$. Let Φ be the lift of the flow φ to the leaves of $\mathcal{F}_\rho$. If the bundle $E_{\tilde{\rho}}$ admits a section σ^+ of largest expansion then $\sigma^+_*(\lambda)$ is an SRB-measure of the flow Φ, whose basin has total Lebesgue measure in M_ρ.*

3. The geodesic flow (and the foliated geodesic flow) have a symmetry: denote by I the involution map on the unit tangent bundle sending each vector v to $-v$ and $\tilde{I}$ the involution $\tilde{I}(w_v) = -w_v$ on $T^1\mathcal{F}_{\tilde{\rho}}$. Then I is a conjugation between the geodesic flow and its inverse $I \circ \varphi_t \circ I = \varphi_{-t}$. This shows that $\sigma^- = \tilde{I} \circ \sigma^+ \circ I$ is a section of largest expansion for the negative geodesic flow, and $\mu^- = \sigma^-_*(dLiouv)$ will be an SRB-measure for the negative orbits of the geodesic flow. Then Lebesgue almost every orbit in $T^1\mathcal{F}$ has negative average converging to μ^- and positive average converging to μ^+.

Proposition 3.4. *Let $E_{\tilde{\rho}} = F \oplus G$ be a $\tilde{\Phi}$-invariant measurable splitting giving rise to a section of largest expansion $\sigma^+ := proj(F)$, then the decomposition is measurably unique (i.e. over a set of full Liouville measure in T^1S).*

Proof: Let $E_{\tilde{\rho}} = F_1 \oplus G_1$ be a $\tilde{\Phi}$-invariant measurable splitting giving rise to a section of largest expansion, $\sigma_1^+ := proj(F_1)$. The line bundle F_1 is not contained in G, for if it were contained, then the order of growth of σ^+ would be larger than the order of growth of σ_1^+. But then G_1 would not be a subset of G and any initial condition in $G_1 - G$ has the same order of growth than σ^+, which is larger than the order of growth of sections in G, like σ_1^+, contradicting that the order of growth of σ_1^+ is larger than the order of growth of any section in G_1.

Assume that $F \neq F_1$. For $\varepsilon > 0$ define the subset

$$H_\varepsilon := \{v \in T^1S \;/\; dist(\sigma^+(v), G_v) > \varepsilon \;,\; dist(\sigma_1^+(v), G_v) > \varepsilon \;,\; dist(\sigma^+(v), \sigma_1^+(v)) > \varepsilon \;\}$$

where the distances are measured in the Fubini-Study metrics of $Proj(E_v)$. For small ε the set H_ε will have positive Liouville measure. But since the Liouville measure is ergodic, almost all points in H_ε are recurrent. But this cannot be, since both σ^+ and σ_1^+ are invariant and as time increases the component in F_v grows much more than the component on G_v so that in $Proj(E_v)$ the sections σ^+ and σ_1^+ are getting closer which contradicts the condition $dist(\sigma^+(v), \sigma_1^+(v)) > \varepsilon$. Hence we must have $F = F_1$ (Liouville almost everywhere), as well as $\sigma^+ = \sigma_1^+$. Now G is uniquely determined by σ^+, since any section outside G has the same order of growth as σ^+, and those on G have smaller order of growth. □

4 Using Oseledec's Theorem

4.1 A Corollary of Oseledec's Theorem

Let

$$f : B \to B \qquad , \qquad A : B \to GL(n, \mathbb{C})$$

be measurable maps. For any $n \in \mathbb{N}$ and any $x \in B$ we denote

$$A^n(x) = A(f^{n-1}(x)) \cdots A(f(x))A(x) \text{ and } A^{-n}(x) = [A^n(f^{-n}(x))]^{-1}.$$

One says that the family $\{A^n\}$ form a multiplicative cocycle over f.

Definition 4.1. *A point $x \in B$ has Lyapunov exponents for the multiplicative cocycle $\{A^n\}$ over f if there exists $0 < k \leq n$ and for all $i \in \{1, \ldots, k\}$ there is $\lambda_i \in \mathbb{R}$ and a subspace F_i of $\mathbb{R}^n$ such that:*

1. $\mathbb{R}^n = \bigoplus_i F_i$
2. *For any i and any non zero vector $v \in F_i$ one has*

$$\lim_{n \to \pm\infty} \frac{1}{n} \log(|A^n(v)|) = \pm\lambda_i$$

Oseledec's Multiplicative Ergodic Theorem ([6, p.666-667]): Let $f : B \to B$ be an invertible measurable transformation, μ an $f-$invariant probability measure and A a mesurable multiplicative cocycle over f. Assume that the functions $\log^+ \|A\|$ and $\log^+ \|A^{-1}\|$ belong to $\mathcal{L}^1(\mu)$. Then the set of points for which the Lyapunov exponents of A are well defined has μ-measure 1. If μ is ergodic the Lyapunov exponents are independent of the point in a set of total $\mu-$measure.

The Lyapunov exponents and the Lyapunov spaces above depend measurably of $x \in B$ on a set of $\mu-$total measure (see [6, p.666-667]). When the measure μ in Oseledec's Theorem is ergodic, we can then speak of the *Lyapunov exponents of the measure μ*.

We want to use Oseledec's Theorem for flows when the base manifold is non-compact. Let φ be a complete flow on the manifold B, $\pi : E \to B$ a vector bundle over B and $\tilde{\Phi}$ be a flow on E inducing a multiplicative cocycle as in (2.4) over φ.

Definition 4.2. *We say that the Lyapunov exponents of $\tilde{\Phi}$ are well defined at a point $v \in B$ if there is a continuous Euclidean or Hermitian metric on the bundle E, a finite sequence $\lambda_1 < \cdots < \lambda_k$ and a $\tilde{\Phi}$-invariant splitting $E(v) = F_1(v) \oplus \cdots \oplus F_k(v)$ such that, for any non zero vector $w \in F_i(v)$, any compact $K \subset B$ and any sequence $\{t_n\}_{n \in \mathbb{Z}}$ with $lim_{n \to \pm\infty} t_n = \pm\infty$ and $\varphi(v, t_n) \in K$ one has:*

$$\lim_{n \to \pm\infty} \frac{1}{t_n} \log(|\tilde{\Phi}(w, t_n)|) = \pm\lambda_i.$$

The existence and the value of the Lyapunov exponents does not depend of the continuous metric on the vector bundle E; moreover we can allow the metric to be discontinuous if the change of metric to a continuous reference metric is bounded on compact sets of the basis B.

Lemma 4.3. *With the notation above the Lyapunov exponents of $v \in B$ for the flow $\tilde{\Phi}$ are well defined if and only if they are well defined for the multiplicative cocycle $\{\tilde{A}_1^n\}$ over φ_1 defined by the diffeomorphism $\tilde{\Phi}_1$. Moreover the Lyapunov exponents and spaces are equal for the flow and the diffeomorphism.*

Proof: One sense is evident, so we will assume that the diffeomorphism Φ_1 has Lyapunov exponents on v. As the flow φ is complete, for any compact set $K \subset B$ the union $K_1 = \bigcup_{t\in[-1,1]} \varphi(K,t)$ is compact. Moreover for each t_n such that $\varphi(v,t_n) \in K$, let T_n be the integer part of t_n, then $\varphi_1^{t_n-T_n}(v) \in K_1$. We conclude the proof noticing that

$$\tilde{A}(v,t_n) = \tilde{A}(\varphi_1^{t_N-T_n}(v),T_n)\tilde{A}(v,t_n-T_n)$$

and that the norm of $\tilde{A}(*,T_n)$ is uniformly bounded over K_1 independently of $t_n - T_n \in [0,1]$. □

Definition 4.4. *Let μ be a $\varphi-$invariant probability on B. We say that the flow $\tilde{\Phi}$ defining a measurable multiplicative cocycle* (2.4) *is $\mu-$integrable if there is a continuous norm $|\cdot|$ on the vector bundle E such that the functions $\log^+\|\tilde{A}_1\|$ and $\log^+\|\tilde{A}_{-1}\|$ belong to $\mathcal{L}^1(\mu)$, where $\|\ \|$ is the operator norm on the normed vector spaces.*

The condition of integrability of the norm of the multiplicative cocycle is always verified if the manifold B is compact.

Proof of Corollary 2: Consider $f = \varphi_1$, the time 1 of the geodesic flow on T^1S, and let $\tilde{A}(v)\colon E_v \to E_{f(v)}$ the linear multiplicative cocycle induced on the vector bundle $T^1\mathcal{F}_{\tilde{\rho}}$ by $\tilde{\rho}$ in Oseledec's Theorem. By hypothesis, this multiplicative cocycle is integrable so that the Lyapunov exponent of the multiplicative cocycle $\tilde{A}$ are well defined for a Liouville total measure set by Lemma 4.3. The Lyapunov exponents and spaces depend measurably of $v \in T^1S$ which are invariant respectively by φ and $\tilde{\Phi}$. As the Liouville measure is ergodic, the Lyapunov exponents are constant on a set of total Liouville measure. This ends the proof of item 1.

The proof of item 2 is a direct consequence of the symmetry of the flow Φ: $\tilde{I}\circ\Phi_t\circ\tilde{I} = \Phi_{-t}$ (see item 3 in remark 3.3). With the hypothesis of item 3 the section σ^+ is clearly a section of largest expansion so that item 3 is a direct consequence of Theorem 1. □

A direct corollary of Theorem 1′ and Oseledec's Theorem is the following

Corollary 2′: *Let f be a diffeomorphism of a manifold B, admitting an invariant ergodic probability λ in the class of Lebesgue and let E be an $n-$dimensional vector bundle over the basis B and M the corresponding projective bundle. Assume that $\tilde{\Psi}$ is a diffeomorphism of E leaving invariant the linear fibration, inducing linear maps on the fibers and whose projection on B is the diffeomorphism f. We denote by Ψ the induced diffeomorphism on M.*

Let U_i be a covering of B by trivializing charts of the bundle E: then writing $\tilde{\Psi}$ in these charts we get a multiplicative cocycle $\tilde{A}\colon B \to GL(n,\mathbb{C})$. Assume that $log^+\|\tilde{A}\|$ and $log^+\|\tilde{A}^{-1}\|$ belong to $\mathcal{L}^1(\lambda)$ and that the largest Lyapunov exponent of the measure λ for the multiplicative cocycle $\tilde{A}$ corresponds to a 1 dimensional space. Denote by σ^+ the corresponding measurable section defined on a Lebegue total measure set of B to M.

Then $\sigma_*^+(\lambda)$ is an SRB-measure for Ψ and its basin has total Lebesgue measure in M. □

4.2 Proof of Theorem 3

Proof: Due to Corollary 2 and the Remark 3.2, the only thing that remains to be proved is that, under the integrablity condition (3), if there is a section of largest expansion then the largest Lyapunov exponent is positive and simple.

We begin first with the case that S is compact. So assume that there is a section σ^+ of largest expansion providing a measurable decomposition $E_{\tilde{\rho}} = F \oplus G$, $\sigma^+ := Proj(F)$ and let λ_i and F_i be the Lyapunov exponents and spaces as in Corollary 2. We have $F \subset F_k$, corresponding to the greatest eigenvalue λ_k, and denote by H the measurable bundle $F_k \cap G$ of dimension $n_k - 1$. Assume that the dimension n_k of F_k is at least 2, and we will argue to obtain a contradiction to this assumption.

Since the foliated geodesic flow is leaving invariant the measurable bundle F_k, after a measurable trivialisation we will obtain a measurable cocycle

$$B : T^1S \times \mathbb{R} \to GL(n_k, \mathbb{C})$$

which carries the information of how initial conditions are transformed into final conditions, when starting from the point $v \in T^1S$, $w \in F_{k,v}$, and flowing a time t along the geodesic.

Recall that we have introduced a Hermitian metric on the bundle $E_{\tilde{\rho}}$, by pull back in the bundle $q^*E_{\tilde{\rho}} = T^1\mathcal{F}_{\tilde{\rho}}$ and by restriction into the bundle F_k. Recall also that if we have a $\mathbb{C}$-linear map L between Hermitian spaces, the determinant $det(L, W)$ of L on a subspace W is by definition the quotient of the volumes of the paralelograms determined by $Lw_1, \ldots, Lw_m, iLw_1, \ldots, iLw_m$ and $w_1, \ldots w_m, iw_1, \ldots, iw_m$ corresponding to any $\mathbb{C}$-basis $w_1, \ldots, w_m$ of W. Define

$$\Delta^m : T^1S \to \mathbb{R} \qquad , \qquad \Delta^m(v) := \frac{det(B(v,m), F_v)^{n_k-1}}{det(B(v,m), H_v)}$$

and note that the cocycle condition (2.4) for B and the $\tilde{\Phi}$-invariance of H and F gives the multiplicative condition

$$\Delta^m(v) = \Delta(\varphi(v, m-1))\Delta(\varphi(v, m-2)) \cdots \Delta(v) \quad , \quad \Delta := \Delta^1. \tag{4.1}$$

The volume in H has exponential rate of growth $(n_k - 1)\lambda_k$, since it is the Lyapunov exponent of $\Lambda^{n_k-1}H$. The exponential rate of growth of F is λ_k, hence

$$\int_{T^1S} \log(\Delta) dLiouv = (n_k - 1)\lambda_k - \lambda_k - \ldots - \lambda_k = 0. \tag{4.2}$$

Now we need the following corollary of a general statement from Ergodic Theory, (see [7], Corollary 1.6.10):

Corollary 4.5. *Let $\varphi : B \to B$ be a measurable transformation preserving a probability measure ν in B, and $g : B \to \mathbb{R}$ a ν-integrable function such that $\lim_{n\to\infty} \sum_{j=0}^{n}(g \circ \varphi^j) = \infty$ at ν-almost every point, then $\int_B g d\nu > 0$.*

Proof: Consider the set

$$A := \{v \in T^1S \ / \ \sum_{j=0}^{\ell}(g \circ \varphi^j)(v) > 0, \ \forall \ell \geq 0\},$$

and for $v \in A$ let

$$S_*g(v) := \inf_{\ell}\{\sum_{j=0}^{\ell}(g \circ \varphi^j)(v)\}.$$

A has a strictly positive ν measure since almost any orbit will have a point in A, and S_*g is a measurable function on A which is strictly positive. By Corollary 1.6.10 in [7] we have

$$\int_B g d\nu = \int_A S_*g d\nu,$$

but this last number is strictly positive, since we are integrating a strictly positive function over a set of positive measure. $\square$

We want to apply the above Lemma to $(X, \nu) = (T^1S, dLiouv)$ and $g = \log\Delta$. Note that the multiplicative relation (4.1) implies

$$\sum_{j=0}^{m-1} \log\Delta(\varphi_j(v)) = \log\Delta^m(v) \tag{4.3}$$

The hypothesis on the growth of the section σ^+ implies that $\lim_{n\to\infty} \log\Delta^m(v) \to \infty$. But on using (4.3) this is the hypothesis in the Lemma, so as a conclusion of it we obtain that

$$\int_{T^1S} \log(\Delta) dLiouv > 0,$$

which contradicts (4.2). Hence F_k has dimension 1, so that the largest Lyapunov exponent is simple.

Assume now that S is not compact. According to Lemma 4.3, it is sufficient to consider the integrability condition for the time 1 flow φ_1. Let K be a compact set of positive Liouville measure in T^1S and partition

$$K_m := \{v \in K \ / \ \varphi^j(v) \notin K, j = 1, \cdots, m-1, \ \varphi^m(j) \in K\}$$

according to the time of the first return to K. Define the multiplicative cocycle generated by

$$C : K \to GL(n, \mathbb{C}) \quad , \quad C(v) := \tilde{A}_1^m(v) \quad , \quad v \in K_m$$

corresponding to the first return map to K. Since

$$C(v) = \tilde{A}_1(\varphi^{m-1}(v)) \ldots \tilde{A}_1(\varphi(v))\tilde{A}_1(v),$$

we have

$$\log^+(\|C(v)\|) \leq \log^+(\|\tilde{A}_1(\varphi^{m-1}(v))\|) + \ldots + \log^+(\|\tilde{A}_1(\varphi(v))\|) + \log^+(\|\tilde{A}_1(v)\|)),$$

and hence on K we obtain

$$\sum_{m=1}^{\infty} \int_{K_m} \log^+(\|C(v)\|) \leq \sum_{m=1}^{\infty} [\log^+(\|\tilde{A}_1(\varphi^{m-1}(v))\|) + \ldots + \log^+(\|\tilde{A}_1(\varphi(v))\|) + \log^+(\tilde{A}_1(v))] \leq$$

$$\leq \int_{T^1S} \log^+(\|\tilde{A}_1(v)\|)$$

since the sets

$$\varphi_j(K_m) \quad , \quad j = 0, \ldots, m-1, \quad , \quad m = 1, \ldots$$

are disjoint. Hence the cocycle generated by C is integrable, and we may repeat the argument presented for the case that T^1S is compact.

□

5 Using Oseledec's Theorem in the Non-compact case

The objective of this paragraph is to prove Theorem 4. The proof of the parts "if" and "only if" are given by some estimates over the punctured disc $\mathbb{D}^*$. As both proofs are long, we will treat them separately. The common argument is the following estimate about the geodesic flow of $\mathbb{D}^*$.

5.1 Estimates on the Geodesic Flow on a Punctured Disc

Denote by $\mathbb{D}^*$ the punctured disc endowed with the usual complete metric of curvature -1, that is, its universal cover is the Poincaré half plane $\mathbb{H}^+$ with covering group generated by the translation $T(z) = z+1$ and define

$$D^* := \frac{\{z \in \mathbb{H}^+ \ / \ Im(z) > 1\}}{(T^n)} \subset \mathbb{D}^* \qquad , \qquad S^1 := \partial D^* = \frac{\{z \in \mathbb{H}^+ \ / \ Im(z) = 1\}}{(T^n)} \subset \mathbb{D}^*.$$

$$\bar{D}^* := \frac{\{z \in \mathbb{H}^+ \ / \ Im(z) \geq 1\}}{(T^n)} \subset \mathbb{D}^*$$

A unit vector $u \in T^1D^*$ at a point $z \in D^*$ is called a *radial vector* if $u \in \mathbb{R}w\frac{\partial}{\partial w}$. Note that for any non-radial vector $u \in T^1D^*$ the geodesic γ_u through u in $\bar{D}^*$ is a compact segment γ_u whose extremities are on the circle S^1. We will denote the tangent vector of the geodesic γ_u on S^1 by $\alpha(u)$ (the incoming) and $\omega(u)$ (the outgoing), and let $t(u)$ be the lenght of γ_u. The set of radial vectors has zero Lebesgue measure. We will denote by M the set of nonradial unit vectors on $T^1\mathbb{D}^*|_{\bar{D}^*}$ and by N the subset of M over the circle S^1. We denote N^+ the set of vectors in N pointing inside D^* and by

$$\mathcal{A} = \{(u,t), u \in N^+, t \in [0, t(u)]\} \subset N^+ \times [0, +\infty[.$$

The geodesic flow φ on $T^1\mathbb{D}^*$ induces a natural map $F \colon \mathcal{A} \to M$ defined by $F(u,t) = \varphi(u,t)$. The unit tangent bundle over S^1 admits natural coordinates : If u is a unit vector at w we will denote $\theta(u)$ the argument of w, and $\eta(u)$ the angle between u and the radial vector $-z\partial/\partial z$. We denote by μ the measure on $\mathcal{A}$ defined by $d\mu = \cos(\eta) \cdot d\theta \wedge d\eta \wedge dt$

Lemma 5.1. *The Liouville measure on T^1D^* is $F_*(d\mu)$ (up to a multiplicative constant).*

Proof: The measure $F_*^{-1}(dLiouv) := hd\theta \wedge d\eta \wedge dt$ for a certain function h. Since the Liouville measure is invariant under the geodesic flow, and in M the geodesic flow has the expression $\frac{\partial}{\partial t}$, then h is independent of t. Since the Liouville measure is invariant under rotations in θ then h is also independent of θ. Hence h is only a function of η. To compute the value of h it is enough to compute for an arbitrary η at a point in N^+. We have $F_*(d\theta \wedge dt) = h(\eta)dArea$. The variable θ is parametrizing according to geodesic length and since the angle between the vertical and the geodesic at $Im(z) = 1$ is η, we project the tangent vector to the geodesic to the vertical direction to obtain the weight $cos(\eta)$. □

We will denote by μ_0 the measure on N^+ defined by $d\mu_0 = d\theta \wedge d\eta$.

Proposition 5.2. *Let $\tilde{A}_t : T^1D^* \times \mathbb{R} \to GL(n, \mathbb{C})$ be a linear multiplicative cocycle over the geodesic flow of D^*. For every unit vector $u \in N^+$, we denote*

$$B : N^+ \to GL(n, \mathbb{C}) \qquad , \qquad B(u) = \tilde{A}_{t(u)}(u)$$

the matrix corresponding to the geodesic γ_u of length $t(u)$ going from $\alpha(u)$ to $\beta(u)$. Then the two following sentences are equivalent:

1. *There is a Hermitian metric $|\cdot|$ on the vector bundle over T^1D^* such that the multiplicative cocycle $\tilde{A}_1$ is integrable for Liouville, that is*

$$\int_{T^1D^*} \log^+ \|\tilde{A}_{\pm 1}\| dLiouv < +\infty. \tag{5.1}$$

2. The function $\log^+(\|B\|)$ *belongs to* $\mathcal{L}^1(\mu_0)$*, that is*

$$\int_{N^+} \log^+(\|B(u)\|)d\mu_0 < +\infty. \tag{5.2}$$

Remark 5.3. (5.2) *does not depend of the choice of the continuous Euclidean metric : Two continuous Hermitian metrics* $|\cdot|_1$ *and* $|\cdot|_2$ *on the bundle over* $T^1\mathbb{D}^*|_{\partial D^*}$ *are equivalent because* ∂D^* *is compact, so that the difference* $|\log(\|B(u)\|_1)| - |\log(\|B(u)\|_2)|$ *is uniformly bounded on* N^+.

Proof: For every $u \in N^+$ set $t_u := t(u)$, and divide the interval $[0, t_u]$ in

$$[0,1] \cup [1,2] \cup \cdots \cup [E(t_u)-1, E(t_u)] \cup [E(t_u), t_u],$$

so that if u is a vector at a point $x \in \partial D^*$ one gets on setting $\varphi := \varphi_1$ the geodesic flow at time 1:

$$B(u) = \tilde{A}_{t_u - E(t_u)}(\varphi^{E(t_u)}(u) \circ \prod_{0}^{E(t_u)-1} \tilde{A}_1(\varphi^i(u))$$

So for any Hermitian norm $|\cdot|$ we get

$$\|B(u)\| \leq \|\tilde{A}_{t_u - E(t_u)}(\varphi^{E(t_u)}(u))\| \prod_{0}^{E(t_u)-1} \|\tilde{A}_1(\varphi^i(u))\|$$

So

$$\log^+(\|B(u)\|) \leq \log^+ \|\tilde{A}_{t_u - E(t_u)}(\varphi^{E(t_u)}(u))\| + \sum_{0}^{E(t_u)-1} \log^+ \|\tilde{A}_1(\varphi^i(u))\|$$

Remark that $\log^+ \|\tilde{A}_{t_u - E(t_u)}(\varphi^{E(t_u)}(u))\|$ is uniformly bounded by a constant K depending on $\tilde{A}$ and $|\cdot|$, because $t_u - E(t_u) \in [0,1[$ and $\varphi^{E(t_u)}(u) = \varphi_{E(t_u)-t_u}(\varphi_{t_u}(u))$ remains in a compact set (recall that $\varphi_{t_u}(u) \in \partial D^*$). So we get that there is a constant K_1 such that for every $u \in N^+$ one has

$$\log^+(\|B(u)\|) \leq K_1 + \int_0^{t_u} \log^+ \|\tilde{A}_1(\varphi_t(x))\| dt$$

Notice now that, for any $\varepsilon \in [0,1[$ there is $\delta > 0$ such that if $\cos(\eta) \leq \varepsilon$ then $t_u \leq \delta$. So it is equivalent that the function $\log^+(\|B\|)$ is integrable for the measure $d\mu_0$ or for $\cos(\eta)d\theta \wedge d\eta$.

Hence we obtain that if $\int_{N^+} \log^+(\|B\|)d\mu_0 = +\infty$ then for any Riemannian metric $|\cdot|_2$ the function $\log^+(\|\tilde{A}_1\|_2)$ is not Liouville integrable. We have proven that item 1 $\implies$ item 2.

For the other implication, choose a continuous Riemannian metric on the bundle over N, assume the integrability condition (5.2) and let $v \in T^1\mathbb{D}|_{\bar{D}^*}$. If v is a radial vector, then push forward the metric over $\alpha(v)$ along the geodesic using the flat structure of the bundle. If v is not a radial vector then push forward the metric on $\alpha(v)$ on the first third of γ_v, on the last third of the geodesic push forward the metric on $\omega(u)$ and on the middle third of γ_u put the corresponding convex combination of the metrics on $\alpha(u)$ and $\omega(u)$. This produces a continuous metric on the bundle over $T^1\mathbb{D}|_{\bar{D}^*}$ such that $\|\tilde{A}_{\pm 1}\|$ does not expand except in the middle part, and there it expands in a constant way. Hence for this metric the integral (5.2) coincides with (5.1). □

To use Proposition 5.2 we will need to estimate $\|B(u)\|$, $u \in N^+$. For that we will use the following estimate of t_u and the estimate of the variation of the argument along the geodesic γ_u:

Proposition 5.4. *1. There is a constant T such that $t_u \in [-2\log|\eta| - T, -2\log|\eta| + T]$.*

2. Denote by a_u the variation of the argument along γ_u. Then $a_u = 2\frac{\cos\eta}{\sin\eta}$

Proof: The easiest way is to look at the universal cover $\mathbb{H}$. Recall that in this model the geodesic for the hyperbolic metric are circles or straight lines (for the Euclidean metric) orthogonal to the real line. Let $u \in E_1^+$ at a point $x \in \partial D^*$. Denote by u the corresponding vector at a point $\tilde{x} \in \mathbb{H}$, $Im(x) = 1$, where $\tilde{x}$ is a lift of x. The angle $\eta(u)$ is the angle between the vector and the vertical line. Consider the geodesic $\tilde{\gamma}_u$ throught u. The Euclidean radius R_u of this circle verifies $1 = |\sin(\eta)| \cdot R_u$. Now denote by $\tilde{y} \neq \tilde{x}$ the intersection point of $\tilde{\gamma}_u$ with the boundary $Im(z) = 1$ of D^*. Then $a_u = \tilde{y} - \tilde{x} = 2\frac{\cos(\eta)}{\sin(\eta)}$. So the second item of Proposition 5.4 is proved.

To give an estimate of t_u let us consider the following curve σ_u joining the points $\tilde{x}$ and $\tilde{y}$: $\tilde{\sigma}_u$ is the union of the vertical segment σ_u^1 joining $\tilde{x} = (\mathcal{R}e(\tilde{x}), 1)$ to $(\mathcal{R}e(\tilde{x}), R_u)$ the horizontal segment σ_2^u joining $(\mathcal{R}e(\tilde{x}), R_u)$ to $(\mathcal{R}e(\tilde{y}), R_u)$ and the vertical segment σ_u^3 joining $(\mathcal{R}e(\tilde{y}), R_u)$ to $(\mathcal{R}e(\tilde{y}), 1) = \tilde{y}$.

The hyperbolic length of the vertical segments is $\log(R_u)$. The hyperbolic length of the horizontal segment is $\frac{|a_u|}{R_u} = 2cos(\eta)$. So we get:

$$\ell(\tilde{\gamma}_u) < \ell(\sigma_u) = -2\log(|\sin(\eta)|) + 2\cos(\eta)$$

On the other hand, consider the point $z_u \in \gamma_u$ whose imaginary part is R_u. This point is the middle of the horizontal segment of σ_u. Denote by γ_u^0 the segment of γ_u joining $\tilde{x}$ to z_u and σ_u^0 the segment of σ_u^2 joining zu to the point $(\mathcal{R}e(\tilde{y}), R_u)$. The union of these 2 segments is a segment joining the two extremities of σ_u^1 which is a geodesic. So we get

$$-\log(|\sin(\eta)|) < \ell(\gamma_u^0) + \ell(\sigma_u^0) = \frac{1}{2}\ell(\tilde{\gamma}_u) + \cos(\eta).$$

So we get

$$t_u = \ell(\tilde{\gamma}_u) \in [-2\log(|\sin(\eta)|) - 2\cos(\eta), -2\log(|\sin(\eta)|) + 2\cos(\eta)]$$

So

$$t_u \in [-2\log(|\sin(\eta)|) - 2, -2\log(|\sin(\eta)|) + 2]$$

To conclude the first item it is enought to note that $|\log(|\eta|) - \log(|\sin(\eta)|)|$ is bounded for $\eta \in [-\pi/2, \pi/2]$. □

5.2 The Parabolic Case

Proposition 5.5. *If for each i all the eigenvalues of $\rho(\gamma_i)$ have modulus 1, then the multiplicative cocycle flow is integrable.*

As the function $\log^+|\tilde{A}_1|$ is continuous, it is integrable for the Liouville measure over every compact set of T^1S. So the problem is purely local, in the neighbourhood of the punctures of S. So it is enough to look at a multiplicative cocycle $\tilde{A}_t$ over the geodesic flow of the punctured disc D^*. The proposition is a direct corollary of the following proposition:

Proposition 5.6. *Let $B \in GL(n, \mathbb{C})$ be a matrix and $\mathcal{F}_B$ be the corresponding suspension foliation over $\mathbb{D}^*$ (as B is isotopic to identity the foliation $\mathcal{F}_B$ is on $\mathbb{D}^* \times \mathbb{C}^n$), and denote by $\tilde{A}_t$ the linear multiplicative cocycle over the geodesic flow φ of $\mathbb{D}^*$ induced by $\mathcal{F}_B$. Assume that all the eigenvalues of B have modulus equal to 1. Then the functions $\log^+(\|\tilde{A}_{\pm 1}\|)$ are in $\mathcal{L}^1(dLiouv|_{D^*})$.*

We begin the proof of Proposition 5.6 by the following remarks allowing us to reduce the proof to an easier case:

Remark 5.7. 1. If two matrices B_1 and B_2 are conjugated then the corresponding cocycles are both integrable or both non-integrable.

2. If B is a matrix on $\mathbb{C}^k \times \mathbb{C}^m$ leaving invariant $\mathbb{C}^k \times \{0\}$ and $\{0\} \times \mathbb{C}^m$, then the multiplicative cocycle induce by B is integrable if and only if the cocycles induced by the restrictions of B to $\mathbb{C}^k \times \{0\}$ and $\{0\} \times \mathbb{C}^m$ are both integrable.

3. As a consequence of item 2, we can assume that B is a matrix which doesn't leave invariant any splitting of $\mathbb{C}^n$ in a direct sum of non-trivial subspaces. In particular B has a unique eigenvalue λ_B and by hypothesis $|\lambda_B| = 1$. Moreover two such matrices are conjugate: their Jordan form is

$$\begin{pmatrix} \lambda_B & 1 & & \cdots & 0 & 0 \\ 0 & \lambda_B & 1 & \cdots & & 0 \\ \cdots & & & & & \cdots \\ 0 & 0 & \cdots & 0 & \lambda_B & 1 \\ 0 & 0 & \cdots & 0 & 0 & \lambda_B \end{pmatrix}$$

Using the remarks above, it is enough to prove Proposition 5.6 for the matrices B_θ define as follows. Let

$$A_\theta = \begin{pmatrix} i\theta & 1 & 0 & \cdots & 0 & 0 \\ 0 & i\theta & 1 & 0 & \cdots & 0 \\ \cdots & & & & & \cdots \\ 0 & 0 & \cdots & 0 & i\theta & 1 \\ 0 & 0 & \cdots & 0 & 0 & i\theta \end{pmatrix}.$$

We define $B_\theta = exp(A_\theta)$. Notice that

$$exp(t \cdot A_\theta) = e^{it\theta} \begin{pmatrix} 1 & t & & \cdots & \frac{t^{n-2}}{(n-2)!} & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \cdots & & \frac{t^{n-2}}{(n-2)!} \\ \cdots & & & & & \cdots \\ 0 & 0 & \cdots & 0 & 1 & t \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

Consider the holomorphic foliation defined by the linear equation

$$\begin{pmatrix} \dot{z} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & A_\theta \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}$$

on $\mathbb{D}^* \times \mathbb{C}^n$ such that the holonomy map from $\{e^{-2\pi}\} \times \mathbb{C}^n \to \{z\} \times \mathbb{C}^n$ with $z \in S^1$ is $exp(arg(z)A_\theta)$. The monodromy of this foliation is $B_\theta = e^{2i\pi\theta} exp(2\pi A_0)$.

Lemma 5.8. *The multiplicative cocycle $\tilde{A}_t$ obtained by lifting the geodesic flow of $\mathbb{D}^*$ on the leaves of $\tilde{\mathcal{F}}_\theta$ is integrable over $T^1\mathbb{D}|_{D^*}$.*

Proof: For any $u \in N^+$ one has $B(u) = A_{t_u}(u) = exp(\frac{a_u}{2\pi} \cdot A_\theta)$, so that there is a constant K such that $\|B(u)\| < K(1 + a_u^{n-1})$, so that $\log^+ \|B(u)\|$ is integrable if and only if $\log^+(|a_u|)$ is integrable for μ_0.

By Proposition 5.4 one has $a_u = 2\cos(\eta)/\sin(\eta)$ so that $a_u < 2/\eta$. As $\int_{-1}^{1} |\log(|1/x|)|dx < +\infty$, we get easily that $\int_{N_1^+} \log^+(|a_u|)d\mu_0 < +\infty$, concluding the proof. $\square$

5.3 The Hyperbolic Case

Proposition 5.9. *If there is i such that the matrix $B = \rho(\gamma_i)$ has an eigenvalue with modulus different from* 1*, then the multiplicative cocycle is not integrable.*

If $B \in GL(n, \mathbb{C})$ has an eigenvalue with modulus different from 1, we may suppose that its modulus is greater than 1, since the suspension of B and B^{-1} are isomorphic. As in the parabolic case the proof of Proposition 5.9 follows directly from a local argument in a neighbourhood of the puncture corresponding to γ_i.

Proposition 5.10. *Let $B \in GL(n, \mathbb{C})$ having an eigenvalue $\lambda > 1$ and $\mathcal{F}_B$ the suspension folition on D^*. Then the multiplicative cocycle $\tilde{A}_t$ induced by $\mathcal{F}_B$ over the geodesic flow φ of D^* is not integrable.*

Proof: We begin by an estimative of the norm of the multiplicative cocycle corresponding to the "in-out" map :

Lemma 5.11. *There is a constant $K > 0$ such that for any $u \in N^+$ one has:*

$$|\tilde{A}_{t_u}(u)| \geq K \cdot \lambda^{a_u/2}.$$

So $\log^+ |\tilde{A}_{t_u}(u)| \geq \log K + \frac{|a_u|}{2} \log \lambda$. One deduces that $\log^+ |\tilde{A}_{t_u}(u)|$ cannot be μ_0-integrable if $|a_u|$ is not integrable. By Proposition 5.4 one knows that $a_u = 2\frac{\cos(\eta)}{\sin(\eta)}$ and this function is not integrable for $d\mu_0 = d\eta \wedge d\theta$. From Proposition 5.2 we get that the multiplicative cocycle $\tilde{A}_1$ is not integrable for Liouville, finishing the proof the Proposition 5.10. $\square$

Remark: If $\rho : \pi_1(S) \to PGL(n, \mathbb{C})$ is a representation that does not admit a lifting to a representation in $GL(n, \mathbb{C})$ we may still define a flat bundle over S but with fibres $\mathbb{C}^n/\mathbb{Z}_n$ and transition coordinates in $SL(n, \mathbb{C})/\mathbb{Z}_n \cdot Id$, and hence a foliation $\mathcal{F}_{\tilde{\rho}}$ on this singular bundle, where $\mathbb{Z}_n$ is the group of n roots of unity. We may introduce a continuous Hermitian norm on this bundle (locally induced from a Hermitian norm in $\mathbb{C}^n$ as well as choosing a trivialisation of the generator of the discrete dynamics $\tilde{A}_1$, and the statements and arguments given in the text extend to this situation.

References

[1] Ch. Bonatti and X. Gómez-Mont *Sur le comportement statistique des feuilles de certains feuilletages holomorphes,* preprint 222, U. Bourgogne, 2000, to appear in *L脱ns. Math.*.

[2] Ch. Bonatti, X. Gómez-Mont and M. Viana *Généricité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices,* preprint 241, U. Bourgogne, 2001.

[3] R. Bowen and D. Ruelle, *The ergodic theory of Axiom A flows,* Inventiones Math., **29**, 181-202, (1975).

[4] Coddington, E., Levinson, N. *Theory of ordinary differential equations,* McGraw-Hill, New York, (1955).

[5] E.Ghys and P. de la Harpe *Sur les groupes hyperboliques d'apres Mikhael Gromov,* Progress in Mathematics, Volume 83, Birkhäuser, Zurich (1990).

[6] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems,* Cambridge University Press, Encyclopedia of Mathematics and its Applications, Volume 54 (1995).

[7] U. Krengel, *Ergodic Theorems,* De Gruyter studies in Meth. vol. 6, Berlin, 1985.

[8] B. Lawson *The Qualitative Theory of Foliations,* CBMS Regional Conference in Math. 27. (1978).

[9] B. Maskit, *Kleinian Groups,* Springer Verlag, Grundlehren 287, New York. (1987)

[10] D. Ruelle, *A measure associated with Axiom A attractors,* Amer. J. Math., **98**. 619-654, (1976).

[11] Y. Sinaï, *Markov partitions and C-diffeomorphisms,* Func. Anal. and its Appl., **2**, 64-89, (1968).

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