ISOVARIANT BORSUK-ULAM TYPE RESULTS AND THEIR CONVERSE

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0. THE BORSUK-ULAM THEOREM

In this note, we first make a brief survey of Borsuk-Ulam type theorems, and next introduce some results on the isovariant Borsuk-Ulam theorem and its converse from [22, 23].

K. Borsuk (1905-82) showed the following three results in 1933.

Theorem 0.1 ([21]).

(B1) If $f : S^n \to S^n$ is antipodal, i.e., $f(-x) = -f(x)$ for all $x \in S^n$, then $f$ is essential, i.e., $f$ is not null-homotopic.

(B2) For any continuous map $f : S^n \to \mathbb{R}^n$, there exists $x_0 \in S^n$ such that $f(x_0) = f(-x_0)$.

(B3) Suppose $S^n = \bigcup_{i=0}^{n} F_i$: nonempty closed sets. Then some $F_i$ contains an antipodal pair; $\{x_0, -x_0\} \subset F_i$. (Lusternik-Schnirelmann 1930)

The second result was conjectured by S. Ulam; so it is usually called the Borsuk-Ulam theorem. It is known that the Borsuk-Ulam theorem has various equivalent statements; indeed, the above statements (B1)–(B3) are equivalent, and in addition, the following statements are also equivalent to the Borsuk-Ulam theorem.

(B4) If $f : S^n \to \mathbb{R}^n$ is antipodal, then $f^{-1}(0) \neq \emptyset$.

(B5) If $f : S^n \to S^m$ is antipodal, then $n \leq m$.

0.1. Generalization. Each of (B1)–(B5) has various generalizations and related topics. Indeed (B1) says that the degree of $f$ is nonzero; in fact, it is well known that $\deg f$ is odd. Thus (B1) is related to the degree of (equivariant) maps or degree theory. Recently Hara [11] and Inoue [13] obtained a natural extension of (B1) for equivariant maps between Stiefel manifolds with standard $O(n)$- or $\mathbb{Z}_p^k$-action.

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Statements (B2) and (B4) are related to coincidence theory or fixed point theory, and there are various researches in this field; see, for example, Gonçalves-Jaworowski-Pergher [8], Gonçalves et al. [9], Gonçalves-Wong [10].

Statement (B3) is related to the Lusternik-Schnirelmann category or Lusternik-Schnirelmann theory, which provides lower estimate for the number of critical points of a smooth function. For example, (B3) implies cat $\mathbb{R}P^n \geq n$ and so we obtain cat $\mathbb{R}P^n = n$, where cat $X$ denotes the Lusternik-Schnirelmann category of $X$, i.e., cat $X := \min\{n | X = \bigcup_{i=0}^{n} F_i, \text{each } F_i \text{ is closed and contractible in } X\}$.

0.2. **Equivariant generalization.** From the viewpoint of transformation groups, (B5) can be rephrased as follows: If there is a $\mathbb{Z}_2$-map $f : S^n \to S^m$, then $n \leq m$ holds, where $\mathbb{Z}_2$ acts antipodally on the spheres. This formulation has a lot of equivariant generalizations; see, for example, Jaworowski [14], Dold [6], Fadell-Husseini [7], Marzantowicz [18], Bartsch [1], Komiya [16], Hara-Minami [12], etc. We recall some well-known equivariant generalizations. A direct generalization of (B5) is the following.

**Theorem 0.2.** Suppose that $G \neq 1$ acts freely on $S^n, S^m$. If there is a $G$-map $f : S^n \to S^m$, then $n \leq m$ holds. (Dold [6], Kobayashi [15], Laitinen [17] etc.)

The proof of Theorem 0.2 is reduced to the case $G = \mathbb{Z}_p$. An important fact is that the degree of a self $G$-map $f : S^n \to S^n$ is nonzero; in fact $\text{deg } f \equiv 1 \pmod p$.

Remark. This result still holds for free finite $G$-$\text{CW}$ complexes homotopy equivalent to spheres.

In nonfree case, the following is known.

**Theorem 0.3.** If there is a $\mathbb{Z}_p^k$-map (or $T^k$-map) $f : S^n \to S^m$, where $\mathbb{Z}_p$ or $T^k$ acts fixed-point-freely on spheres, then $n \leq m$ holds. (Fadell-Husseini [7], Marzantowicz [18], etc.) Moreover this result still holds for $\mathbb{Z}_p$ (or $\mathbb{Q}$)-homology spheres. (Clapp-Puppe [4].)

A euclidean space $V$ with linear $G$-action is called a $G$-representation. We may suppose that the action is orthogonal. Let $SV$ denote the unit sphere of a $G$-representation $V$. In this case, we say that $G$ acts linearly on $SV$ or that $SV$ is a linear $G$-sphere.

A fundamental question is: For which finite groups does a Borsuk-Ulam type result hold? T. Bartsch [1] answered this question as follows.

**Theorem 0.4 ([1]).** Suppose that $G$ is a finite group. The "weak" Borsuk-Ulam theorem for linear $G$-spheres holds if and only if $G$ is a $p$-group. Namely $G$ has the following property $(W)$ if and only if $G$ is a $p$-group.
(W): There exists a monotonely increasing function $\varphi_G$ diverging to infinity such that for any linear $G$-spheres $SV$, $SW$ ($V^G = W^G = 0$) with a $G$-map $f: SV \to SW$, the inequality $\varphi_G(\dim SV) \leq \dim SW$ holds.

By Theorem 0.3, one can take the identity map as $\varphi_G$ for $G = \mathbb{Z}_p^k$, which is the best possible function satisfying (W); such a function $\varphi_G$ is called the Borsuk-Ulam function. In general, it is difficult to determine the Borsuk-Ulam function, but a few results are known; see [1] for relevant results.

For other topics on the Borsuk-Ulam theorem, see also Steinlein [25, 26], Matoušek [19].

1. **The Isovariant Borsuk-Ulam Theorem**

Let $G$ be a compact Lie group. Let $X, Y$ be $G$-spaces, and $V, W$ $G$-representations.

Definition 1. A continuous map $f: X \to Y$ is called $G$-isovariant (or isovariant) if $f$ is $G$-equivariant and preserves the isotropy groups, i.e., $G_{f(x)} = G_x$ for any $x \in X$.

A. G. Wasserman [27] first studied an isovariant version of the Borsuk-Ulam theorem. Using the Borsuk-Ulam theorem for free $\mathbb{Z}_p$-actions, one can obtain the following result.

**Theorem 1.1** (Isovariant Borsuk-Ulam theorem). Let $G$ be a solvable compact Lie group. If there is an isovariant map $f: SV \to SW$, then

$$\dim SV - \dim SV^G \leq \dim SW - \dim SW^G.$$  

We note that this result still holds for semilinear actions on spheres.

Definition 2. The smooth $G$-action on a (homotopy) sphere $M$ is called semilinear if for any $H \leq G$, $M^H$ is a (homotopy) sphere or $\emptyset$. We call such a $G$-manifold $M$ a semilinear $G$-sphere.

**Theorem 1.2** ([21]). Let $G$ be a solvable compact Lie group and let $M, N$ be semilinear $G$-spheres. If there is an isovariant map $f: M \to N$, then

$$\dim M - \dim M^G \leq \dim N - \dim N^G.$$  

It is still open whether Theorem 1.1 holds for an arbitrary compact Lie group, but Theorem 1.2 does not hold if $G$ is nonsolvable.

**Theorem 1.3** ([21]). Let $G$ be a nonsolvable compact Lie group. There are fixed-point-free semilinear $G$-spheres $M_n, n \geq 1$, with $\lim_{n \to \infty} \dim M_n = \infty$ and a representation sphere $SW$ such that there is an isovariant maps $f_n: M_n \to SW$ for every $n$. 

Consequently, we obtain a Bartsch type result for semilinear actions; namely, the isovariant Borsuk-Ulam theorem for semilinear $G$-spheres holds if and only if $G$ is solvable.

Remark. Bartsch’s result, Theorem 0.4, still holds for semilinear $G$-spheres.

2. THE CONVERSE OF THE ISOVARINT BORSUK-ULAM THEOREM

Let $G$ be a solvable compact Lie group. A subgroup means a closed subgroup. As mentioned in the previous section, the isovariant Borsuk-Ulam theorem holds for $G$. We would like to consider the converse.

If there is an isovariant map $f : SV \to SW$, then $f^H : SV^H \to SW^H$, $H \triangleleft K \leq G$, is $K/H$-isovariant. Since $K/H$ is also solvable, we can apply the isovariant Borsuk-Ulam theorem to $f^H$. Hence we have

**Proposition 2.1.** Let $G$ be a solvable compact Lie group. If there is an isovariant map $f : SV \to SW$, then

$$(C_{V,W}) : \dim SV^H - \dim SV^K \leq \dim SW^H - \dim SW^K \text{ for any pair of closed subgroups } H \triangleleft K.$$ 

We formulate the converse problem of the isovariant Borsuk-Ulam theorem as follows.

Question. Let $G$ be a solvable compact Lie group. Suppose that a pair $(V, W)$ of $G$-representations satisfies

(a) $\text{Iso } SV \subset \text{Iso } SW$,
(b) $(C_{V,W})$.

Is there a $G$-isovariant map $f : SV \to SW$ (or $f : V \to W$)?

Remark. (1): The condition (a) is obviously necessary. However if $G$ is abelian, then one can see that the condition (b) implies (a); so the condition (a) can be omitted.

(2) Note that there exists an isovariant map $f : SV \to SW$ if and only if there exists an isovariant map $f : V \to W$.

Definition 3. If this question is affirmative for $G$, we say that $G$ has the complete Borsuk-Ulam property (or $G$ is a complete Borsuk-Ulam group).

Unfortunately the complete answer is not known yet, but there are some partial results. In this note, we would like to give the outline of proof of the following theorem; the full detail will appear in [23].

**Theorem 2.2.** The following groups have the complete Borsuk-Ulam property.

(1) finite abelian $p$-group,
(2) \( \mathbb{Z}_{pq} \),
(3) \( \mathbb{Z}_{pq} \).

where \( p, q, r \) are prime numbers.

Let \( T_k, k \in \mathbb{Z}, \) be the irreducible \( S^1 \)-representation given by \( t \cdot z := t^k z, t \in S^1 (\subset \mathbb{C}), z \in T_k (\subset \mathbb{C}) \). Restricting \( T_k \) to \( \mathbb{Z}_n \subset S^1 \), we have a \( \mathbb{Z}_n \)-representation, denoted by the same symbol \( T_k \). For simplicity we here treat only complex representations.

2.1. Proof of Theorem 2.2 (1) (outline). Let us consider the case \( G = \mathbb{Z}_p \). Then \( T_k, 0 \leq k \leq p - 1 \), are all irreducible \( \mathbb{Z}_p \)-representations. We may suppose \( V^G \) = \( W^G \) = 0. In fact, one can see that there exists an isovariant map \( f : V \to W \) if and only if there exists an isovariant map \( f : V_G \to W_G \), where \( V_G \) denotes the orthogonal complement of \( V^G \) in \( V \). Therefore we may set \( V = T_{k_1} \oplus \cdots \oplus T_{k_n}, W = T_{l_1} \oplus \cdots \oplus T_{l_m}, \) where \( k_i, l_i \) are prime to \( |G| \).

An isovariant map \( f : T_k \to T_l \) is defined by \( f_{k,l}(z) = \xi^{k'l}z \), where \( k'k \equiv 1 \mod |G| \). Since condition \( (C_{V,W}) \) implies \( n \leq m \), one can construct an isovariant map \( f : V \to W \) using \( f_{k,l} \).

For a general abelian \( p \)-group, a similar argument shows Theorem 2.2 (1).

2.2. Proof of Theorem 2.2 (2) (outline).

Definition 4. A pair of representations \( (V, W) \) is called primitive if \( V \) and \( W \) cannot be decomposed into \( V = V_1 \oplus V_2, W = W_1 \oplus W_2 \) such that \( (V_i, W_i) \neq (0, 0) \) satisfies \( (C_{V_i,W_i}); i = 1, 2 \).

If there are isovariant maps \( f_1 : V_1 \to W_1 \), then \( f_1 \oplus f_2 : V_1 \oplus V_1 \to V_2 \oplus W_2 \) is also isovariant; therefore it suffices to construct an isovariant map between each primitive pair.

Let us consider \( G = \mathbb{Z}_{pq} \) for example. Clearly \( (0, T_s) \) and \( (T_k, T_t), (k, |G|) = (l, |G|), \) are primitive, and one can easily construct isovariant maps between these representations as in the proof of (1). In addition, a new primitive pair \( (T_1, T_p \oplus T_q) \) appears for \( G = \mathbb{Z}_{pq} \). In this case an isovariant map exists; for example, the map defined by \( f : z \mapsto (z^p, z^q) \) is isovariant. These pairs mentioned above are essentially all primitive pairs for \( \mathbb{Z}_{pq} \). Therefore \( \mathbb{Z}_{pq} \) has the complete isovariant Borsuk-Ulam property.

For \( \mathbb{Z}_{p^n q^m} \), other primitive pairs appear, but one can directly define isovariant maps in a similar way. For example, \( (T_p \oplus T_q, T_{p^2} \oplus T_{pq} \oplus T_{q^2}) \) is primitive for \( \mathbb{Z}_{p^n q^m}, n, m \geq 2 \). In this case there is an isovariant map; for example \( f : (z_1, z_2) \mapsto (z_1^p, z_1^q + z_2^q, z_2^q) \) is isovariant. Thus one can see that \( \mathbb{Z}_{p^n q^m} \) has the complete isovariant Borsuk-Ulam property.
2.3. Proof of Theorem 2.2 (3) (outline). Next consider the case of $\mathbb{Z}_{pqr}$. The proof is more complicated.

For all primitive pairs except one type, one can directly define isovariant maps as before. The exception is the following type of primitive pair:

$$(T_p \oplus T_q \oplus T_r, T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{pr}).$$

If there is an isovariant map for this pair, it turns out that $\mathbb{Z}_{pqr}$ has the complete isovariant Borsuk-Ulam property. It seems, however, difficult to directly define an isovariant map; so we would like to use equivariant obstruction theory.

The question is the following:

**Question.** Is there a $\mathbb{Z}_{pqr}$-isovariant map

$$f : T_p \oplus T_q \oplus T_r \rightarrow T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{pr}?$$

The answer is yes. Actually we shall show the existence of an $S^1$-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \rightarrow S(T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{pr}).$$

Therefore we see that $\mathbb{Z}_{pqr}$ has the complete Borsuk-Ulam property.

3. The existence of an isovariant map

We shall discuss the above question in a more general setting. Let $G = S^1$ and let $M$ be a rational homology sphere with pseudofree $S^1$-action.

**Definition 5 (Montgomery-Yang).** An $S^1$-action on $M$ is pseudofree if

1. the action is effective, and
2. the singular set $M^{>1} := \bigcup_{1 \neq H \leq S^1} M^H$ consists of finitely many exceptional orbits.

Here an orbit $G(x)$ is called exceptional if $G(x) \cong S^1/C$, $(1 \neq C < S^1)$.

**Example 3.1.** Let $V = T_p \oplus T_q \oplus T_r$. Then the $S^1$-action on $SV$ is pseudofree. Indeed it is clearly effective, and

$$SV^{>1} = ST_p \coprod ST_q \coprod ST_r \cong S^1/Z_p \coprod S^1/Z_q \coprod S^1/Z_r$$

**Remark.** There are many "exotic" pseudofree $S^1$-actions on high-dimensional homotopy spheres. (Montgomery-Yang [20], Petrie [24].)

Let $SW$ be any $S^1$-representation sphere. We consider an $S^1$-isovariant map $f : M \rightarrow SW$.

The result is the following:
Theorem 3.2. With the above notation, there is an $S^1$-isovariant map $f : M \to SW$ if and only if

1. $\text{Iso } M \subset \text{Iso } SW$,
2. $\dim M - 1 \leq \dim SW - \dim SW^H$ when $1 \neq H \leq C$ for some $C \in \text{Iso } M$,
3. $\dim M + 1 \leq \dim SW - \dim SW^H$ when $1 \neq H \leq C$ for every $C \in \text{Iso } M$.

3.1. Examples. We give some examples. Let $p$, $q$, $r$ be pairwise coprime integers greater than 1.

Example 3.3. There is an $S^1$-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

Proof. (PF1) and (PF2) are fulfilled. One can see $\text{Iso } M = \{1, \mathbb{Z}_p, \mathbb{Z}_q, \mathbb{Z}_r\}$ and $\text{Iso } SW = \{1, \mathbb{Z}_p, \mathbb{Z}_q, \mathbb{Z}_r, \mathbb{Z}_{pq}, \mathbb{Z}_{qr}, \mathbb{Z}_{rp}\}$; hence $\text{Iso } M \subset \text{Iso } SW$. \Box

Example 3.4. There is not an $S^1$-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

Proof. (PF1) is not fulfilled. \Box

Remark. There is an $S^1$-equivariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

By Example 3.3, we see that $\mathbb{Z}_{pq}$ has the complete Borsuk-Ulam property.

3.2. Proof of Theorem 3.2 (outline). We shall give the outline of Theorem 3.2. The full detail will appear in [22]. Set $Y := SW \setminus SW^{>1}$. Note that $S^1$ acts freely on $Y$. Let $N_i$ be an $S^1$-tubular neighborhood of each exceptional orbit in $M$. By the slice theorem, $N_i$ is identified with $S^1 \times C_i D_1$ (1 ≤ $i$ ≤ $r$), where $C_i$ is the isotropy group of the exceptional orbit and $U_i$ is the slice $C_i$-representation. Set $X := M \setminus (\bigcup_i \text{int } N_i)$. Note that $S^1$ acts freely on $X$.

The “only if” part is proved by the (isovariant) Borsuk-Ulam theorem. Indeed we can show (PF1) as follows. Take a point $x \in M$ with $G_x = C$ and a $C$-invariant closed neighborhood $B$ of $x$ $C$-diffeomorphic to some unit disk $DV$. Hence we obtain an $H$-isovariant map $f : SV \to SW$. Applying the isovariant Borsuk-Ulam theorem to $f$, we have (PF1).

We next show (PF2). Since $f$ is isovariant, $f$ maps $M$ into $SW \setminus SW^H$, and since $SW \setminus SW^H$ is $S^1$-homotopy equivalent to $SW_H$, we obtain an $S^1$-map $g : M \to SW_H$. By Theorem 0.3, we obtain (PF2).

To show the converse, we begin with the following lemma.

Lemma 3.5. There is an $S^1$-isovariant map $\tilde{f}_i : N_i \to SW$. 

Proof. Let \( N_i = N = S^1 \times_C DV \), where \( C \) is the isotropy group of the exceptional orbit and \( V \) is the slice representation. Similarly take a closed \( S^1 \)-tubular neighborhood \( N' \) of an exceptional orbit with isotropy group \( C \), and set \( N' = S^1 \times_C DV' \). By (PF1), we see that \( \dim SV + 1 \leq \dim SV' - \dim SV'' \). Since \( C \) acts freely on \( SV \), by obstruction theory, there is an \( C \)-map \( g : SV \to SV' \setminus SV'' \subset SW \), and so we obtain a \( C \)-isovariant map \( g : SV \to SW \). Taking a cone, we have a \( C \)-isovariant map \( \tilde{g} : DV \to DV' \); hence there is an \( S^1 \)-isovariant map \( \tilde{f} = S^1 \times_C \tilde{g} : N \to N' \subset SW \).

Set \( f_i := \tilde{f}_i|_{\partial N_i} : \partial N_i \to Y \), and \( f := \bigsqcup_i f_i : \partial X \to Y \). If \( f \) is extended to an \( S^1 \)-map \( F : X \to Y \), by gluing the maps, we obtain an \( S^1 \)-isovariant map

\[
F \cup (\bigsqcup_i \tilde{f}_i) : M \to SW.
\]

Thus it suffices to investigate the following question:

(Q) Is there an extension \( F : X \to Y \) of \( f : \partial X \to Y \)?

Since \( S^1 \) acts freely on \( X \) and \( Y \), the obstruction to an extension lies in

\[
H^*(X/S^1, \partial X/S^1; \pi_{k-1}(Y)).
\]

Set \( k = \dim SW - \dim SW^{>1} \). A standard computation shows

\textbf{Lemma 3.6.} \quad (1) \( Y \) is \( (k-2) \)-connected and \( (k-1) \)-simple.

(2) \( \pi_{k-1}(Y) \cong H_{k-1}(Y) \cong \oplus_{H \in A} \mathbb{Z} \), where \( A := \{ H \in \text{Iso } SW : \dim SW^H = \dim SW^{>1} \} \), and generators are represented by \( SW^H \), \( H \in A \).

Note that \( \dim M - 1 \leq k \) by (PF1) and (PF2). We divide into two cases.

Case I: \( \dim M - 1 < k \) (i.e., \( \dim X/S^1 < k \)). In this case, we see that

\[
H^*(X/S^1, \partial X/S^1; \pi_{k-1}(Y)) = 0
\]

by dimensional reason. Hence the obstruction vanishes and there exists an extension \( F : X \to Y \).

Case II: \( \dim M - 1 = k \) (i.e., \( \dim X/S^1 = k \)). The obstruction \( \gamma_{S^1}(f) \) to an extension lies in

\[
H^k(X/S^1, \partial X/S^1; \pi_{k-1}(Y)) \cong \oplus_{H \in A} \mathbb{Z}.
\]

\[
(H^i(X/S^1, \partial X/S^1; \pi_{k-1}(Y)) = 0, l \neq k)
\]

To detect the obstruction, we introduce the multidegree.

\textbf{3.3. Multidegree.} Let \( N = S^1 \times_C DU \subset M \), \( 1 \neq C \in \text{Iso } (M) \), \( \dim M - 1 = \dim U = k \), and \( f : \partial N \to Y : S^1 \)-map, \( \tilde{f} = f|_{SU} : SU \to Y \). \( C \)-map.

\textbf{Definition 6.} \( \text{Deg } f := \tilde{f}_*([SU]) \in \oplus_{H \in A} \mathbb{Z}, \tilde{f}_* : H_{k-1}(SU) \to H_{k-1}(Y) \), under identifying \( H_{k-1}(Y) \) with \( \oplus_{H \in A} \mathbb{Z} \).

Then the obstruction \( \gamma_{S^1}(f) \) is described by the multidegrees.
Proposition 3.7. Let $F_0 : X \to Y$ be a fixed $S^1$-map (not necessary extending $f$). Set $f_{0,i} = F_0|_{\partial N_i}$. Then
\[ \gamma_{S^1}(f) = \sum_{i=1}^{r}(\text{Deg} f_i - \text{Deg} f_{0,i})/|C_i|. \]

Remark. (1) There always exists $F_0$.

(2) $\text{Deg} f_i - \text{Deg} f_{0,i} \in \oplus_{H \in A}|C_i|Z$ by the equivariant Hopf type result. (See the next section.)

Using this proposition and equivariant Hopf type results in the next section, we can choose $S^1$-isovariant maps $\tilde{f}_i : N_i \to SW$ so that $\gamma_{S^1}(f) = 0$.

4. EQUIVARIANT HOPF TYPE RESULTS

Let $N = S^1 \times_C DU (\subset M)$, dim $M - 1 = k$ as before. Then the following Hopf type theorem holds.

Theorem 4.1 ([22]).
(1) $\text{Deg} : [\partial N, Y]_{S^1} \to \oplus_{H \in A}Z$ is injective.

(2) The image of $\text{Deg} - \text{Deg} f_0$ coincides with $\oplus_{H \in A}|C|Z$, where $f_0$ is any fixed $S^1$-map.

The next result shows the extendability of $f : \partial N = S^1 \times_C SU \to Y$. Set $\text{Deg} f = (\text{deg}_H(f))_{H \in A} \in \oplus_{H \in A}Z$.

Theorem 4.2 ([22]).
(1) $f : \partial N \to Y$ is extendable to an $S^1$-isovariant map $\tilde{f} : N \to SW$ if and only if $d_H(f) = 0$ for any $H \in A$ with $H \not\leq C$.

(2) For any extendable $f$ and for any $(a_H) \in \oplus_{H \in A}|C|Z$ satisfying $a_H = 0$ for $H \in A$ with $H \not\leq C$, there exists an $S^1$-map $f' : \partial N \to Y$ such that $f'$ is extendable to an $S^1$-isovariant map $\tilde{f}' : N \to SW$ and $\text{Deg} f' = \text{Deg} f + (a_H)$.

4.1. Example of multidegrees. Finally we give some examples. Take $S^1$-representations $V = T_p \oplus T_q \oplus T_r$ and $W = T_l \oplus T_{pq} \oplus T_{qr} \oplus T_{rp}$, where $p, q, r$ are distinct primes. Let us consider linear spheres $SV, SW$. Let $N_i$ be a closed $S^1$-tubular neighborhood of the exceptional orbit $ST_i \cong S^1/Z_i$ in $SV$, where $i = p, q, r$, and then $N_i$ is identified with $ST_i \times D(T_j \oplus T_k) \cong S^1 \times_{Z_i} D(T_j \oplus T_k)$. Thus we may set

\[ N_p = \{(z_1, z_2, z_3) \in V \mid |z_1| = 1, ||(z_2, z_3)|| \leq 1\}, \]
\[ N_q = \{(z_1, z_2, z_3) \in V \mid |z_2| = 1, ||(z_1, z_3)|| \leq 1\}, \]
\[ N_r = \{(z_1, z_2, z_3) \in V \mid |z_3| = 1, ||(z_1, z_2)|| \leq 1\}. \]

We have $A = \{Z_p, Z_q, Z_r\}$; hence we can set
\[ \text{Deg} f = (d_{z_p}(f), d_{z_q}(f), d_{z_r}(f)) \in Z^3. \]

Take positive integers $\alpha, \beta, \gamma, \delta, \xi, \eta$ such that $\alpha p - \beta q = 1, \gamma q - \delta r = 1, \xi r - \eta p = 1$. 
Example 4.3. We define $g_i : V \to W$ as follows:

\[ g_p(z_1, z_2, z_3) = (z_3^\xi z_1^\eta, z_2^{\alpha}, z_2^{\beta}, z_1^{\gamma}), \]
\[ g_q(z_1, z_2, z_3) = (z_1^\alpha z_2^\beta, z_2^{\gamma}, z_2^{\gamma}, z_3^\delta), \]
\[ g_r(z_1, z_2, z_3) = (z_2^\gamma z_3^\gamma, z_1^\alpha, z_3^\delta, z_3^\delta). \]

Restricting $g_i$ to $N_i$, we obtain an $S^1$-map $h_i := g_i : N_i \to W$. Since $h_i^{-1}(0) = \emptyset$, we have an $S^1$-map $\tilde{f}_i := h_i/\|h_i\| : N_i \to SW$. Moreover $\tilde{f}_i$ is an $S^1$-isovariant.

Set $f_i = \tilde{f}_i|_{\partial N_i}$. Then $d_{\mathbb{Z}_p}(f_p)$ is equal to the degree of the map $f'_p : S(T_q \oplus T_r) \to S(T_1 \oplus T_{qr})$:

\[ (z_2, z_3) \mapsto (z_2^\xi z_1^\eta, z_2^{\gamma})/\|(z_2^\xi z_1^\eta, z_2^{\gamma})\|, \]

where $z_1$ is any fixed nonzero number. Hence we have $d_{\mathbb{Z}_p}(f_p) = \xi r = 1 + \eta p$. Similarly one can see that $d_{\mathbb{Z}_q}(f_p) = d_{\mathbb{Z}_r}(f_p) = 0$. Thus we obtain

\[
\begin{align*}
\text{Deg } f_p &= (1 + \eta p, 0, 0).
\end{align*}
\]

In a similar way, we have

\[
\begin{align*}
\text{Deg } f_q &= (0, 1 + \beta q, 0), \\
\text{Deg } f_r &= (0, 0, 1 + \delta r).
\end{align*}
\]

Example 4.4. Next we consider the following $S^1$-maps $g'_i : V \to W$:

\[ g'_p(z_1, z_2, z_3) = (z_2^\gamma z_3^\delta, z_1^\gamma, z_1^{\gamma} + z_2^{\gamma}, z_1^{\gamma}), \]
\[ g'_q(z_1, z_2, z_3) = (z_2^\gamma z_1^\gamma, z_2^{\gamma}, z_2^{\gamma} + z_1^{\gamma}), \]
\[ g'_r(z_1, z_2, z_3) = (z_2^\gamma z_3^\gamma, z_1^\gamma, z_3^{\gamma}, z_3^{\gamma}). \]

Then by restriction and normalization, we obtain $S^1$-isovariant maps $\tilde{f}'_i : N_i \to SW$ and $f'_i : \partial N_i \to SW$, respectively. In this case, one can see that

\[
\begin{align*}
\text{Deg } f'_p &= (1, 0, 0), \\
\text{Deg } f'_q &= (0, 1, 0), \\
\text{Deg } f'_r &= (0, 0, 1).
\end{align*}
\]

In fact, for example, $d_{\mathbb{Z}_p}(f'_p) = 1$ is showed as follows. Consider the map $\psi : T_q \oplus T_r \setminus 0 \to T_1 \oplus T_{qr} \setminus 0; (z_2, z_3) \mapsto (z_2^\gamma z_3^\delta, z_1^\gamma, z_1^{\gamma} + z_2^{\gamma})$. One can see that $\psi^{-1}(1, 0) = \{((-1)^\delta, (1)^\gamma)\}$ and the Jacobian is $\gamma q + r \delta > 0$; hence $(1, 0) \in T_1 \oplus T_{qr} \setminus 0$ is a regular value, and so deg $\psi = 1$.

$Y = SW \setminus SW^{>1}$. Let $[\partial N_i, Y]^t$ for $i = p, q, r$, denote the set of $S^1$-homotopy classes of $S^1$-maps extended to $S^1$-isovariant maps from $N_i$ to $SW$. By Theorems 4.1 and 4.2, we see the following.

Proposition 4.5. The map $D_i : [\partial N_i, Y]^t \to \{ f \mapsto (d_{\mathbb{Z}_i}(f) - 1)/i \}$, is a bijection for $i = p, q, r$. 

For the above maps, we have $D_{p}(f_{p}) = \eta$ and $D_{p}(f'_{p}) = 0$.

**Example 4.6.** We next define another $S^{1}$-map $f_{0,i}$ as follows. Define an $S^{1}$-map $g_{0} : V \to W$ by setting

$$g_{0}(z_{1}, z_{2}, z_{3}) = (z_{1}^{\alpha}z_{2}^{\beta} + z_{2}^{\gamma}z_{3}^{\delta} + z_{3}^{\xi}z_{1}^{\eta}, z_{1}^{\eta}, z_{2}^{\rho}, z_{3}^{\nu}).$$

Since $g_{0}$ maps the free part of $V$ into the free part of $W$, by restriction and normalization, we have an $S^{1}$-map $f_{0,i} : \partial N_{i} \to Y$. In this case we have

$$\text{Deg} f_{0,p} = (1 + \eta p, -\beta p, 0),$$

$$\text{Deg} f_{0,q} = (0, 1 + \beta q, -\delta q),$$

$$\text{Deg} f_{0,r} = (-\eta r, 0, 1 + \delta r).$$

By Theorem 4.2, each $f_{0,i}$ cannot be isovariantly extended on $N_{i}$.

However, restricting $g_{0}$ on $X = SV \setminus \text{int}(N_{p} \cup N_{q} \cup N_{r})$, one can regard $g_{0}$ as an $S^{1}$-map from $X$ to $Y$. Consequently it turns out that $\bigsqcup f_{0,i}$ can be extended on $X$. Consider the $S^{1}$-maps $f = \bigsqcup f_{i} : \partial N_{i} \to Y$ and $f' = \bigsqcup f'_{i} : \partial N_{i} \to Y$ in Examples 2 and 3. By Proposition 3.7, the obstruction $\gamma S^{1}(f)$ to an extension on $X$ is described as $\gamma S^{1}(f) = (\eta, \beta, \delta)$ and $\gamma S^{1}(f') = (0, 0, 0)$; hence $f$ cannot be extended on $X$, but $f'$ can.

We also note the following.

**Proposition 4.7.** An $S^{1}$-isovariant map $\tilde{h} = \bigsqcup h_{i} : \bigsqcup N_{i} \to SV$ is isovariantly extended on $SV$ if and only if $\text{Deg} h_{p} = (1, 0, 0)$, $\text{Deg} h_{q} = (0, 1, 0)$ and $\text{Deg} h_{r} = (0, 0, 1)$, where $h_{i} = \tilde{h}_{i}\mid \partial N_{i}$.

**Proof.** One can set $\text{Deg} h_{p} = (1 + \eta p, 0, 0)$, $\text{Deg} h_{q} = (0, 1 + \beta q, 0)$ and $\text{Deg} h_{r} = (0, 0, 1 + \delta r)$. Then one can see $\gamma S^{1}(h) = (n, m, l)$, and so $\gamma S^{1}(h) = 0$ if and only if $(n, m, l) = (0, 0, 0)$. \qed

**References**


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