On the diffeomorphism group of a smooth orbifold and its application

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§1. Introduction

Let \( \mathcal{D}(M) \) denote the group of diffeomorphisms of an \( n \)-dimensional smooth manifold \( M \) which are isotopic to the identity through compactly supported isotopies. In [TH], Thurston proved that the group \( \mathcal{D}(M) \) is perfect, which means \( \mathcal{D}(M) \) coincides with its commutator subgroup. There are many analogous results on the group of a smooth manifold \( M \) preserving a geometric structure of \( M \).

In this note we shall study the case when \( M \) is a smooth orbifold. Since a smooth orbifold is locally diffeomorphic to the orbit space of a smooth \( G \)-manifold with finite group \( G \), first we shall consider in the case of a representation space \( V \) of a finite group \( G \). Let \( \mathcal{D}_G(V) \) denote the group of equivariant smooth diffeomorphisms of \( V \) which are \( G \)-isotopic to the identity through compactly supported equivariant smooth isotopies. In general the group \( \mathcal{D}_G(V) \) is not perfect. Then we calculate the first homology group \( H_1(\mathcal{D}_G(V)) \).

We shall prove that \( \mathcal{D}_G(V) \) is perfect if \( \dim V^G > 0 \) and \( H_1(\mathcal{D}_G(V)) \) is isomorphic to \( H_1(\text{Aut}_G(V)_0) \) if \( \dim V^G = 0 \). Here \( \text{Aut}_G(V)_0 \) is the identity component of the group of \( G \)-equivariant linear automorphisms of \( V \), and \( V^G \) is the fixed point set of \( G \) on \( V \) ([AF5]).

Secondly we apply the above result to the case of smooth orbifold and also smooth \( G \)-manifold. Using the result by Biestone [BI1] and Schwarz [SC1], we see that \( H_1(\mathcal{D}_G(V)) \) is isomorphic to \( H_1(\mathcal{D}(V/G)) \). Combining those results and the fragmentation lemma we can determine the structure of \( H_1(\mathcal{D}(N)) \) of the diffeomorphism group \( \mathcal{D}(N) \) for any smooth orbifold \( N \). Then we see that \( H_1(\mathcal{D}(N)) \) describes a geometric structure around the isolated singularities.

Let \( M \) be a smooth \( G \)-manifold for a finite group \( G \). Then \( H_1(\mathcal{D}_G(M)) \) is isomorphic to \( H_1(\mathcal{D}(M/G)) \), and we see that \( H_1(\mathcal{D}_G(M)) \) describes the properties of the isotropy representations at the isolated fixed points of \( M \). We can also apply the above results to a smooth \( G \)-manifold when \( G \) is a compact Lie group. If \( M \) is a principal \( G \)-manifold with \( G \) a compact Lie group, then we proved that the group \( \mathcal{D}_G(M) \) is perfect for \( \dim(M/G) > 0 \) (Banyaga [BA1] and Abe and Fukui [AF1]). In [AF2] we calculated \( H_1(\mathcal{D}_G(M)) \) when \( M \) is a smooth \( G \)-manifold with codimension
one orbit. We shall apply the above result to the case of a locally free $U(1)$-action on the 3-sphere, and calculate $H_1(D_{U(1)}(S^3))$ ([AF5]).

Thirdly we shall apply the results to the modular group. Let $\Gamma$ be the modular group which acts on the the upper half complex plane $\mathcal{H}$ by the Möbius transformations. Then the orbit space $\mathcal{H}/\Gamma$ is a smooth orbifold. Let $R_\Gamma$ be the compactified space of $\mathcal{H}/\Gamma$ by adjoining the point $*$ which corresponds to the $\Gamma$-equivalence class of the parabolic cusps. With the canonical smooth coordinate around $*$, we shall calculate the group $H_1(D(R_\Gamma))$, which describes the elliptic points and the cusp point. We can also calculate the group for the case of the congruence subgroups of $\Gamma$.

We can apply the above results to the case of foliation preserving diffeomorphism groups. We studied for the similar problem in the Lipschitz category ([AF3], [AF4], [AF6], [AFM]).

§2. Recent results on the diffeomorphism groups on smooth orbifolds

Let $G$ be a finite group and let $M$ be a smooth connected $G$-manifold. Let $D_G(M)$ denote the group of $G$-equivariant smooth diffeomorphisms of $M$ which are $G$-isotopic to the identity through isotopies with compact support.

First we shall calculate $D_G(V)$ for a finite dimensional $G$-module $V$. Let $V^G$ be the subspace of the fixed point set of $V$. Let $A_G(V)$ denote the set of $G$-invariant automorphisms of $V$ and let $A_G(V)_0$ be the identity component of $A_G(V)$. Then we have the following.

Theorem 1
(1) If $\dim V^G > 0$, then $D_G(V)$ is perfect.
(2) If $\dim V^G = 0$, then $H_1(D_G(V)) \cong H_1(A_G(V)_0)$.

We can decompose $V = \oplus_{i=1}^d k_iV_i$, where $V_i$ runs over the inequivalent irreducible representation space of $G$ and $k_i$ is a positive integer. Let $End_G(V_i)$ denote the set of $G$-invariant endmorphisms of $V_i$. Then $\dim End_G(V_i) = 1, 2$ or 4.

Corollary 2 If $\dim V^G = 0$, then

$$H_1(D_G(V)) \cong \mathbb{R}^d \times U(1) \times \cdots \times U(1),$$

where $d_2$ is the number of $V_i$ with $\dim End_G(V_i) = 2$. 

Definition 3 (smooth orbifold)

A paracompact Hausdorff space $M$ is called a smooth orbifold if there exists an open covering $\{U_i\}_{i \in \Lambda}$ of $M$, closed under finite intersections, satisfying the following.

1. There exist an open subset $\tilde{U}_i$ in $\mathbb{R}^n$ such that a finite group $\Gamma_i$ acts effectively on $\tilde{U}_i$ and a homeomorphism $\phi_i : \tilde{U}_i/\Gamma_i \rightarrow U_i$.
2. Whenever $U_i \subset U_j$, there exists a smooth embedding $\phi_{ij} : \tilde{U}_i \rightarrow \tilde{U}_j$ such that
   \[ \begin{array}{ccc}
   \tilde{U}_i & \xrightarrow{\phi_{ij}} & \tilde{U}_j \\
   \pi_i & & \pi_j \\
   \tilde{U}_i/\Gamma_i & \xrightarrow{\phi_{ij}} & \tilde{U}_j/\Gamma_j \\
   \phi_i^{-1} & & \phi_j^{-1} \\
   U_i & \xrightarrow{c} & U_j.
   \end{array} \]

$(U_i, \phi_i)$ is called a local chart of $M$.

Here we define the smooth maps between smooth orbifolds (c.f. [BI1]). $f : M \rightarrow \mathbb{R}$ is said to be smooth if for any local chart $(U_i, \phi_i)$ of $M$, $\tilde{U}_i \xrightarrow{\pi_i} \tilde{U}_i/\Gamma_i \xrightarrow{\phi_i} U_i \xrightarrow{f} \mathbb{R}$ is smooth. $h : M \rightarrow M$ is said to be smooth if for any smooth function $f : M \rightarrow \mathbb{R}$, $f \circ h$ is smooth. $h : M \rightarrow M$ is called a diffeomorphism if $h$ and $h^{-1}$ are smooth. Let $\mathcal{D}(M)$ denote the group of diffeomorphisms of $M$ which are isotopic to the identity through isotopies with compact support.

$p \in M$ is said to be an isolated singular point of $M$ if there exists a local chart $(U_i, \phi_i)$ around $p$ such that $\tilde{p}$ is the isolated fixed point of $\tilde{U}_i$ with $\pi_i(\tilde{p}) = p$. Here $\phi_i : U_i \rightarrow \tilde{U}_i/\Gamma_i$ and $\pi_i : \tilde{U}_i \rightarrow U_i$ are the maps defined in Definition 3.

Let $(U_i, \phi_i), (U_j, \phi_j)$ be local charts of $M$ around an isolated singular point $p$ of $M$. Then we can assume that $\tilde{U}_i$ and $\tilde{U}_j$ are invariant open neighborhoods around the origin of linear representation spaces of $\Gamma_i$ and $\Gamma_j$, respectively. By the result of Strub [ST], the groups $\Gamma_i$ and $\Gamma_j$ are isomorphic and the corresponding representations are equivalent. Then the isolated singular point $p$ determines the equivalence class of the linear representation space $V_p$ of a finite group $\Gamma_p$.

Theorem 4 If a smooth orbifold $M$ has $\{p_1, \ldots, p_k\}$ as the isolated singular point set, then

$$H_1(\mathcal{D}(M)) \cong H_1(A_{\Gamma_{p_1}}(V_{p_1})_0) \times \cdots \times H_1(A_{\Gamma_{p_k}}(V_{p_k})_0).$$
We can apply Theorem 4 to the case of smooth $G$-manifold with finite group $G$.

**Theorem 5** Let $G$ be a finite group and $M$ a smooth $G$-manifold. If the orbit space $M/G$ has $\{G \cdot p_1, \ldots, G \cdot p_k\}$ as the isolated singular points, then

$$H_1(D_G(M)) \cong H_1(A_{G_{p_1}}(T_{p_1}M)_0) \times \cdots \times H_1(A_{G_{p_k}}(T_{p_k}M)_0).$$

**Corollary 6** Let $\tilde{\mathbb{R}}$ be the non-trivial one dimensional representation space of $\mathbb{Z}_2$. Then

$$H_1(D_{\mathbb{Z}_2}(\tilde{\mathbb{R}})) \cong H_1(D(\tilde{\mathbb{R}}/\mathbb{Z}_2)) \cong \mathbb{R}.$$

We can apply Corollary 6 to a smooth $U(1)$-action on $S^3$. Let

$$S^3 = \{(w_1, w_2) \in \mathbb{C}^2 \mid |w_1|^2 + |w_2|^2 = 1\}$$

with $U(1)$-action given by

$$z \cdot (w_1, w_2) = (zw_1, z^2w_2), \quad z \in U(1).$$

Then it has two orbit types $\{(1), (\mathbb{Z}_2)\}$ and the orbit space $S^3/U(1)$ is homeomorphic to the space known as the tear drop which is the two dimensional sphere with one isolated singular point.

**Theorem 7** $H_1(D_{U(1)}(S^3)) \cong \mathbb{R} \times U(1)$.

**Remark 8** If we restrict the above action to $\mathbb{Z}_n$, then $D_{\mathbb{Z}_n}(S^3)$ is perfect.

§3. Application to the modular group

In this section we shall apply the results to the modular group. Let $\mathcal{H}$ be the upper half complex plane. Let $SL(2, \mathbb{R})$ be the group of real matrix with determinant 1. Then $SL(2, \mathbb{R})$ acts on $\mathcal{H}$ as follows.

$$g \cdot z = \frac{az + b}{cz + d} \text{ for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \ z \in \mathcal{H}.$$

Then $SL(2, \mathbb{R})$ acts transitively on $\mathcal{H}$ and the isotropy subgroup at $i = \sqrt{-1}$ is $SL(2, \mathbb{R})_i = SO(2)$. The kernel of the action is $\mathbb{Z}_2 = \{\pm 1\}$ and $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm 1\}$ acts effectively on $\mathcal{H} \cong SL(2, \mathbb{R})/SO(2)$. 
The action can be extended to the Riemannian sphere: \( \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \).

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \ z \in \overline{\mathbb{C}},
\]

\[
g \cdot z = \begin{cases} \ \ \ \ \frac{a z + b}{c z + d} & (z \neq -\frac{d}{c}, \infty) \\
\infty & (z = -\frac{d}{c}, \ z = d = 0) \\
\frac{\infty a}{c} & (z = \infty)
\end{cases}
\]

Set
\[
R_1 = \left\{ \pm \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0 \right\},
\]

\[
R_2 = \left\{ \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right\}.
\]

Then each \( g \in SL(2, \mathbb{R}) \) is conjugate to one of the elements of \( SO(2) \cup R_1 \cup R_2 \), and \( g \neq \pm 1 \) is called elliptic, hyperbolic and parabolic if \( g \) is conjugate of an element in \( SO(2) \), \( R_1 \) and \( R_2 \), respectively.

Let \( \Gamma = SL(2, \mathbb{Z}) \) be the group of the integral matrices with determinant 1. Then \( \bar{\Gamma} = \Gamma/\{\pm 1\} \) acts properly on \( \mathcal{H} \) (i.e. for each \( z \in \mathcal{H} \), there exists open neighborhood \( U \) of \( z \) such that \( \bar{\Gamma}_U = \{ g \in \bar{\Gamma} \mid g \cdot U = U \} \) is a finite group and if \( \gamma \cdot U \cap U \neq \emptyset \) for \( \gamma \in \bar{\Gamma} \), then \( \gamma \in \bar{\Gamma}_U \).

\( z \in \mathcal{H} \) is called elliptic point if there exits an elliptic element \( g \in \Gamma \) such that \( g \cdot z = z \). \( x \in \mathbb{R} \cup \{\infty\} \) is called cusp point if there exists a parabolic element \( g \in \Gamma \) such that \( g \cdot z = z \).

**Proposition 9**

(1) If \( z \) is a elliptic point, then \( \Gamma_z \) is a cyclic group which is conjugate to a cyclic subgroup of \( SO(2) \).

(2) If \( x \) is a cusp point, then \( \Gamma_x \) is isomorphic to \( \mathbb{Z} \) which is conjugate to the group

\[
\Gamma_{\infty} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}.
\]

(3) \( \Gamma \) acts transitively on the set of cusp points which is coincides with \( Q \cup \{\infty\} \), where \( Q \) is the set of rational numbers.

Set

\[
\mathcal{H} = \mathcal{H} \cup Q, \quad \mathcal{R}_\Gamma = \mathcal{H}^*/\Gamma = \mathcal{H}/\Gamma \cup \{*\}.
\]

We give the set

\[
\{*\} \cup \cup_{c>0}\{ z \in \mathcal{H} \mid \exists z > c \}
\]

as a fundamental system of open neighborhood of the point \(*\). Then \( \mathcal{R}_\Gamma \) is homeomorphic to \( S^2 \).
Proposition 10  There exists a $\Gamma_\infty$-invariant open neighborhood $\tilde{U}$ of $*$ satisfying the following.
(1) $\Gamma_\infty = \{ g \in \Gamma | g \cdot \tilde{U} \cap \tilde{U} \neq \phi \}$.
(2) Let $\varphi : \tilde{U}/\Gamma_\infty \rightarrow \mathbb{C}$ be the map given by $\varphi(\Gamma_\infty \cdot z) = \exp(2\pi \sqrt{-1}z)$ for $z \in \tilde{U}$.
Then $\varphi$ is a homeomorphism into an open set $U$ of $\mathbb{C}$.

Let $\iota : \tilde{U}/\overline{\Gamma}_\infty \rightarrow \mathcal{R}_\Gamma$ be the natural map. Put $U = \iota(\tilde{U}/\overline{\Gamma}_\infty)$. By Proposition 10 $U$ is an open neighborhood of $*$ and the homeomorphism $\phi = \varphi \circ \iota^{-1} : U \rightarrow \tilde{U}/\overline{\Gamma}_\infty$ is regarded as a local coordinate of $\mathcal{R}_\Gamma$.

Theorem 11
(1) $H_1(D_\Gamma(H^2)) \cong H_1(D(H^2/\Gamma)) \cong \mathbb{R}^2 \times U(1)$.
(2) $H_1(D(\mathcal{R}_\Gamma)) \cong U(1) \times \mathbb{R}^3$.

The orbifold $H^2/\Gamma$ has two isolated singular points which correspond to the elliptic subgroups of $\Gamma$ with orders 2 and 3, which induces the isomorphism in Theorem 11, (1). In addition to those singular points, $\mathcal{R}_\Gamma$ has the singular point $*$ corresponding to the cusp point, which induces the isomorphism in Theorem 11, (2).

Let $\Gamma(N)$ denote the principal congruence subgroup of level $N$. Then
$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma | a \equiv d \equiv 1, \ b \equiv c \equiv 0 \ \mod N\mathbb{Z} \right\}.$$
Similarly to the case of the modular group, we have the following.

Theorem 12  $H_1(D(\mathcal{R}_\Gamma(N))) \cong \mathbb{R}^{t(N)}$, where $t(N)$ is the number of cusps of $\mathcal{H}/\Gamma(N)$.

The number $t(N)$ is known as:
$$t(1) = 1, \quad t(2) = 3,$$
\[
t(N) = \frac{1}{2N}(N : \Gamma(N)) \quad (N \geq 3),
\]
\[
(N : \Gamma(N)) = N^3 \prod_{p|N}(1 - \frac{1}{p^2}).
\]

We can also apply Theorem 1 to calculate the first homology group of the foliation preserving diffeomorphism group for a compact Hausdorff foliation.

\section*{§4. Outline of the proof of Theorem 1}

First we prove Theorem 1 (1). Let \( G \) be a finite group and let \( V \) be a \( G \)-module with \( \dim V^G > 0 \). Then there exists a \( G \)-module \( W \) with \( \dim W^G = 0 \) such that \( V = W \oplus \mathbb{R}^q \). We prove \( D_G(V) \) is perfect by induction of the order of \( G \). If \( G = \{1\} \), then \( D_G(V) \) is perfect by the result of Thurston [TH]. Assume that Theorem 1 (1) holds for any finite subgroup \( H \) with \(|H| < |G|\).

To investigate the group structure of \( D_G(V) \), we give \( \mathcal{C}^\infty \)-topology on \( D_G(V) \). For the proof we need the following fragmentation lemma.

\textbf{Lemma 13 (fragmentation lemma)}

Let \( M \) be a smooth \( G \)-manifold and let \( \{U_i\} \) be a \( G \)-invariant open covering of \( M \). Let \( N \) be a neighborhood of the identity in \( D_G(M) \). Then, for any \( f \in D_G(M) \), there exist \( \{f_j \in N | 1 \leq j \leq k\} \) such that

1. \( f_j \) is equivariantly isotopic to the identity through \( G \)-diffeomorphisms with the support contained in \( U_j \),
2. \( f = f_1 \circ \cdots \circ f_k \).

Let \( f \in D_G(V) \). In order to prove \( f \in [D_G(V), D_G(V)] \), by the fragmentation lemma, we can assume \( f \) is sufficiently close to the identity. Then we can find \( g_1, g_2 \in D_G(V) \) satisfying

1. \( g_1(x, y) = (x, \hat{g}_1(x)(y)) \) with \( \hat{g}_1(x) \in \mathcal{D}(\mathbb{R}^q) \),
2. \( g_2(x, y) = (\hat{g}_2(y)(x), y) \) with \( \hat{g}_2(y) \in D_G(W) \) for \( x \in W, \ y \in \mathbb{R}^q \),
3. \( f = g_2 \circ g_1 \).

By the result of Tsuboi [TS], we see that \( g_1 \in [D_G(V), D_G(V)] \).

In the next we shall prove that \( g_2 \in [D_G(V), D_G(V)] \). Let \( \alpha_{g_2} : \mathbb{R}^q \to Aut_G(W)_0 \) be a group homomorphism defined by \( \alpha_{g_2}(y) = d\hat{g}_2(y)_0 \), where \( d\hat{g}_2(y)_0 \) is the differential of \( \hat{g}_2(y) \) at \( 0 \). Then \( \alpha_{g_2} \) is a smooth map with compact support \( \{p \in \mathbb{R}^q | \alpha_{g_2}(p) \neq e\} \), where \( e \) is the unit element in \( Aut_G(W)_0 \).

If we take \( f \) close to the identity, then \( \alpha_{g_2} \) is sufficiently close to the constant map \( e \). Then applying [AF1], Lemma 4, we have
(a) \( \exists \varphi_i \in \mathcal{D}(\mathbb{R}^q), \alpha_i \in C^\infty(\mathbb{R}^q, Aut_G(W)_0) \) \( (i = 1, \ldots, r = \dim Aut_G(W)_0) \),
(b) \( \alpha_g = (\alpha_1^{-1} \cdot (\alpha_1 \circ \varphi_1)) \cdots (\alpha_r^{-1} \cdot (\alpha_r \circ \varphi_r)) \).

Let \( | \cdot | \) be a \( G \)-invariant norm of \( W \). Let \( \mu : W \to [0, 1] \) be a \( G \)-invariant smooth function satisfying
(i) \( \mu(x) = 1 \) for \( |x| \leq \frac{1}{2} \),
(ii) \( \mu(x) = 0 \) for \( |x| \geq 1 \).

Define \( h_i, F_i \in \mathcal{D}_G(V) \) \( (i = 1, \ldots, r) \) by
\[
\begin{align*}
  h_i(x, y) & = (\mu(x)\alpha_i(y)(x) + (1 - \mu(x))x, y), \\
  F_i(x, y) & = (x, \mu(x)\varphi_i(y) + (1 - \mu(x))y)
\end{align*}
\]
for \( x \in W, y \in \mathbb{R}^q \).

Lemma 14
\[
(h_i^{-1} \circ F_i^{-1} \circ h_i \circ F_i)(x, y) = ((\alpha_i^{-1} \cdot (\alpha_i \circ \varphi_i))(y)(x), y),
\]
for \( x \in W, y \in \mathbb{R}^q \) with \( |x| \leq \frac{1}{2} \).

Set \( g_3 = \prod_{i=1}^{r} (h_i^{-1} \circ F_i^{-1} \circ h_i \circ F_i)^{-1} \circ g_2 \).

Then \( g_3 \) is written of the form \( g_3(x, y) = (\tilde{g}_3(x)(y), y) \) with \( \tilde{g}_3(x) \in \mathcal{D}_G(W) \) and \( \alpha_{g_3} = e \).

For \( 0 < c < 1 \), let \( \psi_c \in \mathcal{D}_G(V) \) such that, for \( x \in W, y \in \mathbb{R}^q \),
\[
\psi_c(x, y) = \begin{cases} (cx, y) & (|x| \leq 1), \\
(x, y) & (|x| \geq 2).
\end{cases}
\]

Applying the result of Sternberg [S2], there exists \( R \in \mathcal{D}(V) \) such that
(1) \( R \) is of the form \( R(x, y) = (\tilde{R}(y)(x), y) \)
with \( \tilde{R}(y) \in \mathcal{D}(W, 0) \) and \( \alpha_{\tilde{R}} = e \).
(2) \( R \circ (g_3 \circ \psi_c) \circ R^{-1} = \psi_c \) on a neighborhood \( U_0 \) of \( \{0\} \times \mathbb{R}^q \).

Set \( \tilde{R}(x, y) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot R(g \cdot x, y) \) for \( x \in W, y \in \mathbb{R}^q \).

Then
\[
\psi_c \circ \tilde{R} = \tilde{R} \circ g_3 \circ \psi_c \quad \text{on} \quad U_0.
\]
Since $\tilde{R}$ is $G$-equivariant diffeomorphic on a neighborhood of $\{0\} \times \mathbb{R}^q$, we can find $\tilde{R}_1 \in D_G(V)$ such that $\tilde{R}_1 = \tilde{R}$ on a neighborhood $U \subset U_0$ of $\{0\} \times \mathbb{R}^q$. Put
\[
g_4 = g_3 \circ (\tilde{R}_1^{-1} \circ \psi_c \circ \tilde{R}_1 \circ \psi_c^{-1})^{-1}.
\]
Then $g_4 = 1$ on $U$.

There exist a finite point $\{p_i \in V \setminus U \mid 1 \leq i \leq k\}$ and an open disk neighborhood $U(p_i)$ at $p_i$ $(1 \leq i \leq k)$ such that
1. $U(p_i)$ is a slice at $p_i$,
2. $\text{supp}(g_4) \subseteq \bigcup_{i=1}^k G \cdot U(p_i)$.

By the fragmentation lemma there exist $h_j \in D_G(V)$ $(1 \leq j \leq \ell)$ such that
(a) $h_j$ is equivariantly isotopic to the identity through $G$-diffeomorphisms with the support contained in $G \cdot U(p_j)$,
(b) $g_4 = h_1 \circ \cdots \circ h_\ell$.

Since $U(p_j)$ is a slice at $p_j$, the isotropy subgroup $G_{p_j}$ acts on $U(p_j)$ and $G \cdot U(p_j)$ is a disjoint union of $|G/G_{p_j}|$ disks. Then from the above condition (a)
\[
h_j(g \cdot U(p_j)) = g \cdot U(p_j) \quad \text{for } g \in G.
\]

We assumed that $D_H(V)$ is perfect when $H$ is a finite group with $|H| < |G|$ and $\dim V^H > 0$. Therefore each $h_j$ can be written as a commutator in $D_G(V)$ and Theorem 1 (1) follows.

Secondary we prove Theorem 1 (2). Let $V$ be a $G$-module with $\dim V^G = 0$. Let $\Phi : D_G(V) \to Aut_G(V)_0$ be a group homomorphism defined by $\Phi(f) = (df)_0$. Since
\[
1 \to \text{Ker}\Phi \xrightarrow{\Phi} D_G(V) \xrightarrow{\Phi} Aut_G(V)_0 \to 1
\]
is a short exact sequence, we have the exact sequence.
\[
\text{Ker}\Phi/[\text{Ker}\Phi, D_G(V)] \xrightarrow{\Phi} H_1(D_G(V)) \xrightarrow{\Phi} H_1(Aut_G(V)_0) \to 1
\]
Then Theorem 1 (2) follows from the following.

**Proposition 15**  
$\text{Ker}\Phi = [D_G(V), D_G(V)]$

**Proof.** Let $f \in \text{Ker}\Phi$. For $0 < c < 1$, let $\psi_c \in Aut_G(V)_0$ as before. Applying the result by Sternberg [S2] there exists $R \in D(V, 0)$ such that $(dR)_0 = 1_V$ and
$R \circ f \circ \psi_c \circ R^{-1} = \psi_c$ on a neighborhood of 0. Set

$$\tilde{R}(x) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot R(g \cdot x)$$

for $x \in \mathbb{R}^n$, where $|G|$ is the order of $G$. Since $\tilde{R}$ is equivariant diffeomorphism on a neighborhood $U$ of 0, we can find $\hat{R} \in \mathcal{D}_G(V)$ such that $\hat{R} = \tilde{R}$ on an open neighborhood $U_1 \subset U$ of 0. Then

$$f = \hat{R}^{-1} \circ \psi_c \circ \hat{R} \circ \psi_c^{-1} \text{ on } U_1.$$ 

Put

$$g = f \circ (\hat{R}^{-1} \circ \psi_c \circ \hat{R} \circ \psi_c^{-1})^{-1}.$$ 

Then $g = 1$ on $U_1$. By the parallel way as in the proof of the case Theorem 1, (1), we can prove that $g$ is written as a commutator in $\mathcal{D}_G(V)$.

References


