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KYOTO UNIVERSITY
GENERIC STRUCTURES AND CONTROL FUNCTIONS
(A COMMENTARY ON EVANS' PREPRINT)

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ABSTRACT. We survey the results in "Some remarks on generic structures" [E] written by Evans, and give some detailed proofs which are omitted in his note.

1. INTRODUCTION

In simplicity theory, Hrushovski’s generic constructions yield various results. As in his $\omega$-categorical stable pseudoplane, he constructed an $\omega$-categorical, simple, rank one, non-locally modular theory by amalgamating finite graphs whose local rank is controlled by an increasing unbounded convex function. In [E1], Evans gave a sufficient condition on control functions for constructing $\omega$-categorical simple generic structures. We review this in fifth section. In [E], Evans gave an $\omega$-categorical non-simple generic structure by carefully setting a control function (In this note, sixth section). This non-simple generic structure has 3-strong order property. For any $n \geq 3$, $n$-strong order property was introduced by Shelah. (See [Sh] and third section in this note.) Strict order property implies $n$-strong order property, and $n + 1$-strong order property implies $n$-strong order property for any $n \geq 3$. Evans showed that generic structures given by control functions do not have 4-strong order property, we follow this result in fourth section.

In [P], Pourmahdian conjectured that generic structures without control function, so-called $(K_0, \prec)$-generic structure, will be non-simple. In [P], Pourmahdian considered a natural expanded inductive (incomplete) theory $T_{nat}$ of a universal theory $T_0$ only axiomatizing that any finite substructure has non-zero positive local rank. Pourmahdian showed that $T_{nat}$ is a Robinson theory and its universal domain is simple as a structure, and $T_{nat}$ does not have model companion. (Natural expansioned structure of $(K_0, \prec)$-generic structure is an existentially closed model of $T_{nat}$.) Evans gave an example of $(K_0, \prec)$-generic structure having strict order property, we discuss this issue in second section.

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I would like to thank David M. Evans for his permission to submit this note.
This note is organized as follows.

Section 2: We will follow the proof that $\text{Th}(M_0)$ has strict order property, where $M_0$ is $(K_0, <)$-generic structure with one ternary relation.

Section 3: Review of [Sh].

Section 4: We will follow the proof that $\text{Th}(\Lambda I_{0})$ has SOP, where $\Lambda I_{0}$ is a $(K_0, <)$-generic structure with one ternary relation.

Section 5: Review of [E].

Section 6: We will follow the proof that for some control function $f$, $\text{Th}(\Lambda I_{f})$ has SOP, where $\Lambda I_{f}$ is a $(K_f, <)$-generic structure and $K_f$ is the class of finite graph $A$ satisfying with $\delta(A) \geq f(|A|)$ and control function $f$ is a convex increasing unbounded function from $\mathbb{N}$ to $\mathbb{R}$.

Section 7, 8: Long appendices for Section 6, which are omitted in [E1].

2. $\text{Th}(M_0)$ has SOP. (Definable correspondence between graphs and ternary hypergraphs)

Let $\mathfrak{A}$ be a ternary relation. For finite ternary-hypergraph $\mathfrak{A}$, we define the predimension as follows.

$$\delta(\mathfrak{A}) = |\mathfrak{A}| - |\mathfrak{A}^\mathfrak{A}|$$

For finite $\mathfrak{A} \subseteq \mathfrak{B}$ we define a partial order $<$ as follows

$$\mathfrak{A} < \mathfrak{B} \iff \delta(\mathfrak{X}) > \delta(\mathfrak{A})(\mathfrak{X} \subset \mathfrak{B}).$$

For possibly infinite $\mathfrak{A} \subseteq \mathfrak{B}$ we define

$$\mathfrak{A} < \mathfrak{B} \iff \mathfrak{X} \cap \mathfrak{A} = \mathfrak{A}(\forall \mathfrak{X} \subseteq \mathfrak{B}).$$

Note that $\mathfrak{A} < \mathfrak{A}$. For possibly infinite $\mathfrak{A} \subseteq \mathfrak{B}$, there exists the $<$-closure $\text{cl}_{\mathfrak{B}}(\mathfrak{A})$ of $\mathfrak{A}$ in $\mathfrak{B}$. $K_0$ is the class of finite 3-hypergraphs defined by

$$\mathfrak{A} \in K_0 \iff \emptyset < \mathfrak{A}$$

$K_0$ is the class of 3-hypergraphs whose finite sub-hypergraph is all in $K_0$. $M_0$ denotes the $(K_0, <)$-generic structure.

**Notation 2.1.** Let $(A, R)$ be a graph, where $R$ is the binary relation for the graph. We define the following ternary graph $(\mathfrak{H}_A, \mathfrak{A})$.

- $\mathfrak{H}_A = A \cup R^A \cup \{x_A, y_A\}$, where $x_A, y_A$ are new elements.
- $(\mathfrak{A})^{\mathfrak{H}_A} = \{(x_A, y_A, a) : a \in A\} \cup \{(a, b, (a, b)) : (a, b) \in R^A\}$

$(\mathfrak{H}_A, \mathfrak{A})$ is definable in $(A, R)$ with two new constants.

**Lemma 2.2.** Let $(A, R)$ be a graph. Then

1. $\mathfrak{H}_A \subseteq \overline{K_0}$.
2. $\mathfrak{H}_A = \text{cl}_{\mathfrak{H}_A}(x_A, y_A)$. 


Proof. Let $\mathfrak{X} \subseteq \mathfrak{H}_{A}$, and let $V(\mathfrak{X})$ be the vertex set of $\mathfrak{X}$. Then $V(\mathfrak{X}) \subseteq A \cup R^{A} \cup \{x_{A}, y_{A}\}$ follows. If $x_{A}, y_{A} \in \mathfrak{X}$, then $\delta(\mathfrak{X}) = |V(\mathfrak{X})| - (|V(\mathfrak{X}) \cap R^{A}| + |V(\mathfrak{X}) \cap A|) > 0$, since $V(\mathfrak{X}) = \{x_{A}, y_{A}\} \cup (V(\mathfrak{X}) \cap R^{A}) \cup (V(\mathfrak{X}) \cap A)$. Otherwise, $\delta(\mathfrak{X}) = |V(\mathfrak{X})| - (|V(\mathfrak{X}) \cap R^{A}|) > 0$, since $|V(\mathfrak{X}) \cap R^{A}| > 0$ implies $|V(\mathfrak{X}) \cap A| > 0$.

Let $c \in A$. Then $\delta(c/x_{A}, y_{A}) = 0$. So, if $c \not\in \text{cl}_{A}(x_{A}, y_{A})$, then $0 < \delta(c/\text{cl}_{A}(x_{A}, y_{A})) \leq \delta(c/x_{A}, y_{A}) = 0$, a contradiction. Next, let $c = (a, b) \in R^{A}$. Then $\delta(c/a, b) = 0$. By the above argument, we see $c \in \text{cl}_{A}(a, b)$. As $a, b \in \text{cl}_{A}(x_{A}, y_{A})$, we see that $c \in \text{cl}_{A}(x_{A}, y_{A})$.

Next, for any symmetric 3-hypergraph having at least two vertices, we construct a graph as follows.

Notation 2.3. Let $(\mathfrak{A}, \mathfrak{R})$ be a symmetric 3-hypergraph having at least two vertices. Fix two vertices $a, b \in \mathfrak{A}$. We define the following graph $G_{(\mathfrak{A}, a, b)} = (G_{\mathfrak{A}}, R)$ as follows.

- $G_{\mathfrak{A}} = \{c \in \mathfrak{A} : \mathfrak{A} \models \mathfrak{R}(c, a, b)\}$
- $R^{\mathfrak{A}} = \{(c, d) \in \mathfrak{A}^{2} : \mathfrak{A} \models \mathfrak{R}(c, a, b) \cap \mathfrak{R}(d, a, b) \cap \exists x \mathfrak{R}(x, c, d)\}$

$G_{(\mathfrak{A}, a, b)} = (G_{\mathfrak{A}}, R)$ is definable in $(\mathfrak{A}, \mathfrak{R})$ with parameters $a, b \in \mathfrak{A}$.

Remark 2.4. (1) $\mathfrak{A} \not\models \mathfrak{R}(c, a, b)$, where $a, b \in \mathfrak{A}$. (If $\mathfrak{A} \models \neg\mathfrak{R}(d, a, b)$, $d$ will not appear in the righthand.)

(2) $A \simeq G_{(\mathfrak{A}, a, b)}$.

Proof. Clearly, $G_{\mathfrak{A}} = A$ and $R^{\mathfrak{A}} = R^{A}$, as desired.

Lemma 2.5. Let $(\mathfrak{A}, \mathfrak{R})$ be a symmetric 3-hypergraph having at least two vertices. Then

1. $a, b \not\in G_{\mathfrak{A}} \subseteq \text{cl}_{\mathfrak{A}}(a, b)$
2. If $(c, d) \in R^{\mathfrak{A}}$, then $\mathfrak{R}^{\text{cl}_{\mathfrak{A}}(a, b)} \models \exists x \mathfrak{R}(x, c, d)$
3. If $\mathfrak{A} < \mathfrak{B}$, then $G_{(\mathfrak{B}, a, b)} = G_{(\mathfrak{A}, a, b)}$

Proof. If $\mathfrak{A} \models \mathfrak{R}(c, a, b)$, then $c \in \text{cl}_{\mathfrak{A}}(a, b)$. (1),(2) follow. If $\mathfrak{A} < \mathfrak{B}$, then $\text{cl}_{\mathfrak{A}}(a, b) = \text{cl}_{\mathfrak{B}}(a, b)$. So, (3) follows.

Notation 2.6. Let $\varphi$ be a sentence in the language of graphs with binary relation symbol $R(x_{1}, x_{2})$. We construct a formula $\sigma_{\varphi}$ having free variable $y, z$ in the the language of 3-hypergraphs with ternary relation symbol $\mathfrak{R}(x_{1}, x_{2}, x_{3})$ as follows.

- Replace all atomic subformulas $R(x_{1}, x_{2})$ by $\mathfrak{R}(x_{1}, y, z) \land \mathfrak{R}(x_{2}, y, z) \land \exists w \mathfrak{R}(w, y, z)$
- Replace $\forall x(\psi(\bar{x}))$, $\exists x(\psi(\bar{x}))$ by $\forall x(\mathfrak{R}(x, y, z) \rightarrow \psi(\bar{x}))$, $\exists x(\mathfrak{R}(x, y, z) \land \psi(\bar{x}))$. 

**Remark 2.7.** Let \((\mathfrak{A}, \mathfrak{R}) \in \mathcal{K}_0, a, b \in \mathfrak{A}\) and \(\varphi\) be a sentence in the language of graphs. Then
\[
G_{(\mathfrak{A}, a, b)} \models \varphi \iff \mathfrak{A} \models \psi_\varphi(a, b)
\]

The above remark follows from "REDUCTION THEOREM", a (non-onto) map from \(G_{(\mathfrak{A}, a, b)}\) to \(\mathfrak{A}\), and the way of replacement of quantifiers. Reduction theorem needs a onto map, but our \(\psi_\varphi\)'s quantifiers are bounded in \(\mathfrak{A}(\ast, a, b)\). So we need not a onto map, here.

**Fact 2.8.** Let \(M\) be an \(L\)-structure, and \(N\) be an \(L'\)-structure. Suppose that
- there exists a partial onto map \(f\) from \(M^n\) to \(N\) (for some \(n < \omega\))
- for every positive atomic \(L\)-formula \(\theta\), there exists an \(L'\)-formula \(\psi_\theta\) such that \(M \models \theta(\overline{a}) \iff N \models \psi_\theta(f(\overline{a}))\)

THEN, by induction on the complexity of formulas, for every \(L\)-formula \(\varphi\), there exists an \(L'\)-formula \(\psi_\varphi\) such that \(M \models \varphi(\overline{a}) \iff N \models \psi_\varphi(f(\overline{a}))\).

**Lemma 2.9.** Let \(\varphi\) be a sentence in the language of graphs. THEN, "there exists a finite graph \(A \models \varphi\) iff \(M_0 \models \exists yz \psi_\varphi(y, z)\).

**Proof.** \((\Rightarrow):\) We may assume that \(\mathfrak{H}_A < M_0\). So, by Remark 2.4, \(A \simeq G(\mathfrak{H}_A, x_A, y_A) \simeq G(M_0, x_A, y_A)\). Therefore, \(M_0 \models \psi_\varphi(x_A, y_A)\).

\((\Leftarrow):\) \(G(M_0, a, b) \models \exists yz \psi_\varphi(y, z)\) and \(G(M_0, a, b) \subseteq \text{cl}_M(a, b) \subset M_0\)

**Proposition 2.10.** Let \(\varphi\) be a sentence in the language of graphs. Suppose that \(\varphi\) has arbitralily large finite model. Then there exists an infinite model, definable in some model of \(\text{Th}(M_0)\).

**Proof.** By our assumption, for any \(n < \omega\), there exists a finite graph \(A_n\) such that \(A_n \models \varphi\) and \(|A_n| \geq n\). As \(A_n \simeq G(\mathfrak{H}_{A_n}, x_{A_n}, y_{A_n})\) (by Remark 2.4) and \(\omega > |\mathfrak{H}_{A_n}| \geq |A_n| \geq n\), for any \(n < \omega\),

\[
|\mathfrak{H}_{A_n}| \models \psi_\varphi(x_{A_n}, y_{A_n}) \land |\mathfrak{R}_{0, A_n}(\ast, x_{A_n}, y_{A_n})| \geq n.
\]

As \(M_0\) is \((\mathcal{K}_0, <)\)-generic, there exists \(\mathfrak{H}_{A_n} \simeq \mathfrak{A} < M_0\). Since \(G(\mathfrak{H}_{A_n}, x_{A_n}, y_{A_n}) \simeq G(\mathfrak{A}, a, b) = G(M_0, a, b)\), where \(x_{A_n} A_n \mapsto ab\),

\[
\text{Th}(M_0) \models \exists yz \psi_\varphi(y, z) \land |\mathfrak{R}(\ast, y, z)| \geq n.
\]

By compactness, there exist infinite \(M \models \text{Th}(M_0)\) and \(a', b' \in M\) such that \(G(M, a', b') \models \varphi\), where \(G(M, a', b')\) is definable in \(M\).

**Theorem 2.11.** \(\text{Th}(M_0)\) has strict order property.

**Proof.** Let \(A_n\) be the graph as follows;
- Vertices: \(\{b_i : i < n\} \cup \{c_i : i < n\}\)
- Edges: \(\{(b_i, c_j) : 0 \leq i < j < n\}\)
Let \( a_i = (b_i, c_i) \), and \( \varphi(xy, zw) \equiv R(x, y) \land R(z, w) \land R(x, w) \land \neg R(x, z) \land \neg R(y, w) \land \neg R(y, z) \). Then \( A_n \models \varphi(a_i, a_j) \Leftrightarrow i < j < n \).

By Lemma 2.9, we can find a linear (uniformly definable) ordering of arbitrarily finite length in \( M_0 \). By compactness, we see that \( \text{Th}(M_0) \) has the strict order property. 

\[ \square \]

3. Review of Strong Order Property

This section consists of Shelah’s results in [Sh].

**Definition 3.1.** A complete theory \( T \) has \( n \)-strong order property, denoted \( \text{SOP}_n \), if there exists a formula \( \varphi(x, y) \) (\( \text{lh}(x) = \text{lh}(y) \)) and a sequence \( (a_i : i < \omega) \) in some model \( N \) of \( T \) such that

1. \( N \models \varphi(a_i, a_j) \) for \( i < j < \omega \)
2. there is no \( n \)-\( \varphi \)-loops;

\[ N \models \neg \exists x_0, x_1, \ldots, x_{n-1} \varphi(x_0, x_1) \land \varphi(x_1, x_2) \land \cdots \land \varphi(x_{n-2}, x_{n-1}) \]

**Fact 3.2.**

1. \( \text{SOP} \) implies \( \text{SOP}_n \).
2. \( \text{SOP}_{n+1} \) implies \( \text{SOP}_n \).
3. If \( T \) has \( \text{SOP}_3 \), then \( T \) has the tree property.

**Proof.** (1): By way of contradiction, suppose that \( T \) has \( \text{SOP} \) and \( \text{NSOP}_n \). So, there exist \( \varphi(x, y) \), \( N \models T \) and \( (a_i : i < \omega) \subset N \) such that \( \forall x(\varphi(x, a_i) \rightarrow \varphi(x, a_j)) \land \exists x(\neg \varphi(x, a_i) \land \varphi(x, a_j)) \) for \( i < j < \omega \). Let \( \psi(x_0, x_1) = \forall x(\varphi(x, x_0) \rightarrow \varphi(x, x_1)) \land \exists x(\neg \varphi(x, x_0) \land \varphi(x, x_0)) \). As \( T \) has \( \text{NSOP}_n \), there exists \( n \)-\( \psi \)-loop, but it is impossible.

(2): Let \( \varphi(x, y) \), a model \( M \), and \( (a_i : i < \omega) \in M \) be witness for \( \text{SOP}_{n+1} \). We may assume that \( (a_i : i < \omega) \) is indiscernible. We divide the arugment into two cases, whether

\[ M \models \exists x_0, \ldots, x_{n-1}[x_0 = a_1 \land x_{n-1} = a_0 \land \bigwedge_{i,j<n,k\equiv l+1(\text{mod} n)} \varphi(x_i, x_j)] \]

or not.

- The case that \( M \models \exists x_0, \ldots, x_{n-1}[x_0 = a_1 \land x_{n-1} = a_0 \land \bigwedge_{i,j<n,k\equiv l+1(\text{mod} n)} \varphi(x_i, x_j)] \)

As \( a_1 = a_0 \), we have \( M \models \exists x_0, \ldots, x_{n-1}[x_0 = a_2 \land x_{n-1} = a_0 \land \bigwedge_{i,j<n,k\equiv l+1(\text{mod} n)} \varphi(x_i, x_j)] \).

Let \( a_2, c_1, \ldots, c_{n-2}, a_0 \) be the witness for \( x_0, \ldots, x_{n-1} \). By the way, \( M \models \varphi(a_1, a_2) \land \varphi(a_0, a_1) \), so \( a_1, a_2, c_1, \ldots, c_{n-2}, a_0 \) is an \( (n+1) \)-\( \varphi \)-loop, a contradiction.

- The case that \( M \models \neg \exists x_0, \ldots, x_{n-1}[x_0 = a_1 \land x_{n-1} = a_0 \land \bigwedge_{i,j<n,k\equiv l+1(\text{mod} n)} \varphi(x_i, x_j)] \)

Put \( \psi(x, y) \equiv \varphi(x, y) \land \neg \exists x_0, \ldots, x_{n-1}[x_0 = x \land x_1 = y \land \bigwedge_{i,j<n,k\equiv l+1(\text{mod} n)} \varphi(x_i, x_j)] \)

Then \( M \models \psi(a_i, a_i)(i < j < \omega) \), and \( n \)-\( \psi \)-loops never exist.

(3): Let \( \kappa = \text{cf}(\kappa) > |T| \) and \( \lambda > \kappa \) be such that \( \text{cf}(\lambda) = \kappa \) and "\( \mu < \lambda \) implies \( 2^\mu < \lambda \)" (strongly limit singular cardinal of cofinality \( \kappa \)). Put \( J = ^\kappa \lambda \) and
I \subset J be such that \eta \in I iff \eta(i) = 0 for every i < \kappa large enough.

Let \varphi(x, y) be the witness for SOP₃. By compactness, there exist a sequence \langle a_\eta : \eta \in I \rangle in some model M such that M \models \varphi(a_\eta, a_\nu) for any \eta < \nu. The lexicographic order on I is as usual; if i is the least such that \eta|i = \nu|i, then \eta(i) = \nu(i).

We may assume that M is \kappa^+-saturated, and |M| \geq \lambda. Fix an \eta \in \kappa(\lambda \setminus \{0\}) \setminus I. We will define a_\eta as follows.

Put \rho = \{\varphi(a_\eta|_i a_\nu|_i, x) \wedge \varphi(a_\eta|_{i+1} a_\nu|_{i+1}, x) : i < \kappa\}.

Note that (\eta|i)0_{(i, \kappa)}, (\eta|i, \eta(i) + 1)0_{(i, \kappa)} \in I, and

a\eta = \varphi(a_\eta|_{i+1} a_\nu|_{i+1}, x) \wedge \varphi(a_\eta|_i a_\nu|_i, x).

As M is \kappa^+-saturated, there exists a realization of p_\eta in M, say a_\eta.

**Claim.** If \eta_1 \neq \eta_2 \in \kappa(\lambda \setminus \{0\}), then p_{\eta_1} \cup p_{\eta_2} is inconsistent.

Suppose that \eta_1 < \eta_2. Then there exists i such that \eta_1|i = \eta_2|i, \eta_1(i) < \eta_2(i). Take \nu < \rho \in I be with \eta_1 < \nu < \rho < \eta_2 as follows.

\eta_1|i = \eta_2|i = \nu|i = \rho|i, \nu(i) = \eta_1(i) + 1, \rho(i) = \nu_2(i), \nu(j) = 0(j > i), \rho(j) = \nu_2(i+1).

As \varphi(x, a_\nu) \in p_{\eta_1}, \varphi(x, a_\rho) \in p_{\eta_2}, and M \models \varphi(a_\nu, a_\rho), if we found the realization of p_{\eta_1} \cup p_{\eta_2}, say c, then c, a_\nu, a_\rho would be the 3-loop, a contradiction.

We also have |p_\eta| = \kappa, |\{Dom(p_\eta) : \eta \in \kappa(\lambda \setminus \{0\})\}| \leq \lambda (as \bigcup\{Dom(p_\eta) : \eta \in \kappa(\lambda \setminus \{0\})\} \subseteq \{a_\nu : \nu \in I\})

By 7.7(3) and 7.6(2) on p.141 of Shelah’s 2nd edition book, \lambda = \lambda^{<\kappa} > 2^{|T|} (by cf(\lambda) = \kappa < \lambda) and \kappa > |T| imply that T has the tree property. □

It is conjectured that SOP₄ is a good dividing line for existence of universal models, i.e. if T does not have SOP₄, it will have universal models of cardinality \lambda > |T| (Shelah showed that if T is simple and \lambda > |T|, then there exists universal models of cardinality \lambda^{++}. As the above, simplicity implies NSOP₃.)

4. \text{Th}(M_f) does not have SOP₄

Let \delta be a local rank on relational finite structures such that \delta(A/B) \leq \delta(A/A \cap B), where \delta(A/B) = \delta(AB) - \delta(B). Let f : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0} be upper unbounded and monotone increasing. Let K_0 = \{A \in K_0 : \delta(X) \geq f(|X|) (\forall X \subseteq A)\} and \beta(x) = \min\{\delta(X/A) : A < X \in K_0, A \neq X, |X| \leq x\}.

**Fact 4.1.** Suppose that

\[ f'(x) \leq \frac{\beta(x)}{x}. \]
Then $K_f$ is closed under free amalgamation, so $(K_f, \prec)$-generic $M_f$ exists, $cl = acl$ in $M_f$ and $Th(M_f)$ is $\omega$-categorical. ($\omega$-categoricity follows from $|cl(*)| \leq f^{-1}(\delta(*))$ for finite graphs.)

Proof. Let $A < B_1, B_2 \in K_f$ and let $C = B_1 \otimes_A B_2$. We need to show that if $X \subseteq C$, then $\delta(X) \geq f(|X|)$. We may assume that $X < C$, because $\delta(X) \geq \delta(cl(X))$ and $f(|cl(X)|) \geq f(|X|)$.

Let $X_i = X \cap B_i (i = 1, 2)$ and let $X_0 = X \cap A$. Suppose that
\[
\frac{\delta(X_1) - \delta(X_0)}{|X_1| - |X_0|} \leq \frac{\delta(X) - \delta(X_0)}{|X| - |X_0|} \leq \frac{\delta(X_2) - \delta(X_0)}{|X_2| - |X_0|}.
\]
As $X_0 < X_1$, $\beta(|X_1|) \leq \delta(X_1/X_0)$. Therefore $\frac{\delta(X_1) - \delta(X_0)}{|X_1| - |X_0|} \geq \frac{\beta(|X_1|)}{|X_1|} \geq f'(|X_1|)$. So, the line between $(|X_0|, \delta(X_0))$ and $(|X_1|, \delta(X_1))$ lies above $f$. As $f'$ is decreasing and $\delta(X_1) \geq f(|X_1|)$, $\delta(X) \geq f(|X|)$ follows. \qed

Let $d(A) = \delta(cl(A))$, and $d(a/A) = \delta(cl(aA)/cl(A))$. For possibly infinite $B$, let $d(a/B) = \inf\{d(a/B_0) : B_0 \subset_\omega B\}$.

**Fact 4.2.** Let $\mathcal{M}$ be a relational structure having $\delta$-rank. Let $a, A, B \subset_\omega \mathcal{M}$. Suppose that $A < B < \mathcal{M}$ and $cl(aA) \subset_\omega \mathcal{M}$. Then $d(a/B) = d(a/A)$ iff $cl(aA) \cap B = A, cl(aA)B = cl(aA) \otimes_A B$ and $d(aB) = \delta(cl(aA)B)$ (i.e. $cl(aA)B \leq cl(aB)$).

Proof. As $A < cl(aA) \cap B$ or $A = cl(aA) \cap B$, we have $\delta(A) \leq \delta(cl(A) \cap B)$. So, $\delta(cl(aA)/cl(A)) \cap B \leq \delta(cl(aA)/A)$. Therefore
\[
d(a/B) \leq \delta(cl(aA)/B) \leq \delta(cl(aA)/cl(aA) \cap B) \leq \delta(cl(aA)/A) = d(a/A).
\]
Now we can see the conclusion. \qed

By Fact 4.2, for $a, b, A \subset_\omega \mathcal{M}$,
\[
d(a/Ab) = d(a/A) \Leftrightarrow d(b/Aa) = d(b/A).
\]
(By $d(a/Ab) = d(a/A) \Leftrightarrow "cl(aA) \cap cl(bA) = cl(A), cl(aA)cl(bA) = cl(aA) \otimes cl(aA) cl(bA) \leq cl(abA)".

From now on, we assume that the control function $f$ satisfies $"f'(x) \leq \frac{\beta(x)}{x}"$. Let $\overline{K}_f$ be the class of possibly infinite structures whose finite substructures are all in $K_f$. Let $T_f = \{ \forall \bar{x} \exists \bar{y} \Diag_A(\bar{X}) : \delta(A) < f(|A|), |A| < \omega \}$. Then $M \models T_f \iff M \in \overline{K}_f$. Let $\mathcal{M}$ be a big model of $M_f$. Note that if $A \subset_\omega \mathcal{M}$, then $A \in \overline{K}_f$.

**Proposition 4.3.** Suppose that, in $\mathcal{M}$, if $A = acl(A)$, $d(a/A) = d(a/Ab)$, $acl(aA) \cap acl(bA) = A$, then there exists $A_0 \subset_\omega A$ such that $d(a/A_0b) = d(a/A_0)$. THEN $Th(M_f)$ has NSOP$^4$. 
Proof. Let \((a_i : i < \omega)\) be an infinite indiscernible sequence in \(\mathcal{M}\). Put \(p(x_0x_1) = \text{tp}(a_0a_1)\). We will show that
\[p(x_0x_1) \cup p(x_1, x_2) \cup p(x_2x_3) \cup p(x_3x_0)\]
is consistent.

**Claim.** There exists \(B \subset_\omega \mathcal{M}\) such that \((a_i : i < \omega)\) is \(B\)-indiscernible, and \(d(a_2/Ba_0a_1) = d(a_2/Ba_1) = d(a_2/B)\). (Then \(a_2 \equiv_{a_0} a_1\), \(d(a_2/Ba_0a_1) = d(a_2/Ba_2)\) follows.)

Extend \((a_i : i < \omega)\) to \((a_i : i < \mathbb{Z})\). As \((a_i : i \geq 0)\) is indiscernible over \((a_i : i < 0)\), \((a_i : i \geq 0)\) is indiscernible over \(\text{acl}(a_i : i < 0) =: A_0\). As \(a_{<i} \equiv_{a_i} a_{<0}\), we see that \(d(a_i/A_0a_{<i}) = d(a_i/A_0)\).

By extending \((a_i : i \geq 0)\) over \(A_0\) and applying Erdos-Rado Theorem, we may assume that \(\text{acl}(A_0a_{<i}) \cap \text{acl}(A_0a_{<j}) =: C\) is constant for any \(i < j < k\), and \((a_i : i \geq 0)\) is indiscernible over \(C\).

Now, by our assumption, take \(B \subset_\omega C\) such that \(d(a_2/Ba_0a_1) = d(a_2/B)\), as desired. The claim is proven.

As \(d(a_2/Ba_0a_1) = d(a_2/B)\), we have
\[\text{cl}(a_2) \subset \text{cl}(a_0a_1)B = \text{cl}(a_2) \otimes_{\text{cl}(B)} \text{cl}(a_0a_1B) \leq \text{cl}(a_0a_1a_2B).
\]
As \(\text{cl}(a_0a_1a_2B) \in K_f\), we may assume that
\[\text{cl}(a_0a_1a_2B) \leq M_f.
\]
So, we can work inside \(M_f\). (i.e. we have \(a_0, a_1, a_2, B \subset_\omega M_f\) such that \((a_0, a_1, a_2)\) is \(B\)-indiscernible and \(d_{M_f}(a_2/Ba_0a_1) = d_{M_f}(a_2/B)\).)

Let \(C_{i,j} = \text{cl}(a_ia_jB)\), \(C_i = \text{cl}(a_iB)\). By \(d(a_2/Ba_0a_1) = d(a_2/Ba_1)\) and Fact 4.2, we see that \(C := C_{0,1}C_{1,2} = C_{0,1} \otimes_{C_1} C_{1,2}\). And \(C_{0,1} \cap C_{0,2} = C_0\) and \(C_{1,2} \cap C_{0,2} = C_2\) follow by \(d(a_2/Ba_0a_1) = d(a_2/Ba_1), d(a_1/Ba_0a_2) = d(a_1/Ba_2)\) and Fact 4.2. So we have
\[C \cap C_{0,2} = C_0C_2 = C_0 \otimes_B C_2 \subset C.
\]

Let \(f : C_0C_2 \rightarrow C_0C_2\) be an isomorphism over \(B\) sending \(a_0a_2\) to \(a_2a_0\), and let \(g : C_0C_2 \rightarrow C\) be the inclusion map. Put \(g' = g \circ f\). As \(K_f\) is closed under free amalgamation, there exist \(D \in K_f\) and \(h, h' : C \rightarrow D\) such that \(h \circ g(C_0C_2) = h' \circ g'\), \(h \circ g(C_0C_2) = h' \circ g'\) and \(D = h(C) \otimes_{h_0g(C_0C_2)} h'(C)\). We may assume that \(D < M_f\). Put \(a_0' = h \circ g(a_0), a_1' = h(a_1), a_2' = h' \circ g'(a_2), a_3' = h'(a_1)\).

**Claim.** \(a_0' a_1', a_1' a_2', a_2' a_3', a_3' a_0' \models p = \text{tp}(a_0a_1)\). (This proposition is proven.)

Note that
\[
h(a_0a_1) = a_0' a_1', h(a_1a_2) = a_1' a_2', h'(a_0a_1) = (h' \circ g'(a_2))a_3' = a_3' a_4',
\]
\[
h'(a_1a_2) = a_3'(h' \circ g'(a_0)) = a_3'(h \circ g(a_0)) = a_2'h(a_0) = a_3'a_0'.
\]
On the other hand,

\[ h(C_{0,1}), h(C_{1,2}) < h(C) < D < M_f, \]
\[ h'(C_{0,1}), h'(C_{1,2}) < h'(C) < D < M_f. \]

Put \( B' = h \circ g(B) = h' \circ g'(B) \). Then

\[ h(\text{cl}(a_0a_1B)) = h(C_{0,1}) = \text{cl}(a'_0a'_1B'), \]
\[ h'(\text{cl}(a_0a_1B)) = h'(C_{0,1}) = \text{cl}(a'_2a'_3B'), \]
\[ h(\text{cl}(a_1a_2B)) = h(C_{1,2}) = \text{cl}(a'_3a'_4B'). \]

By genericity of \( M_f \), we see that

\[ \text{cl}(a_0a_1B) \equiv \text{cl}(a_1a_2B) \equiv \text{cl}(a'_0a'_1B') \equiv \text{cl}(a'_2a'_3B') \equiv \text{cl}(a'_3a'_4B'). \]

\[ \square \]

**Remark 4.4.** Suppose that for any \( a, A \subset \mathcal{M} \), there exists \( A_0 \subset A \) such that \( d(a/A) = d(a/A_0) \). Then the assumption of Proposition 4.3 holds.

**Proof.** Take \( A_0, A_1 \subset A \) such that \( d(a/Ab) = d(a/A_0b) \) and \( d(a/A) = d(a/A_1) \). Then \( d(a/A_0A_1) = d(a/A_0A_1b) \).

\[ \square \]

5. **Review of Evans’ Paper on Simple \( \omega \)-Categorical Generic Structures**

Let \( \delta \) be a local rank on relational finite structures such that \( \delta(A/B) \leq \delta(A/A \cap B) \), where \( \delta(A/B) = \delta(AB) - \delta(B) \). Let \( f : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0} \) be an upper unbounded, monotone increasing, convex \( (f'(x) \) is monotone decreasing) and \( f'(x) \leq \frac{\beta(x)}{x} \), where \( \beta(x) = \min\{1, \delta(X/A) : A < X \in \mathcal{K}_0, A \neq X, |X| \leq x\} \).

Let \( K_f = \{ A \in \mathcal{K}_0 : f(|X|) \geq f(|X|)(\forall X \subseteq A) \} \).

The following fact is Corollary 2.20 of [E1].

**Fact 5.1.** Let \( M_f \) be \((K_f, <)\)-generic. And suppose the condition on \( \mathcal{M} \) (big model of \( \text{Th}(M_f) \)) as in Proposition 4.3. Furthermore, suppose the following.

1. (d-extension property in \( \mathcal{M} \))
   
   Let \( A \subset B \subset \mathcal{M} \) be algebraically closed and \( c \subset A \). Then there exists \( c' \subset \omega \mathcal{M} \) such that \( \text{tp}(c/A) = \text{tp}(c'/A) \), \( d(c'/B) = d(c/A) \) and \( \text{acl}(c'/A) \cap B = A \).

2. (Independence theorem over finite closed sets in \( M_f \))
   
   Let \( A, B_1, B_2 < M_f \) be finite such that \( B_1 \cap B_2 = A \) and \( d(B_1/B_2) = d(B_1/A) \). Suppose that \( c_1, c_2 \subset \omega \mathcal{M}_f \), \( \text{tp}(c_1/A) = \text{tp}(c_2/A) \) and \( d(c_1/B) = d(c_1/A) \). Then there exists \( c \subset \omega \mathcal{M}_f \) such that \( \text{tp}(c/B_1) = \text{tp}(c_1/B_1) \) and \( d(c/B_1B_2) = d(c/A) \).

**THEN** \( \text{Th}(M_f) \) is simple and \( "c \perp_A B \iff d(c/B) = d(c/A) \) and \( \text{acl}(c/A) \cap B = A \), for \( A, B \) algebraically closed in \( \hat{\mathcal{M}} \)."
We give the proof of the following lemma. (Theorem 3.6 of [E1])

**Lemma 5.2.** Suppose that d-extension property over finite closed sets in $\mathcal{M}$ and $f(3x) \leq f(x) + \beta(x)$. Then the independence theorem over finite closed sets holds in $\mathcal{M}_f$.

**Proof.** Let $c_i, B_i, A$ be as in Fact 5.1. Then $\text{acl}(c_1A) \simeq_A \text{acl}(c_2A)$. Put $E_{12} = \text{acl}(B_1B_2)$, $E_{13} = \text{acl}(c_1B_1)$, $E_{23} = \text{acl}(c_2B_2)$. By considering free amalgamation and copies, we may assume that $B_1 \cap E_{13}, B_2 = E_{12} \cap E_{23}, B_3 := E_{13} \cap E_{23} = \text{acl}(c_iA)$.

Let $E = E_{12}E_{13}E_{23}$. We need to show that $A < E$ and $E \in \mathcal{K}_f$.

**Claim.** $A < E$.

By Fact 4.2, $B_iB_j \leq E_{ij}$. As $E = E_{ij} \otimes_{B_iB_j} E_{ik}E_{jk}$, $E_{ik}E_{jk} \leq E$ follows. We also have $E_{ik}E_{jk} = E_{ik} \otimes_{B_k} E_{jk}$ and $B_k < E_{jk}, E_{ik} < E_{ik}E_{jk}$ follows. Thus $E_{ik} < E$. As $A < B_i < E_{ik}, A < E$ follows.

**Claim.** $E \in \mathcal{K}_f$.

We have $E = E_{ij} \otimes_{B_iB_j} E_{ik}E_{jk}$, but we do not have $B_iB_j < E_{ij}, E_{ik}E_{jk}$. So we can not conclude this claim by using Fact 4.1.

We need to show $\delta(D) \leq f(|D|)$ for any $D \leq E$ as in Fact 4.1. Put $D_{ij} = D \cap E_{ij}$ and $d_{ij} = \delta(D_{ij})$. Suppose that $d_{12}$ is the largest of these.

As $E_{12}E_{23} \in \mathcal{K}_f$, we may assume that $D \neq D_{12}D_{23}$. Put $D^1 = D_{12}D_{13}$. As $E_{12}E_{13} \leq E$, we see that $D^1 \leq D$. As $D^1 = D_{12} \otimes_{D \cap B_1} D_{13}$ and $D \cap B_1 < D_{13}$,

$$\delta(D^1) = d_{12} + \delta(D_{13}/D \cap B_1) \geq d_{12} + \beta(|D_{13}|).$$

As $d_{13} \leq d_{12}, |D_{13}| \leq f^{-1}(d_{13}) \leq f^{-1}(d_{12})$.

So, as $\beta$ is monotone decreasing, $d_{12} \leq \delta(D^1) - \beta(|D_{13}|) \leq \delta(D^1) - \beta(f^{-1}(d_{12}))$.

By our assumption on $f$

$f(3x) \leq f(x) + \beta(x)$, so $3f^{-1}(x) \leq f^{-1}(x + \beta(f^{-1}(x)))$

$$3f^{-1}(d_{12}) = f^{-1}(d_{12} + \beta(f^{-1}(d_{12}))).$$

So, $3f^{-1}(d_{12}) \leq f^{-1}(\delta(D^1))$. As $|D| \leq \sum_{ij} |D_{ij}| \leq \sum_{ij} f^{-1}(d_{ij}) \leq 3f^{-1}(d_{12})$ and $\delta(D^1) \leq \delta(D)$, we see that $|D| \leq f^{-1}(\delta(D))$. 

\[ \square \]
6. Th($M_f$) has SOP$_3$ for some $f$

We work with undirected graphs, and $\delta(A) = 2|A| - e(A)$. Note that $\beta(x) = 1$. The control function $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is an upper unbounded, monotone increasing satisfying the following five conditions:

(F1): $f(0) = 0, f(2) = 2, f(4) = 3, f(8) = 4 < f(10) < \frac{1}{2} < f(12) < 5 < f(14) < \frac{1}{3} < f(16) < f(18) \leq 6.$

(F2): $2f'(2n) \leq \frac{1}{n}$ for $n \geq 7$

(F3): $f\left(\frac{k^2}{2}\right) \leq k$ if $k \geq 6$

(F4): $f(3n) \leq f(n) + 1$ for $n \geq 10$

(F5): $f(10) + 1 \geq f(14), f(12) + 1 \geq f(16)$.

Let $f_1(x) = f(2x)$. So, $f_1'(x) = 2f'(2x)$ and F2: $f_1'(n) \leq \frac{1}{n}$ for $n \geq 7$.

We consider $K_{f_1}$.

**Remark 6.1.**

1. $\delta$(3-cycle)$= 6 - 3 = 3 = f(4) < f(6) = f_1(3)$, so 3-cycle does not belong to $K_{f_1}$. $\delta$(4-cycle)$= 8 - 4 = 4 = f(8) = f_1(4)$, so 4-cycle belongs to $K_{f_1}$.

2. The graph does not belong to $K_{f_1}$, because its $\delta$-rank$= 14 - 9 = 5 < f(14) = f_1(7)$.

3. (F1) and (F2) give the free amalgamation property of $(K_{f_1}, \prec)$.

4. (F1) and (F3) are needed to show that the graphs $G(A_n, B_n, x_0)$ belong to $K_{f_1}$. (Lemma 6.4.)

5. (F4) is needed to show Subclaim 2 in the proof of Lemma 6.7. Lemma 6.7 ensures that the important graphs $E_n$ can be closedly embedded into $M_{f_1}$ and the graphs $E_n$ will give the witness formula for SOP$_3$.

6. (F1), (F2) and (F5) are needed to show Lemma 6.6. (Lemma 6.6 gives a very important key to get Lemma 6.7.)

By the graphs $E_n < M_{f_1}(n \in \omega)$, we will give a formula $\varphi(x, y)$ and infinite sequence $(a_i)_{i<\omega}$ in $M_{f_1}$ such that $M_{f_1} \models \varphi(a_i, a_j)$ whenever $i < j$. But if there were a 3-$\varphi$-loop in some model $N$ of Th($M_{f_1}$), then $N$ would have the
graph as in (2) of Remark 6.1. As any finite graph of $N$ belongs to $\mathbb{K}_{f_{1}}$, so SOP3 follows.

**Lemma 6.2.** $\mathbb{K}_{f_{1}}$ has the free amalgamation property.

**Proof.** Let $A < B_{1}, B_{2} \in \mathbb{K}_{f}$ and let $C = B_{1} \otimes_{A} B_{2}$. We need to show that if $X \subseteq C$, then $\delta(X) \geq f_{1}(|X|)$. We may assume that $X < C$, because $\delta(X) \geq \delta(\text{cl}(X))$ and $f_{1}(\text{cl}(X)) \geq f_{1}(|X|)$.

Let $X_{i} = X \cap B_{i}(i = 1, 2)$ and let $X_{0} = X \cap A_{n}$. Suppose that

$$\frac{\delta(X_{1}) - \delta(X_{0})}{|X_{1}| - |X_{0}|} \leq \frac{\delta(X) - \delta(X_{0})}{|X| - |X_{0}|} \leq \frac{\delta(X_{2}) - \delta(X_{0})}{|X_{2}| - |X_{0}|}, |X_{1}| \geq 7.$$

As $X_{0} < X_{1}$, $\beta(|X_{1}|) \leq \delta(X_{1}/X_{0})$. So, by (F2),

$$\frac{\delta(X_{1}) - \overline{\delta}(X_{0})}{|X_{1}| - |X_{0}|} \geq \frac{1}{|X_{1}|} \geq f_{1}'(|X_{1}|).$$

So, the line between $(|X_{0}|, \overline{\delta}(X_{0}))$ and $(|X_{1}|, \delta(X_{1}))$ lies above $f_{1}$. As $f_{1}'$ is decreasing and $\delta(X_{1}) \geq f_{1}(|X_{1}|)$, $\delta(X) \geq f_{1}(|X|)$ follows. In Appendix 1, we give the proof when $|X_{1}| \leq 6$.

\[\square\]

**Notation 6.3.** Consider the following graphs $G(A_{n}, B_{n}, x_{0})$ for each $n < \omega$.

- Vertex set: $A_{n} \cup B_{n} \cup \{x_{0}\} \cup \{z_{ij} : 0 \leq i < j \leq n\}$, where $A_{n} = \{a_{i} : 0 \leq i \leq n\}, B_{n} = \{b_{i} : 0 \leq i \leq n\}$.
- Edges: $R(x_{0}, a_{i}), R(x_{0}, b_{i})$ for $0 \leq i \leq n$ and $R(z_{ij}, a_{i}), R(z_{ij}, b_{j})$ for $0 \leq i < j \leq n$.

**Lemma 6.4.**

1. $G(A_{n}, B_{n}, x_{0}) \in \mathbb{K}_{f_{1}}$
2. $x_{0}A_{n} < G(A_{n}, B_{n}, x_{0})$
3. $d(A_{n}/B_{n}) = d(A_{n}/x_{0})$, where $d(\cdot) = d_{G\langle A_{11},B_{n},x_{0}\rangle}(\cdot)$.

**Proof.** Put $G = G(A_{n}, B_{n}, x_{0}), A = A_{n}, B = B_{n}, Z = \{z_{ij} : 0 \leq i < j \leq n\}$. (1): It suffices to show that if $X < G$, then $\delta(X) \geq f_{1}(|X|)$. It is clear in case of $|X| = 1$. If $|X| \geq 2$, then $x_{0} \in X$. (If $x_{0} \neq a, b \in X$, then $\delta(x_{0}/ab) = 0$, so $x_{0} \in \text{cl}_{G}(ab) \subseteq X$.)

**Claim.** $a_{i}, b_{j} \in X \Leftrightarrow z_{ij} \in X$.

This claim follows from $\delta(z_{ij}/a_{i}b_{j}) = \delta(a_{i}/x_{0}z_{ij}) = \delta(b_{j}/x_{0}z_{ij}) = 0$ and $X < G$.

Put $X_{A} = X \cap A_{n}, X_{B} = X \cap B, X_{Z} = X \cap Z$ and $m = |X_{A}| + |X_{B}|$. By claim, we see that $\delta(X_{Z}/x_{0}X_{A}X_{B}) = 0$, so we have

$$\delta(X) = \delta(x_{0}X_{A}X_{B}) = 2(m + 1) - m = m + 2 =: k$$
As \(|X_Z| \leq |X_A||X_B| \leq |X_A|(m - |X_A|) = (\frac{m}{2})^2 - (|X_A| - \frac{m}{2})^2 \leq (\frac{m}{2})^2\), we have

\[|X| \leq 1 + m + (\frac{m}{2})^2 = (1 + \frac{m}{2})^2 = \frac{k^2}{4}\]

If \(k \geq 6\), by (F3), \(\delta(X) = k \geq f(\frac{k^2}{2}) = f_1(\frac{k^2}{4}) \geq f_1(|X|)\), as desired.

If \(k \leq 5\), then \(|X_A| + |X_B| \leq 3\).

If \(|X_A| = 3\), then \(X_Z = \emptyset\) and \(\delta(X) = 2\).

If \(|X_A| = 2, |X_B| = 1\), then \(\delta(X) \geq \{1\}\).

If \(|X_A| = 2, |X_B| = 0\), then \(X_Z = \emptyset\) and \(\delta(X) = 2 \cdot 2 - 1 = 3 = f(4) = f_1(2)\).

By symmetry, we see that \(X \in \mathbf{K}_{f_1}\).

(2): Let \(x_0 A \subset X \subset G\). We show that \(\delta(X/x_0 A) > 0\). We may assume \(X < G\). By \(\dagger\) we have

\[
\delta(X/x_0 A) = \delta(x_0 X_A X_B/x_0 A) = \delta(X_B/x_0 A) = 2|X_B| - |X_B| > 0.
\]

(3): It is clear that \(cl_G(x_0) = x_0, cl_G(x_0 A) = x_0 A, cl_G(x_0 B) = x_0 B\), and \(\delta(A/Bx_0) = \delta(A/x_0)\). We also have \(x_0 AB \leq cl_G(x_0 AB) = G\), because \(\delta(Z'/x_0 AB) = \sum_{z \in Z'} \delta(z/x_0 AB) = 0\). So, by Fact 4.2, we are done. \(\square\)

**Notation 6.5.** Suppose that \(C_n = \{c_i : 0 \leq i \leq n\}\) and \(C_n \cap A_n B_n = \emptyset\).

Let \(E_n\) be the free amalgam of \(G(A_n, B_n, x_0), G(B_n, C_n, x_0)\) and \(G(C_n, A_n, x_0)\).

i.e.

\[
\text{Edges} = \text{edges of } G(A_n, B_n, x_0), G(B_n, C_n, x_0) \text{ and } G(C_n, A_n, x_0), \text{ only}.
\]

In particular, we have \(G(A_n, B_n, x_0)G(B_n, C_n, x_0) = G(A_n, B_n, x_0) \otimes_{B_n x_0} G(B_n, C_n, x_0)\),

\(G(B_n, C_n, x_0)G(C_n, A_n, x_0) = G(B_n, C_n, x_0) \otimes_{C_n x_0} G(C_n, A_n, x_0)\)

\(G(C_n, A_n, x_0)G(A_n, B_n, x_0) = G(C_n, A_n, x_0) \otimes_{A_n x_0} G(A_n, B_n, x_0)\).

**Lemma 6.6.** Suppose that \(A, B, C \in \mathbf{K}_{f_1}, |A|, |B|, |C| \leq 4\). Suppose that \(A \cap B < A, B, A \cap C < A, C\) and \(B \cap C < B, C\), and \(AB = A \otimes_{A \cap B} B, AC = A \otimes_{A \cap C} C, BC = B \otimes_{B \cap C} C\). Put \(X = A \cap B \cap C, Z = A \setminus (B \cup C), W = B \setminus (A \cup C), U = C \setminus (A \cup B)\).

Suppose that \(D = ABC \notin \mathbf{K}_{f_1}\)

Then \(D\) is isomorphic to \(\begin{array}{c}
\begin{array}{ccc}
\text{a} & \text{b} & \text{c} \\
\text{x} & \text{y} & \text{z} \\
\text{w} & \text{u} & \text{v}
\end{array}
\end{array}\), where \(a \in A \cap C, b \in A \cap B, c \in B \cap C, x \in X, z \in Z, w \in W, u \in U\).
Proof. See Appendix 2. As $A \cap B < A$, if $c \in A \setminus (A \cap B)$, there is no $a, b \in A \cap B$ such that $R(a, c) \cap R(b, c)$. This easy fact is important for the proof. (F1), (F2) and (F5) are also needed.

Lemma 6.7. 
(1) $E_n \in K_{f_1}$
(2) $E_n < E_{n+1}$, so we may assume $E_n < E_{n+1} < M_{f_1}$ for any $n < \omega$.

Proof. (1): Let $D \subseteq E_n$ and $D_{AB} = D \cap G(A_n, B_n, x_0)$, $D_{BC} = D \cap G(B_n, C_n, x_0)$, $D_{CA} = D \cap G(C_n, A_n, x_0)$, and $D_A = D \cap x_0 A_n$, $D_B = D \cap x_0 B_n$, $D_C = D \cap x_0 C_n$. By way of contradiction, suppose that $\delta(D) < f_1(|D|)$.

Claim. $|D_{AB}|, |D_{BC}|, |D_{CA}| \leq 4$.

Suppose that $\delta(D_{BC}), \delta(D_{CA}) \leq \delta(D_{AB}) =: d_{AB}$. By Fact 6.2, $G(A_n, B_n, x_0)G(C_n, A_n, x_0) \in K_{f_1}$. So we have $D' \neq D$. As $E_n = G(A_n, B_n, x_0)G(C_n, A_n, x_0)\otimes_{B, C_n} G(B_n, C_n, x_0)$ and $B_n C_n x_0 \leq G(B_n, C_n, x_0)$ by (3) of Lemma 6.4, we see

$D' \leq D$.

As $x_0 A_n < G(C_n, A_n, x_0)$ (so $D_A < D_{CA}$) and $D' = D_{AB} \otimes_{D_A} D_{CA}$, so

$\delta(D') \geq d_{AB} + 1$.

Subclaim 1: $f^{-1}(d_{AB} + 1) < 3 f^{-1}(d_{AB})$.

Note that $f^{-1}(d_{AB}) \geq f^{-1}(\delta(D_{**})) \geq 2 |D_{**}|$. Suppose that this subclaim does not hold, then we have

$f^{-1}(d_{AB} + 1) \geq 3 f^{-1}(d_{AB}) \geq 2(|D_{AB}| + |D_{BC}| + |D_{CA}|) \geq 2 |D|$. 

So, we have $\delta(D) \geq \delta(D') \geq d_{AB} + 1 \geq f_1(|D|)$, a contradiction. This subclaim is proven.

Subclaim 2: $d_{AB} < f(10)$.

Otherwise, we have $f^{-1}(d_{AB}) \geq 10$. Thus, by ((F4): $f(3n) \leq f(n) + 1$), we have $3 f^{-1}(d_{AB}) \leq f^{-1}(f^{-1}(d_{AB})) + 1 = f^{-1}(d_{AB} + 1)$, this contradicts subclaim 1. Subclaim 2 is proven.

As $\delta(D_{**}) \leq d_{AB} < f(10)$, and $D_{**} \in K_{f_1}$, we see the claim.

By this claim and Lemma 6.6, we have the following graph $E_n$, where $a \in D_A, b \in D_B, c \in D_C, z \in D_{AB} \setminus D_A D_B, w \in D_{BC} \setminus D_B D_C, u \in D_{CA} \setminus D_A D_C$. But this is impossible by definition of $E_n$.

(2): Let $V = \{z_{i+n+1}, u_{i+n+1}, u_{n+n+1} : 0 \leq i \leq n\}$ be the vertices of $E_{n+1} \setminus (E_n \cup \{a_{n+1}, b_{n+1}, c_{n+1}\})$. Then

$E_{n+1} = E_n \cup \{a_{n+1}, b_{n+1}, c_{n+1}\} \cup V$. 

Let $X \subseteq \{a_{n+1}, b_{n+1}, c_{n+1}\} \cup V$. Then $e(X, E_n) = |X|$, so $\delta(X/E_n) = \delta(X) - |X| = |X| - e(X)$. If $X \cap V = \emptyset$ or $X \cap \{a_{n+1}, b_{n+1}, c_{n+1}\} = \emptyset$, then $e(X) = 0$. Otherwise, $e(X) = |X \cap V| < |X|$, as desired.

**Theorem 6.8.** Th($M_{f_1}$) has SOP$_3$.

**Proof.** Let $\varphi(x_1y_1z_1, x_2y_2z_2) \equiv \bigwedge_{i=1,2}(R(x_0, x_i) \land R(x_0, y_i) \land R(x_0, z_i)) \land \exists z, w, u(R(x_1, z) \land R(z, y_2) \land R(y_1, w) \land R(w, z_2) \land R(z_1, u) \land R(u, x_2))$.

Let $a_n, b_n, c_n$ be as in $E_n (n < \omega)$, and put $d_n = a_n b_n c_n$. Then $\Lambda I_{f_1} \models \varphi(d_i, d_j)$ for $i < j < \mu j$.

By way of contradiction, suppose that there exist $N = Th(M_{f_1})$ and $d_0', d_1', d_2' \in N$ such that $N \models \varphi(d_0', d_1') \land \varphi(d_1', d_2') \land \varphi(d_2', d_0')$. Let $d_i' = a_i'b_i'c_i'$.

Now we have $\overline{d}_i' \in N$. But any substructure of $N$ is in $K_{f_1}$, a contradiction.

**7. Appendix 1 (Free AP of $K_{f_1}$)**

We show Lemma 6.2, when $|X_1| \leq 6$ and

$$\frac{\delta(X_1) - \delta(X_0)}{|X_1| - |X_0|} \leq \frac{\delta(X) - \delta(X_0)}{|X| - |X_0|} \leq \frac{\delta(X_2) - \delta(X_0)}{|X_2| - |X_0|}.$$ 

By assumption and $|X| - |X_1| = |X_2| - |X_0|$, $\delta(X_2/X_0) \geq \delta(X_1/X_0)$ follows.

**Remark 7.1.**

1. $\delta(X) \geq \delta(X_1) + \delta(X_1/X_0)\frac{|X| - |X_1|}{|X_1| - |X_0|}$

2. $f'(x)(\leq \frac{1}{14})$ is decreasing for $x \geq 14$ by (F2).

3. $e(X_1 \backslash X_0, X_0) \leq |X_1 \backslash X_0|$ by $X_0 < X_1$. So we have

$$\delta(X_1/X_0) \geq |X_1 \backslash X_0| - e(X_1 \backslash X_0).$$

4. $X_0, X_1, X_2$ do not contain 3-cycles, since they belong to $K_{f_1}$.

**Proof.** (3): $\delta(X_1/X_0) = \delta(X_1 \backslash X_0) - e(X_1 \backslash X_0, X_0) = \delta(X_1 \backslash X_0) - |X_1 \backslash X_0| - e(X_1 \backslash X_0)$. Now we check $\delta(X) \geq f(2|X|)$ for each case on the size of $X_1 \backslash X_0, X_0$.

Recall (F1): $f(0) = 0, f(2) = 2, f(4) = 3, f(8) = 4 < f(10) < 4\frac{1}{2} < f(12) < 5 < f(14) < 5\frac{1}{3}$. 


The case that $|X_1 \setminus X_0| = 1$

$\bullet \ |X_1 \setminus X_0| = 1, |X_0| = 0$

$\delta(X) \geq 2 + \frac{|X| - 1}{1} = 2|X| \geq f(2|X|)$.

(By $\delta(X_1) = \delta(X_1/X_0) = 2$ and $2x \geq f(2x)$ for $x \geq 2$)

$\bullet \ |X_1 \setminus X_0| = 1, |X_0| = 1$

$\delta(X) \geq (4 - 1) + (2 - 1)\frac{|X| - 2}{1} = 1 + |X| \geq f(2|X|)$.

(By $1 + x \geq f(2x)$ and $\delta(X_1) \geq 4 - 1, \delta(X_1/X_0) \geq 2 - 1$.)

$\bullet \ |X_1 \setminus X_0| = 1, |X_0| = 2$

$\delta(X) \geq (6 - 2) + 1\frac{|X| - 3}{1} = 3 + f(2|X|)$.

(By $\delta(X_1) \geq 6 - 2, \delta(X_1/X_0) \geq 2 - 1$ and $1 + x \geq f(2x)$)

$\bullet \ |X_1 \setminus X_0| = 1, |X_0| = 3$

$\delta(X) \geq (8 - 3) + 1\frac{|X| - 4}{1} = 4 + f(2|X|)$.

(By $\delta(X_1) \geq 8 - 3, \delta(X_1/X_0) \geq 2 - 1$ and $1 + x \geq f(2x)$)

$\bullet \ |X_1 \setminus X_0| = 1, |X_0| = 4$

$\delta(X) \geq (10 - 5) + 1\frac{|X| - 5}{1} = 5 + f(2|X|)$.

(By $\delta(X_1) \geq 10 - 5, \delta(X_1/X_0) \geq 2 - 1$ and $x \geq f(2x)$ if $x \geq 6$)

$\bullet \ |X_1 \setminus X_0| = 1, |X_0| = 5$

$\delta(X) \geq (12 - 6) + 1\frac{|X| - 6}{1} = 6 + f(2|X|)$.

(By $\delta(X_1) \geq 12 - 6, \delta(X_1/X_0) \geq 2 - 1$ and $x \geq f(2x)$ if $x \geq 6$)

The case that $|X_1 \setminus X_0| = 2$

$\bullet \ |X_1 \setminus X_0| = 2, |X_0| = 0$

$\delta(X) \geq 3 + 3\frac{|X| - 1}{1} \geq f(2|X|)$.

(By $\delta(X_1) = \delta(X_1/X_0) \geq 3$ and $3x + 2 \geq f(2x)$.)

$\bullet \ |X_1 \setminus X_0| = 2, |X_0| = 1$

$\delta(X) \geq 4 + 2\frac{|X| - 3}{2} \geq f(2|X|)$.

(By $\delta(X_1) \geq 6 - 2, \delta(X_1/X_0) \geq 3 - 1$ and $x + 1 \geq f(2x)$.)

$\bullet \ |X_1 \setminus X_0| = 2, |X_0| = 2$

$\delta(X) \geq 4 + 1\frac{|X| - 4}{2} \geq f(2|X|)$.

(As $\delta(X_1) \geq 8 - 4, \delta(X_1/X_0) \geq 3 - 2$ and $4 + \frac{x - 4}{2} \geq f(2x)$ if $x \geq 5$.)
The case that $|X_1 \setminus X_0| = 3$

- $|X_1 \setminus X_0| = 3, |X_0| = 0$
  \[ \delta(X) \geq 4 + 4 \frac{|X| - 3}{3} \geq f(2|X|). \]
  (As $\delta(X_1) = \delta(X_1/X_0) \geq 6 - 2$ and $4 + 4 \frac{x - 3}{3} \geq f(2x)$ if $x \geq 4$.)
- $|X_1 \setminus X_0| = 3, |X_0| = 1$
  \[ \delta(X) \geq 5 + 3 \frac{|X| - 4}{3} \geq f(2|X|). \]
  (As $\delta(X_1) \geq 8 - 3$, $\delta(X_1/X_0) \geq 4 - 1$ and $5 + 3 \frac{x - 4}{3} = x + 1 \geq f(2x)$ if $x \geq 5$.)
- $|X_1 \setminus X_0| = 3, |X_0| = 2$
  \[ \delta(X) \geq 5 + 2 \frac{|X| - 5}{3} \geq f(2|X|). \]
  (As $\delta(X_1) \geq 10 - 5$, $\delta(X_1/X_0) \geq 4 - 2$ and $5 + 2 \frac{x - 5}{3} \geq f(2x)$ if $x \geq 6$.)
- $|X_1 \setminus X_0| = 3, |X_0| = 3$
  \[ \delta(X) \geq 5 + 1 \frac{|X| - 6}{3} \geq f(2|X|). \]
  (As $\delta(X_1) \geq 12 - 7$, $\delta(X_1/X_0) \geq 4 - 3$ and $5 + 1 \frac{x - 6}{3} \geq f(2x)$ if $x \geq 7$.)

The case that $|X_1 \setminus X_0| = 4$

- $|X_1 \setminus X_0| = 4, |X_0| = 0$
  \[ \delta(X) \geq 4 + 4 \frac{|X| - 4}{4} \geq f(2|X|). \]
  (As $\delta(X_1) = \delta(X_1/X_0) \geq 8 - 4$ and $4 + 4 \frac{x - 4}{4} = x \geq f(2x)$ if $x \geq 5$.)
- $|X_1 \setminus X_0| = 4, |X_0| = 1$
\[ \delta(X) \geq 5 + 3 \frac{|X| - 5}{4} \geq f(2|X|). \]

(As \( \delta(X_1) \geq 10 - 5 \), \( \delta(X_1/X_0) \geq 4 - 1 \) and \( 5 + 3 \frac{x-5}{4} \geq f(2x) \) if \( x \geq 6 \).)

- \( |X_1 \setminus X_0| = 4, |X_0| = 2 \)

\[ \delta(X) \geq 5 + 2 \frac{|X| - 6}{4} \geq f(2|X|). \]

(As \( \delta(X_1) \geq 12 - 7 \), \( \delta(X_1/X_0) \geq 4 - 2 \) and \( 5 + 3 \frac{x-6}{4} \geq f(2x) \) if \( x \geq 7 \).)

- \( |X_1 \setminus X_0| = 5 \)

\[ \delta(X) \geq 5 + 5 \frac{|X| - 5}{5} = |X| \geq f(2|X|). \]

(The case that \( |X_1 \setminus X_0| = 5 \)

- \( |X_1 \setminus X_0| = 5, |X_0| = 0 \)

\[ \delta(X) \geq 5 + 5 \frac{|X| - 5}{5} = |X| \geq f(2|X|). \]

(As \( \delta(X_1) = \overline{\delta}(X_1/X_0) \geq 10 - 5 \) and \( x \geq f(2x) \) if \( x \geq 6 \).)

- \( |X_1 \setminus X_0| = 5 \), \( |X_0| = 1 \)

\[ \delta(X) \geq 5 + 3 \frac{|X| - 6}{5} \geq f(2|X|). \]

(As \( \delta(X_1) \geq 12 - 7 \), \( \delta(X_1/X_0) \geq 5 - 2 \) and \( 5 + \frac{x-6}{5} \geq f(2x) \) if \( x \geq 7 \).)

8. Appendix 2 (The Proof of Lemma 6.6)

We show the following.

**Lemma 6.6** Suppose that \( A, B, C \in \mathbb{K}_{f_1}, |A|, |B|, |C| \leq 4 \). And suppose that \( A \cap B < A, B, A \cap C < A, C, B \cap C < B, C, \) and \( AB = A \otimes_{A \cap B} B, AC = A \otimes_{A \cap C} C, BC = B \otimes_{B \cap C} C \). Put \( X = A \cap B \cap C, Z = A \setminus (B \cup C), W = B \setminus (A \cup C), U = C \setminus (A \cup B) \).

If \( D = ABC \not\in \mathbb{K}_{f_1} \), then \( D \) is isomorphic to \( \begin{array}{c}
\text{A} \\
\text{B} \\
\text{C}
\end{array} \), where \( a \in A \cap C, b \in A \cap B, c \in B \cap C, x \in X, z \in Z, w \in W, u \in U \).

**Proof.** We use the following easy fact: If \( X < Y, c \in Y \setminus X, a, b \in X, \) then \( R(a, c) \land R(b, c) \) does not hold.
Clearly, $D = BCZ$.

We may assume that $Z, W, U \neq \emptyset$, since, for example, if $Z = \emptyset$, then $D = B \otimes_{B \cap C} C \in K_f$ by free AP. As $|A|, |B|, |C| \leq 4$, we have $|A \cap C| \leq 3$.

$a, a'$ denote elements of $A \cap C$, $b, b'$ denote elements of $A \cap B$, $c, c'$ denote elements of $B \cap C$, $z, z'$ denote elements of $Z$, $w, w'$ denote elements of $W$, $u, u'$ denote elements of $U$ and $x, x'$ denote elements of $X$.

We check each case on the size of $|A \cap C|$.

The case that $|A \cap C| = 3$

We have $6 \leq |D| \leq 9$. As $|A| \leq 4$, $|Z| = 1$ and $A \cap B \setminus X = \emptyset$ follow. So, we have $\delta(Z/BC) \geq 1$. Thus $\delta(D) = \delta(BC) + \delta(Z/BC) \geq f(2|D| - 2) + 1 \geq f(2|D|)$.

The case that $|A \cap C| = 2$

$\bullet |(A \cap C) \setminus X| = 2$ (i.e. $X = \emptyset$.)

Suppose that $|Z| = 2$. So, $6 \leq |D| \leq 10$.

As $A \cap B = \emptyset$, $\delta(Z/BC) \geq 3 - 2$ follows. So, $\delta(D) \geq \delta(BC) + 1 \geq f(2|D| - 4) + 1 \geq f(2|D|)$ by (F5), $f(8) + 1 = 5 \geq f(12)$, $f(14) + 1 \geq 6 \geq f(18)$ and (F2).

Suppose that $|Z| = 1$, so $|A \cap B| \leq 1$.

If $A \cap B = \emptyset$, then $5 \leq |D| \leq 9$, $\delta(Z/BC) \geq 2 - 1$ follows. So, $\delta(D) \geq \delta(BC) + 1 \geq f(2|D| - 2) + 1 \geq f(2|D|)$.

If $|A \cap B| = 1$, then $6 \leq |D| \leq 9$.

If $|D| = 6$, then $D = aa'zbwu$. Then $\delta(D) = 12 - 5 = 7 \geq f(12)$.

If $|D| = 7$, then $D = aa'zbwuw'$, $aa'zbwcu$ or $aa'zbw'u$, because $Z, W, U \neq \emptyset$. Then $\delta(D) \geq 16 - 9 = 7 \geq f(16)$.

If $|D| = 8$, then $D = aa'zbwuw'u'$ or $aa'zbw'u$, because $Z, W, U \neq \emptyset$. Then $\delta(D) \geq 18 - 9 = 9 \geq f(18)$.

$\bullet |(A \cap C) \setminus X| = |X| = 1$.

Suppose that $|A \cap B \setminus X| = 0$. Then $\delta(Z/BC) \geq 1$.

So, $\delta(D) \geq f(2|D| - 2|Z|) + 1$. 

If $|Z| = 1$, then $5 \leq |D| \leq 8$, so $f(2|D| - 2) + 1 \geq f(2|D|)$ holds.

If $|Z| = 2$, then $6 \leq |D| \leq 9$. $f(2|D| - 4) + 1 \geq f(2|D|)$ holds for $|D| = 6, 9$. $(f(8)+1 = 5 \geq f(12)$ and $f(14)+1 \geq 6 \geq f(18).$) For $|D| = 7$, $D = xazz'wcu$, $xazz'wuw$ or $xazz'wuw'u$ and then $\delta(D) \geq 14 - 8 \geq f(14)$ holds. For $|D| = 8$, $D = xazz'wuw'cu$ or $xazz'wuw'u'$ and then $\delta(D) \geq 16 - 10 \geq f(16)$ holds.

Suppose that $|A \cap B \setminus X| = 1$. Then $6 \leq |D| \leq 8$.

If $|D| = 6$, then $D = xazbwu$ and $\delta(D) \geq 12 - 6 \geq f(12)$.

If $|D| = 7$, then $D = xazbwu'$, $xazbwu'u$ or $xazbwcu$. If the former two cases hold, then $\delta(D) \geq 14 - 8 \geq f(14)$.

In the latter case, $D$ is if and only if $\delta(D) = 14 - 9 < f(14)$.

If $|D| = 8$, then $D = xazbwu'u'$ and $\delta(D) \geq 16 - 10 \geq f(16)$.

\[ \bullet \ (|A \cap C \setminus X| = 0, |X| = 2) \]

We have $5 \leq |D| \leq 8$.

If $|D| = 5$, then $D = xx'zwu$ and $\delta(D) \geq 10 - 4 \geq f(10)$.

If $|D| = 6$, then $D = xx'zz'wu, xx'zbwu, xx'zww'u, xx'zwcu$ or $xx'zwuu'$ and $\delta(D) \geq 12 - 7 \geq f(12)$.

If $|D| = 7$, then $D = xx'zz'wu'u, xx'zbwu', xx'zww'u'u, xx'zz'wcu, xx'zz'wu'u'$, $xx'zbwu'u'$ or $xx'zwuu'u'$. If $D \neq xx'zz'wcu, xx'zbwu'$, then $\delta(D) \geq 14 - 8 \geq f(14)$. And we have $\delta(D) = 14 - 9 < f(14)$ if and only if $D$ is or . But this never happens, because $B \cap C < B$ and $A \cap B < B$, so $w$ does not have two edges to $B \cap C$, also to $A \cap B$.

If $|D| = 8$, then $D = xx'zz'wuu'u'$ and $\delta(D) = 16 - 10 \geq f(16)$.

The case that $|A \cap C| = 1$

\[ \bullet \ (|A \cap C\setminus X| = 1 (|X| = 0) \]

By symmetry, we may assume $|A \cap B|, |B \cap C| \leq 1$.

Suppose that $|A \cap B|, |B \cap C| = 1$. Then $6 \leq |D| \leq 9$.

If $|D| = 6$, then $\delta(D) \geq 12 - 6 \geq f(12)$. If $|D| = 7$, then $\delta(D) \geq 14 - 7 \geq f(14)$. If $|D| = 8$, then $\delta(D) \geq 16 - 8 \geq f(16)$. If $|D| = 9$, then $\delta(D) \geq 18 - 9 \geq f(18)$. 

Suppose that $|A \cap B| = 0$ or $|B \cap C| = 1$. By symmetry, we assume that $|A \cap B| = 0$. Then $AC \cap B = B \cap C$. By assumption on $A, B, C$, $B \cap C < C < AC$ and $AC = A \otimes_{AC} C \in K_{f_1}$ by free AP. As $B \cap C < AC$, $B$ and $D = AC \otimes_{AC} B$, we have $D \in K_{f_1}$ by free AP.

• $|(A \cap C) \setminus X| = 0$ and $|X| = 1$.
As we have shown the case that $|A \cap C| = 2, 3$, by symmetry, we may assume that $D = XZWU$. (i.e. $|(A \cap B) \setminus X| = 0$ and $|(B \cap C) \setminus X| = 0$) As $X < XZW = XZ \otimes_X XW \in K_{f_1}$ and $X < XU \in K_{f_1}$, we have $D = XZW \otimes_X XZ \in K_{f_1}$ by free AP.

The case that $|A \cap C| = 0$
As we have shown the case that $|A \cap C| = 1, 2, 3$, by symmetry, we may assume that $D = ZWU$ (i.e. $|A \cap B| = 0$ and $|B \cap C| = 0$.) By free AP, we see $D \in K_{f_1}$.

References

[E] D.M. Evans, Some remarks on generic structures, preprint, July 2003 (working draft)

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