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Kyoto University
GENERIC STRUCTURES AND CONTROL FUNCTIONS
(A COMMENTARY ON EVANS' PREPRINT)

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Abstract. We survey the results in "Some remarks on generic structures" [E] written by Evans, and give some detailed proofs which are omitted in his note.

1. Introduction

In simplicity theory, Hrushovski's generic constructions yield various results. As in his \(\omega\)-categorical stable pseudoplane, he constructed an \(\omega\)-categorical, simple, rank one, non-locally modular theory by amalgamating finite graphs whose local rank is controlled by an increasing unbounded convex function. In [E1], Evans gave a sufficient condition on control functions for constructing \(\omega\)-categorical simple generic structures. We review this in fifth section. In [E], Evans gave an \(\omega\)-categorical non-simple generic structure by carefully setting a control function (In this note, sixth section). This non-simple generic structure has 3-strong order property. For any \(n \geq 3\), \(n\)-strong order property was introduced by Shelah. (See [Sh] and third section in this note.) Strict order property implies \(n\)-strong order property, and \(n+1\)-strong order property implies \(n\)-strong order property for any \(n \geq 3\). Evans showed that generic structures given by control functions do not have 4-strong order property, we follow this result in fourth section.

In [P], Pourmahdian conjectured that generic structures without control function, so-called \((\mathbb{K}_0, <)\)-generic structure, will be non-simple. In [P], Pourmahdian considered a natural expansioned inductive (incomplete) theory \(T_{nat}\) of a universal theory \(T_0\) only axiomatizing that any finite substructure has non-zero positive local rank. Pourmahdian showed that \(T_{nat}\) is a Robinson theory and its universal domain is simple as a structure, and \(T_{nat}\) does not have model companion. (Natural expansioned structure of \((\mathbb{K}_0, <)\)-generic structure is an existentially closed model of \(T_{nat}\).) Evans gave an example of \((\mathbb{K}_0, <)\)-generic structure having strict order property, we discuss this issue in second section.

Date: August 1, 2005.

I would like to thank David M. Evans for his permission to submit this note.
This note is organized as follows.

Section 2: We will follow the proof that $\text{Th}(M_0)$ has strict order property, where $M_0$ is $(K_0, <)$-generic structure with one ternary relation.

Section 3: Review of [Sh].

Section 4: We will follow the proof that $\text{Th}(M_f)$ does not have SOP$_4$, where $M_f$ is a $(K_f, <)$-generic structure and $K_f$ is the class of finite graph $A$ satisfying with $\delta(A) \geq f(|A|)$ and control function $f$ is a convex increasing unbounded function from $\mathbb{N}$ to $\mathbb{R}$.

Section 5: Review of [E].

Section 6: We will follow the proof that for some control function $f$, $\text{Th}(M_f)$ has SOP$_3$, where $M_f$ is a $(K_f, <)$-generic structure and $K_f$ is the class of finite graph $A$ satisfying with $\delta(*) = 2|_*| - c(*)$.

Section 7, 8: Long appendices for Section 6, which are omitted in [E1].

2. $\text{Th}(M_0)$ has SOP. (Definable correspondence between graphs and ternary hypergraphs)

Let $\mathcal{A}$ be a ternary relation. For finite ternary-hypergraph $\mathfrak{A}$, we define the predimension as follows.

$$\delta(\mathfrak{A}) = |\mathfrak{A}| - |\mathfrak{A}^3|$$

For finite $\mathfrak{A} \subseteq \mathfrak{B}$ we define a partial order $<$ as follows

$$\mathfrak{A} < \mathfrak{B} \iff \delta(\mathfrak{A}) > \delta(\mathfrak{B}) \quad (\forall \mathfrak{X} \subseteq \mathfrak{A} \subseteq \mathfrak{B}).$$

For possibly infinite $\mathfrak{A} \subseteq \mathfrak{B}$ we define

$$\mathfrak{A} < \mathfrak{B} \iff \mathfrak{X} \cap \mathfrak{A} < \mathfrak{X} \quad (\forall \mathfrak{X} \subseteq \mathfrak{B}).$$

Note that $\mathfrak{A} < \mathfrak{A}$. For possibly infinite $\mathfrak{A} \subseteq \mathfrak{B}$, there exists the $<$-closure $\text{cl}_< (\mathfrak{A})$ of $\mathfrak{A}$ in $\mathfrak{B}$. $K_0$ is the class of finite 3-hypergraphs defined by

$$\mathfrak{A} \in K_0 \iff \emptyset < \mathfrak{A}$$

$K_0$ is the class of 3-hypergraphs whose finite sub-hypergraph is all in $K_0$.

$M_0$ denotes the $(K_0, <)$-generic structure.

Notation 2.1. Let $(A, R)$ be a graph, where $R$ is the binary relation for the graph. We define the following ternary graph $(\mathcal{A}, \mathcal{R})$.

- $\mathcal{A}_A = A \cup R^A \cup \{x_A, y_A\}$, where $x_A, y_A$ are new elements.
- $(\mathcal{R})^{\mathcal{A}} = \{(x_A, y_A, a) : a \in A\} \cup \{(a, b, (a, b)) : (a, b) \in R^A\}$

$(\mathcal{A}, \mathcal{R})$ is definable in $(A, R)$ with two new constants.

Lemma 2.2. Let $(A, R)$ be a graph. Then

1. $\mathcal{A}_A \in K_0$.
2. $\mathcal{A}_A = \text{cl}_{\mathcal{A}_A}(x_A, y_A)$. 

Proof. Let $\mathcal{X} \subseteq \mathcal{H}_A$, and let $V(\mathcal{X})$ be the vertex set of $\mathcal{X}$. Then $V(\mathcal{X}) \subseteq A \cup R_A \cup \{x_A, y_A\}$ follows. If $x_A, y_A \in \mathcal{X}$, then $\delta(\mathcal{X}) = |V(\mathcal{X})| - (|V(\mathcal{X}) \cap R_A| + |V(\mathcal{X}) \cap A|) > 0$, since $V(\mathcal{X}) = \{x_A, y_A\} \cup (V(\mathcal{X}) \cap R_A) \cup (V(\mathcal{X}) \cap A)$. Otherwise, $\delta(\mathcal{X}) = |V(\mathcal{X})| - (|V(\mathcal{X}) \cap R_A|) > 0$, since $|V(\mathcal{X}) \cap R_A| > 0$ implies $|V(\mathcal{X}) \cap A| > 0$.

Let $c \in A$. Then $\delta(c/x_A, y_A) = 0$. So, if $c \not\in cl_{\mathcal{H}_A}(x_A, y_A)$, then $0 < \delta(c/\text{cl}_{\mathcal{H}_A}(x_A, y_A)) \leq \delta(c/x_A, y_A) = 0$, a contradiction. Next, let $c = (a, b) \in R_A$. Then $\delta(c/a, b) = 0$. By the above argument, we see $c \in \text{cl}_{\mathcal{H}_A}(a, b)$. As $a, b \in cl_{\mathcal{H}_A}(x_A, y_A)$, we see that $c \in \text{cl}_{\mathcal{H}_A}(x_A, y_A)$.

Next, for any symmetric 3-hypergraph having at least two vertices, we construct a graph as follows.

**Notation 2.3.** Let $(\mathfrak{A}, \mathfrak{R})$ be a symmetric 3-hypergraph having at least two vertices. Fix two vertices $a, b \in \mathfrak{A}$. We define the following graph $G_{(\mathfrak{A}, a, b)} = (G_{\mathfrak{A}}, R)$ as follows.

- $G_{\mathfrak{A}} = \{c \in \mathfrak{A} : \mathfrak{A} \models \mathfrak{R}(c, a, b)\}$
- $R^{\mathfrak{A}} = \{(c, d) \in \mathfrak{A}^2 : \mathfrak{A} \models \mathfrak{R}(c, a, b) \land \mathfrak{R}(d, a, b) \land \exists x \mathfrak{R}(x, c, d)\}$

$G_{(\mathfrak{A}, a, b)} = (G_{\mathfrak{A}}, R)$ is defined in $(\mathfrak{A}, \mathfrak{R})$ with parameters $a, b \in \mathfrak{A}$.

**Remark 2.4.**

1. $\mathfrak{A} \not\models \mathfrak{R}_{(\mathfrak{A}, a, b)}$, where $a, b \in \mathfrak{A}$. (If $\mathfrak{A} \models \neg \mathfrak{R}(d, a, b)$, $d$ will not appear in the righthand.)

2. $A \simeq G_{(\mathfrak{H}_A, x_A, y_A)}$.

**Proof.** Clearly, $G_{\mathcal{H}_A} = A$ and $R^{\mathcal{H}_A} = R_A$, as desired.

**Lemma 2.5.** Let $(\mathfrak{A}, \mathfrak{R})$ be a symmetric 3-hypergraph having at least two vertices. Then

1. $a, b \not\in G_{\mathfrak{A}} \subseteq \text{cl}_\mathfrak{A}(a, b)$
2. If $(c, d) \in R^{\mathfrak{A}}$, then $\mathfrak{A}_{\text{cl}(a, b)} \models \exists x \mathfrak{R}(x, c, d)$
3. If $\mathfrak{A} < \mathfrak{B}$, then $G_{(\mathfrak{A}, a, b)} = G_{(\mathfrak{B}, a, b)}$

**Proof.** If $\mathfrak{A} \models \mathfrak{R}(c, a, b)$, then $c \in \text{cl}_\mathfrak{A}(a, b)$. (1),(2) follow. If $\mathfrak{A} < \mathfrak{B}$, then $\text{cl}_\mathfrak{A}(a, b) = \text{cl}_\mathfrak{B}(a, b)$. So, (3) follows.

**Notation 2.6.** Let $\varphi$ be a sentence in the language of graphs with binary relation symbol $R(x_1, x_2)$. We construct a formula $\sigma_\varphi$ having free variable $y, z$ in the language of 3-hypergraphs with ternary relation symbol $\mathfrak{R}(x_1, x_2, x_3)$ as follows.

- Replace all atomic subformulas $R(x_1, x_2)$ by $\mathfrak{R}(x_1, y, z) \land \mathfrak{R}(x_2, y, z) \land \exists w \mathfrak{R}(w, y, z)$
- Replace $\forall x (\psi(x, z)) \land (\exists x (\mathfrak{R}(x, y, z) \rightarrow \psi(x, z)))$ by $\forall x (\mathfrak{R}(x, y, z) \land \psi(x, z))$. 

Remark 2.7. Let \( (\mathfrak{A}, \mathfrak{B}) \in K_0 \), \( a, b \in \mathfrak{A} \) and \( \varphi \) be a sentence in the language of graphs. Then

\[
G_{(\mathfrak{A},a,b)} \models \varphi \iff \mathfrak{A} \models \psi_\varphi(a,b)
\]

The above remark follows from “REDUCTION THEOREM”, a (non-onto) map from \( G_{(\mathfrak{A},a,b)} \) to \( \mathfrak{A} \), and the way of replacement of quantifiers. Reduction theorem needs a onto map, but our \( \psi_\varphi \)'s quantifiers are bounded in \( \mathfrak{A}(\ast,a,b) \). So we need not a onto map, here.

Fact 2.8. Let \( M \) be an \( L \)-structure, and \( N \) be an \( L' \)-structure. Suppose that

- there exists a partial onto map \( f \) from \( M^n \) to \( N \) (for some \( n < \omega \))
- for every positive atomic \( L \)-formula \( \theta \), there exists an \( L' \)-formula \( \psi_\theta \) such that \( M \models \theta(a) \iff N \models \psi_\theta(f(a)) \)

THEN, by induction on the complexity of formulas, for every \( L \)-formula \( \varphi \), there exists an \( L' \)-formula \( \psi_\varphi \) such that \( M \models \varphi(a) \iff N \models \psi_\varphi(f(a)) \)

Lemma 2.9. Let \( \varphi \) be a sentence in the language of graphs. THEN, “there exists a finite graph \( A \models \varphi \)” iff \( M_0 \models \exists yz \psi_\varphi(y,z) \).

Proof. \((\Rightarrow)\): We may assume that \( \mathfrak{A}_A < M_0 \). So, by Remark 2.4, \( A \simeq G_{(\mathfrak{A},x_A,y_A)} \simeq G_{(M_0,x_A,y_A)} \). Therefore, \( M_0 \models \psi_\varphi(x_A,y_A) \).

\((\Leftarrow)\): \( G_{(M_0,a,b)} \models \varphi \) and \( G_{(M_0,a,b)} \subseteq \text{cl}_{M_0}(a,b) \subseteq \omega M_0 \)

Proposition 2.10. Let \( \varphi \) be a sentence in the language of graphs. Suppose that \( \varphi \) has arbitraliy large finite model. Then there exists an infinite model, definable in some model of \( \text{Th}(M_0) \).

Proof. By our assumption, for any \( n < \omega \), there exists a finite graph \( A_n \) such that \( A_n \models \varphi \) and \( |A_n| \geq n \). As \( A_n \simeq G_{(\mathfrak{A}_A,x_A,y_A)} \) (by Remark 2.4) and \( \omega > |\mathfrak{A}_A| \geq |A_n| \geq n \), for any \( n < \omega \),

\( \mathfrak{A}_A \models \psi_\varphi(x_A,y_A) \land |\mathfrak{A}_{A^n}(\ast,x_A,y_A)| \geq n \).

As \( M_0 \) is \((K_0,\text{<})\)-generic, there exists \( \mathfrak{A}_A \simeq \mathfrak{A} < M_0 \). Since \( G_{(\mathfrak{A}_A,x_A,y_A)} \simeq G_{(\mathfrak{A},a,b)} \), where \( x_{A^{\mathfrak{A}}} \mapsto ab \),

\( \text{Th}(M_0) \models \exists yz \psi_\varphi(y,z) \land |\mathfrak{A}(\ast,y,z)| \geq n \).

By compactness, there exist infinite \( M \models \text{Th}(M_0) \) and \( a',b' \in M \) such that \( G_{(M,a',b')} \models \varphi \), where \( G_{(M,a',b')} \) is definable in \( M \).

Theorem 2.11. \( \text{Th}(M_0) \) has strict order property.

Proof. Let \( A_n \) be the graph as follows;

- Vertices: \( \{b_i : i < n\} \cup \{c_i : i < n\} \)
- Edges: \( \{(b_i,c_j) : 0 \leq i < j < n\} \)

...
Let \( a_i = (b_i, c_i) \), and \( \varphi(xy, zw) \equiv R(x, y) \wedge R(z, w) \wedge R(x, w) \wedge \neg R(x, z) \wedge \neg R(y, w) \wedge \neg R(y, z) \). Then \( A_n \models \varphi(a_i, a_j) \iff i < j < n \).

By Lemma 2.9, we can find a linear (uniformly definable) ordering of arbitrarily finite length in \( M_0 \). By compactness, we see that \( \text{Th}(M_0) \) has the strict order property.

3. Review of Strong Order Property

This section consists of Shelah’s results in [Sh].

**Definition 3.1.** A complete theory \( T \) has \( n \)-strong order property, denoted \( \text{SOP}_n \) if there exists a formula \( \varphi(x, y) \) (\( \text{lh}(x) = \text{lh}(y) \)) and a sequence \( (a_i : i < \omega) \) in some model \( N \) of \( T \) such that

1. \( N \models \varphi(a_i, a_j) \) for \( i < j < \omega \)
2. there is no \( n \)-\( \varphi \)-loops;

\[ N \models \neg \exists x_0, x_1, \ldots x_{n-1} \varphi(x_0, x_1) \wedge \varphi(x_1, x_2) \wedge \cdots \wedge \varphi(x_{n-2}, x_{n-1}) \]

**Fact 3.2.**

1. \( \text{SOP} \) implies \( \text{SOP}_n \).
2. \( \text{SOP}_{n+1} \) implies \( \text{SOP}_n \).
3. If \( T \) has \( \text{SOP}_3 \), then \( T \) has the tree property.

**Proof.** (1): By way of contradiction, suppose that \( T \) has \( \text{SOP} \) and \( \text{NSOP}_n \). So, there exist \( \varphi(x, y), N \models T \) and \( (a_i : i < \omega) \subset N \) such that \( \forall x \, \varphi(x, a_i) \rightarrow \varphi(x, a_j), \exists x (\varphi(x, a_i) \wedge \varphi(x, a_j)) \) for \( i < j < \omega \). Let \( \psi(x_0, x_1) = \forall x \, \varphi(x, x_0) \rightarrow \varphi(x, x_1) \). By \( \text{SOP} \), there exists \( n \)-\( \psi \)-loop, but it is impossible.

(2): Let \( \varphi(x, y) \), a model \( M \), and \( (a_i : i < \omega) \in M \) be witness for \( \text{SOP}_{n+1} \). We may assume that \( (a_i : i < \omega) \) is indiscernible. We divide the arugment into two cases, whether

\[ M \models \exists x_0, \ldots x_{n-1} \{ x_0 = a_1 \wedge x_{n-1} = a_0 \wedge \bigwedge_{i,j < n, k \equiv l+1 (\text{mod} \, n)} \varphi(x_i, x_j) \} \]

or not.

- The case that \( M \models \exists x_0, \ldots x_{n-1} \{ x_0 = a_1 \wedge x_{n-1} = a_0 \wedge \bigwedge_{i,j < n, k \equiv l+1 (\text{mod} \, n)} \varphi(x_i, x_j) \} \)

As \( a_1 \equiv a_0, a_2 \), we have \( M \models \exists x_0, \ldots x_{n-1} \{ x_0 = a_2 \wedge x_{n-1} = a_0 \wedge \bigwedge_{i,j < n, k \equiv l+1 (\text{mod} \, n)} \varphi(x_i, x_j) \} \).

Let \( a_2, c_1, \ldots c_{n-2}, a_0 \) be the witness for \( x_0, \ldots x_{n-1} \). By the way, \( M \models \varphi(a_1, a_2) \wedge \varphi(a_0, a_1) \), so \( a_1, a_2, c_1, \ldots c_{n-2}, a_0 \) is an \((n+1)-\varphi\)-loop, a contradiction.

- The case that \( M \not\models \exists x_0, \ldots x_{n-1} \{ x_0 = a_1 \wedge x_{n-1} = a_0 \wedge \bigwedge_{i,j < n, k \equiv l+1 (\text{mod} \, n)} \varphi(x_i, x_j) \} \)

Put \( \psi(x, y) \equiv \varphi(x, y) \wedge \neg \exists x_0, \ldots x_{n-1} \{ x_0 = a_1 \wedge x_{n-1} = a_0 \wedge \bigwedge_{i,j < n, k \equiv l+1 (\text{mod} \, n)} \varphi(x_i, x_j) \} \)

Then \( M \models \psi(a_i, a_j) (i < j < \omega) \), and \( n \)-\( \psi \)-loops never exist.

(3): Let \( \kappa = \text{cf}(\kappa) > |T| \) and \( \lambda > \kappa \) be such that \( \text{cf}(\lambda) = \kappa \) and "\( \mu < \lambda \) implies \( 2^\mu < \lambda \)" (strongly limit singular cardinal of cofinality \( \kappa \)). Put \( J = {}^\kappa \lambda \) and
Let $\varphi(x, y)$ be the witness for SOP$_3$. By compactness, there exist a sequence $(a_\eta : \eta \in I)$ in some model $M$ such that $M \models \varphi(a_\eta, a_\nu)$ for any $\eta < \nu$.

The lexicographic order on $I$ is as usual; if $i$ is the least such that $\eta > i$, then $\eta(i) = \nu(i)$.

We may assume that $M$ is $\kappa^+$-saturated, and $|M| \geq \lambda$. Fix an $\eta \in {}^\langle \lambda \setminus \{0\} \rangle I$. We will define $a_\eta$ as follows.

Put $p_\eta = \{ \varphi(a_{(\eta|i, \eta(i)+1)0_{(i, \kappa)}}, x) : i < \kappa \}$.

Note that $(\eta|i)0_{(i, \kappa)}, (\eta|i, \eta(i) + 1)0_{(i, \kappa)} \in I$, and $a_{(\eta|i, \eta(i)0_{(i, \kappa)})} \models \varphi(a_{(\eta|i, \eta(i)+1)0_{(i, \kappa)}}, x)$.

As $M$ is $\kappa^+$-saturated, there exists a realization of $p_\eta$ in $M$, say $a_\eta$.

Claim. If $\eta_1 \neq \eta_2 \in {}^\langle \lambda \setminus \{0\} \rangle I$, then $p_{\eta_1} \cup p_{\eta_2}$ is inconsistent.

Suppose that $\eta_1 < \eta_2$. Then there exists $i < \kappa$ such that $\eta_1|i = \eta_2|i, \eta_1(i) < \eta_2(i)$. Take $\nu < \rho \in I$ be with $\eta_1 < \nu < \rho < \eta_2$ as follows.

$\eta_1|i = \eta_2|i = \nu|i = \rho|i, \nu(i) = \eta_1(i) + 1, \rho(i) = \nu_2(i), \nu(j) = 0(j > i), \rho(j) = \nu_2(i) + 1, \rho(j) = 0(j > i + 1)$.

As $\varphi(x, a_\nu) \in p_{\eta_1}, \varphi(x, a_\rho) \in p_{\eta_2},$ if we found the realization of $p_{\eta_1} \cup p_{\eta_2}$, say $c$, then $c, a_\nu, a_\rho$ would be the $3$-$\varphi$-loop, a contradiction.

We also have $|p_\eta| = \kappa, |\{ \text{Dom}(p_\eta) : \eta \in {}^\langle \lambda \setminus \{0\} \rangle I \}| \leq \lambda$ (as $\cup \{ \text{Dom}(p_\eta) : \eta \in {}^\langle \lambda \setminus \{0\} \rangle I \} \subseteq \{ \text{Dom}(p_\eta) : \eta \in {}^\langle \lambda \setminus \{0\} \rangle I \}$)

By 7.7(3) and 7.6(2) on p.141 of Shelah’s 2nd edition book, $\lambda = \lambda^{< \kappa} > 2^{|T|}$ (by cf$(\lambda) = \kappa < \lambda$ and $\kappa > |T|$ imply that $T$ has the tree property.)

It is conjectured that SOP$_4$ is a good dividing line for existence of universal models, i.e. if $T$ does not have SOP$_4$, it will have universal models of cardinality $\lambda > |T|$ (Shelah showed that if $T$ is simple and $\lambda > |T|$, then there exists universal models of cardinality $\lambda^{++}$. As the above, simplicity implies NSOP$_3$.)

4. Th$(M_f)$ does not have SOP$_4$

Let $\delta$ be a local rank on relational finite structures such that $\delta(A/B) \leq \delta(A/A \cap B)$, where $\delta(A/B) = \delta(AB) - \delta(B)$. Let $f : \mathbb{R}^\geq 0 \to \mathbb{R}^\geq 0$ be upper unbounded and monotone increasing. Let $K_f = \{ A \in K_\delta : \delta(X) \geq f(|X|)(\forall X \subseteq A) \}$ and $\beta(x) = \min \{ \delta(X/A) : A < X \in K_\delta, A \neq X \}$.

Fact 4.1. Suppose that

$$f'(x) \leq \frac{\beta(x)}{x}.$$
Then \( \mathbf{K}_f \) is closed under free amalgamation, so \( (\mathbf{K}_f, <) \)-generic \( M_f \) exists, \( \text{cl} = \text{acl} \) in \( M_f \) and \( \text{Th}(M_f) \) is \( \omega \)-categorical. (\( \omega \)-categoricity follows from \( |\text{cl}(*)| \leq f^{-1}(\delta(*)) \) for finite graphs.)

**Proof.** Let \( A < B, B_2 \in \mathbf{K}_f \) and let \( C = B_1 \otimes A B_2 \). We need to show that if \( X \subseteq C \), then \( \delta(X) \geq f(|X|) \). We may assume that \( X < C \), because \( \delta(X) \geq \delta(\text{cl}(X)) \) and \( f(|\text{cl}(X)|) \geq f(|X|) \).

Let \( X_1 = X \cap B_i (i = 1, 2) \) and let \( X_0 = X \cap A \). Suppose that

\[
\frac{\delta(X_1) - \delta(X_0)}{|X_1| - |X_0|} \leq \frac{\delta(X) - \delta(X_0)}{|X| - |X_0|} \leq \frac{\delta(X_2) - \delta(X_0)}{|X_2| - |X_0|}.
\]

As \( X_0 < X_1 \), \( \beta(|X_1|) \leq \delta(X_1/X_0) \). Therefore \( \frac{\delta(X_1) - \delta(X_0)}{|X_1| - |X_0|} \geq \frac{\beta(|X_1|)}{|X_1|} \geq f'(|X_1|) \). So, the line between \( (|X_0|, \delta(X_0)) \) and \( (|X_1|, \delta(X_1)) \) lies above \( f \). As \( f' \) is decreasing and \( \delta(X_1) \geq f(|X_1|) \), \( \delta(X) \geq f(|X|) \) follows.

Let \( d(A) = \delta(\text{cl}(A)) \), and \( d(a/A) = \delta(\text{cl}(aA)/\text{cl}(A)) \). For possibly infinite \( B \), let \( d(a/B) = \inf \{d(a/B_0) : B_0 \subset_{\omega} B \} \).

**Fact 4.2.** Let \( \mathcal{M} \) be a relational structure having \( \delta \)-rank. Let \( a, A, B \subset_{\omega} \mathcal{M} \). Suppose that \( A < B < \mathcal{M} \) and \( \text{cl}(aA) \subset_{\omega} \mathcal{M} \). Then \( d(a/B) = d(a/A) \) iff \( \text{cl}(aA) \cap B = A, \text{cl}(aA)B = \text{cl}(aA) \otimes_{A} B \) and \( d(aB) = \delta(\text{cl}(aA)B) \) (i.e. \( \text{cl}(aA)B \leq \text{cl}(aB) \)).

**Proof.** As \( A < \text{cl}(aA) \cap B \) or \( A = \text{cl}(aA) \cap B \), we have \( \delta(A) \leq \delta(\text{cl}(A) \cap B) \). So, \( \delta(\text{cl}(aA)/\text{cl}(aA) \cap B) \leq \delta(\text{cl}(aA)/A) \). Therefore

\[
d(a/B) \leq \delta(\text{cl}(aA)/B) \leq \delta(\text{cl}(aA)/\text{cl}(aA) \cap B) \leq \delta(\text{cl}(aA)/A) = d(a/A).
\]

Now we can see the conclusion. \( \square \)

By Fact 4.2, for \( a, b, A \subset_{\omega} \mathcal{M} \),

\[
d(a/Ab) = d(a/A) \Leftrightarrow d(b/Aa) = d(b/A).
\]

(By \( d(a/Ab) = d(a/A) \leftrightarrow \"\text{cl}(aA) \cap \text{cl}(bA) = \text{cl}(A), \text{cl}(aA) \text{cl}(bA) = \text{cl}(aA) \otimes_{\text{cl}(aA)} \text{cl}(bA) \leq \text{cl}(abA)\"."

From now on, we assume that the control function \( f \) satisfies

\[
\text{"}f'(x) \leq \frac{\beta(x)}{x}\text{"}.
\]

Let \( \mathbf{K}_f' \) be the class of possibly infinite structures whose finite substructures are all in \( \mathbf{K}_f \). Let \( T_f = \{ \forall \tilde{x} \text{Diag}_{A}(\tilde{X}) : \delta(A) < f(|A|), |A| < \omega \} \). Then \( M \models T_f \iff M \in \mathbf{K}_f' \). Let \( \mathcal{M} \) be a big model of \( M_f \). Note that if \( A \subset_{\omega} \mathcal{M} \), then \( A \in \mathbf{K}_f' \).

**Proposition 4.3.** Suppose that, in \( \mathcal{M} \), if \( A = \text{acl}(A) \), \( d(a/A) = d(a/Ab) \), \( \text{acl}(aA) \cap \text{acl}(bA) = A \), then there exists \( A_0 \subset_{\omega} A \) such that \( d(a/A_0b) = d(a/A_0) \). THEN \( \text{Th}(M_f) \) has \( \text{NSOP}_4 \).
Proof. Let \((a_i : i < \omega)\) be an infinite indiscernible sequence in \(\mathcal{M}\). Put 
\[ p(x_0x_1) = \text{tp}(a_0a_1). \]
We will show that 
\[ p(x_0x_1) \cup p(x_1, x_2) \cup p(x_2x_3) \cup p(x_3x_0) \]
is consistent.

Claim. There exists \(B \subseteq \omega \mathcal{M}\) such that \((a_i : i < \omega)\) is \(B\)-indiscernible, and 
\[ d(a_2/Ba_0a_1) = d(a_2/Ba_1) = d(a_2/B). \]
(Then \(a_2 \equiv_{a_0} a_1, d(a_2/Ba_0a_1) = d(a_2/Ba_2)\) follows.)

Extend \((a_i : i < \omega)\) to \((a_i : i < \mathbb{Z})\). As \((a_i : i \geq 0)\) is indiscernible over 
\((a_i : i < 0), (a_i : i \geq 0)\) is indiscernible over \(\text{acl}(a_i : i < 0) =: A_0\). As 
\(a_{ci} \equiv_{a_i} a_{<0}\), we see that 
\[ d(a_i/A_0a_{<i}) = d(a_i/A_0). \]
By extending \((a_i : i \geq 0)\) over \(A_0\) and applying Erdos-Rado Theorem, we 
may assume that \(\text{acl}(A_0a_k) \cap \text{acl}(A_0a_la_i) =: C\) is constant for any 
\(i < j < k\), and \((a_i : i \geq 0)\) is indiscernible over \(C\).

Now, by our assumption, take \(B \subseteq \omega C\) such that 
\[ d(a_2/Ba_0a_1) = d(a_2/B), \]
as desired. The claim is proven.

As \(d(a_2/Ba_0a_1) = d(a_2/B)\), we have 
\[ \text{cl}(a_2B)\text{cl}(a_0a_1B) = \text{cl}(a_2B) \otimes_{\text{cl}(B)} \text{cl}(a_0a_1B) \leq \text{cl}(a_0a_1a_2B). \]
As \(\text{cl}(a_0a_1a_2B) \in K_f\), we may assume that 
\[ \text{cl}(a_0a_1a_2B) < M_f. \]
So, we can work inside \(M_f\), i.e. we have \(a_0, a_1, a_2, B \subseteq \omega M_f\) such that \((a_0, a_1, a_2)\) is 
\(B\)-indiscernible and \(d_{M_f}(a_2/Ba_0a_1) = d_{M_f}(a_2/B)\).

Let \(C_{i,j} = \text{cl}(a_iB), C_i = \text{cl}(a_iB)\). By \(d(a_2/Ba_0a_1) = d(a_2/Ba_1)\) and Fact 4.2, we see that 
\(C := C_{0,1}C_{1,2} = C_{0,1} \otimes_{C_1} C_{1,2}\). And \(C_{0,1} \cap C_{0,2} = C_0\) and 
\(C_{1,2} \cap C_{0,2} = C_2\) follow by 
\[ d(a_2/Ba_0a_1) = d(a_2/Ba_1), \]
\(d(a_1/Ba_0a_2) = d(a_1/Ba_2)\) and 
Fact 4.2. So we have 
\[ C \cap C_{0,2} = C_0C_2 = C_0 \otimes_B C_2 < C. \]

Let \(f : C_0C_2 \rightarrow C_2C_0\) be an isomorphism over \(B\) sending \(a_0a_2\) to \(a_2a_0\), and 
let \(g : C_0C_2 \rightarrow C\) be the inclusion map. Put \(g' = g \circ f\). As \(K_f\) is closed 
under free amalgamation, there exist \(D \in K_f\) and \(h, h' : C \rightarrow D\) such that 
\(h \circ g(C_0C_2) = h' \circ g'(C_0C_2)\) and 
\(D = h(C) \otimes_{h_{g_0}(C_0C_2)} h'(C)\). We may assume that 
\(D < M_f\). Put 
\(a_0' = h \circ g(a_0), a_1' = h(a_1), a_2' = h' \circ g'(a_2), a_3' = h'(a_1)\).

Claim. \(a_0'a_1', a_1'a_2', a_2'a_3', a_3'a_0' \models p = \text{tp}(a_0a_1).\) (This proposition is proven.)

Note that 
\[ h(a_0a_1) = a_0'a_1', h(a_1a_2) = a_1'a_2', h'(a_0a_1) = (h' \circ g'(a_2))a_3' = a_2'a_3', \]
\[ h'(a_1a_2) = a_3'(h' \circ g'(a_0)) = a_3'(h \circ g(a_0)) = a_2'h(a_0) = a_3'a_0'. \]
On the other hand,
\[ h(C_{0,1}), h(C_{1,2}) < h(C) < D < M_f, \]
\[ h'(C_{0,1}), h'(C_{1,2}) < h'(C) < D < M_f. \]

Put \( B' = h \circ g(B) = h' \circ g'(B) \). Then
\[
\begin{align*}
&\quad h(\text{cl}(a_0 a_1 B)) = h(C_{0,1}) = \text{cl}(a_0' a_2' B'), \quad h(\text{cl}(a_1 a_2 B)) = h(C_{1,2}) = \text{cl}(a_2' a_3' B'), \\
&\quad h'(\text{cl}(a_0 a_1 B)) = h'(C_{0,1}) = \text{cl}(a_0' a_2' B'), \quad h(\text{cl}(a_1 a_2 B)) = h(C_{1,2}) = \text{cl}(a_3' a_0' B').
\end{align*}
\]

By genericity of \( M_f \), we see that
\[ \text{cl}(a_0 a_1 B) \equiv \text{cl}(a_1 a_2 B) \equiv \text{cl}(a_0' a_2' B') \equiv \text{cl}(a_2' a_3' B') \equiv \text{cl}(a_3' a_0' B'). \]

\[ \square \]

**Remark 4.4.** Suppose that for any \( a, A \subset \mathcal{M} \), there exists \( A_0 \subset_{\omega} A \) such that \( d(a/A) = d(a/A_0) \). Then the assumption of Proposition 4.3 holds.

**Proof.** Take \( A_0, A_1 \subset_{\omega} A \) such that \( d(a/Ab) = d(a/A_0 b) \) and \( d(a/A) = d(a/A_1) \). Then \( d(a/A_0 A_1) = d(a/A_0 A_1 b) \).

\[ \square \]

5. Review of Evans' paper on simple \( \omega \)-categorical generic structures

Let \( \delta \) be a local rank on relational finite structures such that \( \delta(A/B) \leq \delta(A/A \cap B) \), where \( \delta(A/B) = \delta(AB) - \delta(B) \). Let \( f : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0} \) be upper unbounded, monotone increasing, convex \( (f'(x) \) is monotone decreasing) and \( f'(x) \leq \frac{\beta(x)}{x} \), where \( \beta(x) = \min\{1, \delta(X/A) : A < X \in \mathcal{K}_o, A \neq X, |X| \leq x \} \).

Let \( \mathcal{K}_f = \{ A \in \mathcal{K}_o : \delta(X) \geq f(|X|) \forall X \subseteq A \} \).

The following fact is Corollary 2.20 of [E1].

**Fact 5.1.** Let \( M_f \) be \( (\mathcal{K}_f, <) \)-generic. And suppose the condition on \( \mathcal{M} \) (big model of \( \text{Th}(M_f) \)) as in Proposition 4.3. Furthermore, suppose the following.

1. (\( d \)-extension property in \( \mathcal{M} \))
   Let \( A \subseteq B \subseteq \mathcal{M} \) be algebraically closed and \( c \subseteq_{\omega} \mathcal{M} \). Then there exists \( c' \subseteq_{\omega} \mathcal{M} \) such that \( \text{tp}(c/A) = \text{tp}(c'/A) \), \( d(c'/B) = d(c/A) \) and \( \text{acl}(c'/A) \cap B = A \).

2. (Independence theorem over finite closed sets in \( M_f \))
   Let \( A, B_1, B_2 \subseteq M_f \) be finite such that \( B_1 \cap B_2 = A \) and \( d(B_1/B_2) = d(B_1/A) \). Suppose that \( c_1, c_2 \subseteq_{\omega} \mathcal{M}_f \), \( \text{tp}(c_1/A) = \text{tp}(c_2/A) \) and \( d(c_1/B) = d(c_1/A) \). Then there exists \( c \subseteq_{\omega} M_f \) such that \( \text{tp}(c/B_1) = \text{tp}(c_2/B_1) \) and \( d(c/B_1 B_2) = d(c/A) \).

THEN \( \text{Th}(M_f) \) is simple and \( "c \downarrow A \Leftrightarrow d(c/B) = d(c/A) \) and \( \text{acl}(c/A) \cap B = A \), for \( A, B \) algebraically closed in \( \mathcal{M} \).
We give the proof of the following lemma. (Theorem 3.6 of [E1])

**Lemma 5.2.** Suppose that $d$-extension property over finite closed sets in $M$ and $f(3x) \leq f(x) + \beta(x)$. Then the independence theorem over finite closed sets holds in $M_f$.

**Proof.** Let $c_i, B_i, A$ be as in Fact 5.1. Then $acl(c_1A) \simeq_A acl(c_2A)$. Put $E_{12} = acl(B_1B_2), E_{13} = acl(c_1B_1), E_{23} = acl(c_2B_2)$. By considering free amalgamation and copies, we may assume that

$B_1 \cap E_{13}, B_2 = E_{12} \cap E_{23}, B_3 := E_{13} \cap E_{23} = acl(c_3A)$,

$B_1 \cap B_2 \cap B_3 = A, B_1, B_2, B_3$ are $d$-independent over $A, E_{ij}E_{jk} = E_{ij} \otimes_{B_j} E_{jk}$. Let $E = E_{12}E_{13}E_{23}$. We need to show that $A < E$ and $E \in K_f$.

**Claim.** $A < E$.

By Fact 4.2, $B_iB_j \leq E_{ij}$. As $E = E_{ij} \otimes_{B_iB_j} E_{ik}E_{jk}, E_{ik}E_{jk} \leq E$ follows. We also have $E_{ik}E_{jk} = E_{ik} \otimes_{B_k} E_{jk}$. Thus $E_{ik} < E$. As $A < B_i < E_{ik}, A < E$ follows.

**Claim.** $E \in K_f$.

We have $E = E_{ij} \otimes_{B_iB_j} E_{ik}E_{jk}$, but we do not have $B_iB_j < E_{ij}, E_{ik}E_{jk}$. So we cannot conclude this claim by using Fact 4.1. We need to show $\delta(D) \leq f(|D|)$ for any $D < E$ as in Fact 4.1. Put $D_{ij} = D \cap E_{ij}$ and $d_{ij} = \delta(D_{ij}).$ Suppose that $d_{12}$ is the largest of these.

As $E_{12}E_{23} \in K_f$, we may assume that $D \neq D_{12}D_{23}$. Put $D^1 = D_{12}D_{13}$. As $E_{12}E_{13} \leq E$, we see that $D^1 \leq D$. As $D^1 = D_{12} \otimes_{D \cap B_1} D_{13}$ and $D \cap B_1 < D_{13},$

$$\delta(D^1) = d_{12} + \delta(D_{13}/D \cap B_1) \geq d_{12} + \beta(|D_{13}|).$$

As $d_{13} \leq d_{12}, |D_{13}| \leq d_{13} \leq D_{13}/D \cap B_1 \leq d_{12},$

$\delta(D^1) = \delta(D_{13}/D \cap B_1) \leq \delta(D^1)/d_{12} + \beta(|D_{13}|) \leq \delta(D_{13}/D \cap B_1) \leq \delta(D_{13}/D \cap B_1) \leq \delta(D^1) - \delta(D_{13}/D \cap B_1) \leq \delta(D^1) - \delta(D_{13}/D \cap B_1)$.

By our assumption on $f$,

$$f(3x) \leq f(x) + \beta(x),$$

$$3f^{-1}(d_{12}) = f^{-1}(d_{12} + \beta(f^{-1}(d_{12}))).$$

So, $3f^{-1}(d_{12}) \leq f^{-1}(\delta(D^1))$. As $|D| \leq \sum_{ij} |D_{ij}| \leq \sum_{ij} f^{-1}(d_{ij}) \leq 3f^{-1}(d_{12})$ and $\delta(D^1) \leq \delta(D)$, we see that

$$|D| \leq f^{-1}(\delta(D)).$$

$\Box$
6. \( \Th(M_f) \) has \( \SOP_3 \) for some \( f \)

We work with undirected graphs, and \( \delta(A) = 2|A| - e(A) \). Note that \( \beta(x) = 1 \). The control function \( f : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0} \) is an upper unbounded, monotone increasing satisfying the following five conditions:

\[
\text{(F1): } f(0) = 0, f(2) = 2, f(4) = 3, f(8) = 4, f(10) < \frac{1}{2} < f(12) < 5 < f(14) < \frac{1}{3} < f(16) < f(18) \leq 6.
\]

\[
\text{(F2): } 2f'(2n) \leq \frac{1}{n} \text{ for } n \geq 7.
\]

\[
\text{(F3): } f\left(\frac{k^2}{2}\right) \leq k \text{ if } k \geq 6.
\]

\[
\text{(F4): } f(3n) \leq f(n) + 1 \text{ for } n \geq 10.
\]

\[
\text{(F5): } f(10) + 1 \geq f(14), f(12) + 1 \geq f(16).
\]

Let \( f_1(x) = f(2x) \). So, \( f_1'(x) = 2f'(2x) \) and \( f_2'(n) \leq \frac{1}{n} \) for \( n \geq 7 \).

We consider \( K_{f_1} \).

**Remark 6.1.**

1. \( \delta(3\text{-cycle}) = 6 - 3 = 3 < f(6) = f_1(3) \), so 3-cycle does not belong to \( K_{f_1} \). \( \delta(4\text{-cycle}) = 8 - 4 = 4 = f(8) = f_1(4) \), so 4-cycle belongs to \( K_{f_1} \).

2. The graph \( \text{————} \) does not belong to \( K_{f_1} \), because its \( \delta \)-rank = 14 - 9 = 5 < f(14) = f_1(7) \).

3. (F1) and (F2) give the free amalgamation property of \( (K_{f_1}, <) \).

4. (F1) and (F3) are needed to show that the graphs \( G(A_n, B_n, x_0) \) belong to \( K_{f_1} \). (Lemma 6.4.)

5. (F4) is needed to show Subclaim 2 in the proof of Lemma 6.7. Lemma 6.7 ensures that the important graphs \( E_n \) can be closedly embedded into \( M_{f_1} \) and the graphs \( E_n \) will give the witness formula for \( \SOP_3 \).

6. (F1), (F2) and (F5) are needed to show Lemma 6.6. (Lemma 6.6 gives a very important key to get Lemma 6.7.)

By the graphs \( E_n < M_{f_1}(n \in \omega) \), we will give a formula \( \varphi(x,y) \) and infinite sequence \( (a_i)_{i<\omega} \) in \( M_{f_1} \) such that \( M_{f_1} \models \varphi(a_i, a_j) \) whenever \( i < j \). But if there were a 3-\( \varphi \)-loop in some model \( N \) of \( \Th(M_{f_1}) \), then \( N \) would have the
graph as in (2) of Remark 6.1. As any finite graph of \( N \) belongs to \( K_{f_{1}} \), so SOP\(_{3} \) follows.

**Lemma 6.2.** \( K_{f_{1}} \) has the free amalgamation property.

**Proof.** Let \( A < B_{1}, B_{2} \in K_{f} \) and let \( C = B_{1} \otimes_{A} B_{2} \). We need to show that if \( X \subseteq C \), then \( \delta(X) \geq f_{1}(|X|) \). We may assume that \( X < C \), because \( \delta(X) \geq \delta(\text{cl}(X)) \) and \( f_{1}(|\text{cl}(X)|) \geq f_{1}(|X|) \).

Let \( X = X \cap B_{i} \) (\( i = 1, 2 \)) and let \( X_{0} = X \cap A \). Suppose that
\[
\frac{\delta(X) - \delta(X_{0})}{|X| - |X_{0}|} \leq \frac{\delta(X_{2}) - \delta(X_{0})}{|X_{2}| - |X_{0}|},
\]
As \( X_{0} < X \), \( \beta(|X_{1}|) \leq \delta(X_{1}/X_{0}) \). So, by (F2),
\[
\frac{\delta(X) - \delta(X_{0})}{|X| - |X_{0}|} \geq \frac{\delta(X_{1}) - \delta(X_{0})}{|X_{1}| - |X_{0}|} \geq \frac{1}{|X_{1}|} \geq f_{1}'(|X_{1}|).
\]
So, the line between \( (|X_{0}|, \delta(X_{0})) \) and \( (|X_{1}|, \delta(X_{1})) \) lies above \( f_{1} \). As \( f_{1}' \) is decreasing and \( \delta(X_{1}) \geq f_{1}(|X_{1}|), \tilde{\delta}(X) \geq f_{1}(|X|) \) follows. In Appendix 1, we give the proof when \( |X_{1}| \leq 6 \).

\[\square\]

**Notation 6.3.** Consider the following graphs \( G(A_{n}, B_{n}, x_{0}) \) for each \( n < \omega \).
- Vertex set: \( A_{n} \cup B_{n} \cup \{x_{0}\} \cup \{z_{ij} : 0 \leq i < j \leq n\} \), where \( A_{n} = \{a_{i} : 0 \leq i \leq n\} \), \( B_{n} = \{b_{i} : 0 \leq i \leq n\} \).
- Edges: \( R(x_{0}, a_{i}), R(x_{0}, b_{i}) \) for \( 0 \leq i \leq n \) and \( R(z_{ij}, a_{i}), R(z_{ij}, b_{j}) \) for \( 0 \leq i < j \leq n \).

**Lemma 6.4.**

1. \( G(A_{n}, B_{n}, x_{0}) \in K_{f_{1}} \)
2. \( x_{0}A_{n} < G(A_{n}, B_{n}, x_{0}) \)
3. \( \tilde{\delta}(x_{0}X_{A}X_{B}) = 2(m+1) - m = m + 2 : k \)

**Proof.** Put \( G = G(A_{n}, B_{n}, x_{0}), A = A_{n}, B = B_{n}, Z = \{z_{ij} : 0 \leq i < j \leq n\} \).

(1): It suffices to show that if \( X < G \), then \( \delta(X) \geq f_{1}(|X|) \). It is clear in case of \( |X| = 1 \). If \( |X| \geq 2 \), then \( x_{0} \in X \). (If \( x_{0} \neq a, b \in X \), then \( \delta(x_{0}/ab) = 0 \), so \( x_{0} \in \text{cl}_{G}(ab) \subset X \).

**Claim.** \( a_{i}, b_{j} \in X \Leftrightarrow z_{ij} \in X \).

This claim follows from \( \delta(z_{ij}/a_{i}b_{j}) = \delta(a_{i}/x_{0}z_{ij}) = \delta(b_{j}/x_{0}z_{ij}) = 0 \) and \( X < G \).

Put \( X_{A} = X \cap A, X_{B} = X \cap B, X_{Z} = X \cap Z \) and \( m = |X_{A}| + |X_{B}| \). By claim, we see that \( \delta(X_{Z}/x_{0}X_{A}X_{B}) = 0 \), so we have
\[
\delta(X) = \delta(x_{0}X_{A}X_{B}) = 2(m+1) - m = m + 2 =: k \]
As $|X_{Z}| \leq |X_{A}||X_{B}| \leq |X_{A}|(m - |X_{A}|) = (\frac{m}{2})^{2} - (|X_{A}| - \frac{m}{2})^{2} \leq (\frac{m}{2})^{2}$, we have

$$|X| \leq 1 + m + (\frac{m}{2})^{2} = (1 + \frac{m}{2})^{2} = \frac{k^{2}}{4}$$

If $k \geq 6$, by (F3), $\delta(X) = k \geq f(\frac{k^{2}}{2}) = f_{1}(\frac{k^{2}}{4}) \geq f_{1}(|X|)$, as desired.

If $k \leq 5$, then $|X_{A}| + |X_{B}| \leq 3$. If $|X_{A}| = 3$, then $X_{Z} = \emptyset$ and $\delta(X) = 2$.

If $|X_{A}| = 2$, then $|X_{B}| = 0$. If $|X_{A}| = 1$, then $|X_{B}| = 0$ and $\delta(X) = 2 \cdot 2 - 1 = 3 = f(4) = f_{1}(2)$.

By symmetry, we see that $X \in K_{f_{1}}$.

(2): Let $x_{0}A \subset X \subset G$. We show that $\delta(X/x_{0}A) > 0$. We may assume $X < G$. By $\dagger$ we have

$$\delta(X/x_{0}A) = \delta(x_{0}X_{A}X_{B}/x_{0}A) = \delta(X_{B}/x_{0}A) = \delta(X/x_{0}) - 2|X_{B}| > 0.$$

(3): It is clear that $cl_{G}(x_{0}) = x_{0}, cl_{G}(x_{0}A) = x_{0}A, cl_{G}(x_{0}B) = x_{0}B$, and $\delta(A/Bx_{0}) = \delta(A/x_{0})$. We also have $x_{0}AB \leq cl_{G}(x_{0}AB) = G$, because $\delta(Z'/x_{0}AB) = \sum_{z \in Z} \delta(z/x_{0}AB) = 0$. So, by Fact 4.2, we are done.

Notation 6.5. Suppose that $C_{n} = \{c_{i} : 0 \leq i \leq n\}$ and $C_{n} \cap A_{n}B_{n} = \emptyset$.

Let $E_{n}$ be the free amalgam of $G(A_{n}, B_{n}, x_{0}), G(B_{n}, C_{n}, x_{0})$, and $G(C_{n}, A_{n}, x_{0})$, i.e.

Edges = edges of $G(A_{n}, B_{n}, x_{0}), G(B_{n}, C_{n}, x_{0})$, and $G(C_{n}, A_{n}, x_{0})$, only.

In particular, we have $G(A_{n}, B_{n}, x_{0})G(B_{n}, C_{n}, x_{0}) = G(A_{n}, B_{n}, x_{0}) \otimes_{B_{n}x_{0}} G(B_{n}, C_{n}, x_{0}), G(B_{n}, C_{n}, x_{0})G(C_{n}, A_{n}, x_{0}) = G(B_{n}, C_{n}, x_{0}) \otimes_{C_{n}x_{0}} G(C_{n}, A_{n}, x_{0})$ and $G(C_{n}, A_{n}, x_{0})G(A_{n}, B_{n}, x_{0}) = G(C_{n}, A_{n}, x_{0}) \otimes_{A_{n}x_{0}} G(A_{n}, B_{n}, x_{0})$.

Lemma 6.6. Suppose that $A, B, C \in K_{f_{1}}, |A|, |B|, |C| \leq 4$. Suppose that $A \cap B < A, B, A \cap C < A, C$ and $B \cap C < B, C$, and $AB = A \otimes_{A \cap B} B$, $AC = A \otimes_{A \cap C} C$, $BC = B \otimes_{B \cap C} C$. Put $X = A \cap B \cap C, Z = A \setminus (B \cup C), W = B \setminus (A \cup C), U = C \setminus (A \cup B)$.

Suppose that $D = ABC \not\in K_{f_{1}}$.

Then $D$ is isomorphic to $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}a
\begin{array}{c}b
\begin{array}{c}c
\begin{array}{c}x
\begin{array}{c}z
\begin{array}{c}w
\begin{array}{c}u
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}$, where $a \in A \cap C, b \in A \cap B, c \in B \cap C, x \in X, z \in Z, w \in W, u \in U$. 


Proof. See Appendix 2. As $A \cap B < A$, if $c \in A \setminus (A \cap B)$, there is no $a, b \in A \cap B$ such that $R(a, c) \land R(b, c)$. This easy fact is important for the proof. (F1), (F2) and (F5) are also needed.

**Lemma 6.7.**

(1) $E_n \in K_{f_1}$

(2) $E_n < E_{n+1}$, so we may assume $E_n < E_{n+1} < M_{f_1}$ for any $n < \omega$.

**Proof.** (1): Let $D \subseteq E_n$ and $D_{AB} = D \cap G(A_n, B_n, x_0)$, $D_{BC} = D \cap G(B_n, C_n, x_0)$, $D_{CA} = D \cap G(C_n, A_n, x_0)$ and $D_A = D \cap x_0A_n$, $D_B = D \cap x_0B_n$, $D_C = D \cap x_0C_n$. By way of contradiction, suppose that $\delta(D) < f_1(\|D\|)$.

**Claim.** $|D_{AB}|, |D_{BC}|, |D_{CA}| \leq 4$.

Suppose that $\delta(D_{BC}), \delta(D_{CA}) \leq \delta(D_{AB}) =: d_{AB}$. By Fact 6.2, $G(A_n, B_n, x_0)G(C_n, A_n, x_0) \in K_{f_1}$. So we have $D' \neq D$. As $E_n = G(A_n, B_n, x_0)G(C_n, A_n, x_0) \otimes_{B, C_n, x_0} G(B_n, C_n, x_0)$ and $B_nC_n x_0 \leq G(B_n, C_n, x_0)$ by (3) of Lemma 6.4, we see

$$D' \leq D.$$ 

As $x_0A_n < G(C_n, A_n, x_0)$ (so $D_A < D_{CA}$) and $D' = D_{AB} \otimes_{D_A} D_{CA}$, so

$$\delta(D') \geq d_{AB} + 1.$$ 

**Subclaim 1:** $f^{-1}(d_{AB} + 1) < 3f^{-1}(d_{AB})$. 

Note that $f^{-1}(d_{AB}) \geq f^{-1}(\delta(D_{**})) \geq 2|D_{**}|$. Suppose that this subclaim does not hold, then we have

$$f^{-1}(d_{AB} + 1) \geq 3f^{-1}(d_{AB}) \geq 2(|D_{AB}| + |D_{BC}| + |D_{CA}|) \geq 2|D|.$$ 

So, we have $\delta(D) \geq \delta(D') \geq d_{AB} + 1 \geq f_1(\|D\|)$, a contradiction. This subclaim is proven.

**Subclaim 2:** $d_{AB} < f(10)$.

Otherwise, we have $f^{-1}(d_{AB}) \geq 10$. Thus, by ((F4): $f(3n) \leq f(n) + 1$), we have $3f^{-1}(d_{AB}) \leq f^{-1}(f(f^{-1}(d_{AB}) + 1) = f^{-1}(d_{AB} + 1)$, this contradicts subclaim 1. Subclaim 2 is proven.

As $\delta(D_{**}) \leq d_{AB} < f(10)$, and $D_{**} \in K_{f_1}$, we see the claim.

By this claim and Lemma 6.6, we have the following graph $\text{Graph}$, where $a \in D_A, b \in D_B, c \in D_C, z \in D_{AB} \setminus D_AD_B, w \in D_{BC} \setminus D_BD_C, u \in D_{CA} \setminus D_AD_C$. But this is impossible by definition of $E_n$.

(2): Let $V = \{z_{i+1}, w_{i+1}, u_{i+1} : 0 \leq i \leq n\}$ be the vertices of $E_{n+1} \setminus (E_n \cup \{a_{n+1}, b_{n+1}, c_{n+1}\})$. Then $E_{n+1} = E_n \cup \{a_{n+1}, b_{n+1}, c_{n+1}\} \cup V$. 

Let $X \subseteq \{a_{n+1}, b_{n+1}, c_{n+1}\} \cup V$. Then $e(X, E_n) = |X|$, so $\delta(X/E_n) = \delta(X) - |X| = |X| - e(X)$. If $X \cap V = \emptyset$ or $X \cap \{a_{n+1}, b_{n+1}, c_{n+1}\} = \emptyset$, then $e(X) = 0$. Otherwise, $e(X) = |X \cap V| < |X|$, as desired. 

**Theorem 6.8.** Th($M_{f_1}$) has SOP$_3$.

**Proof.** Let $\varphi(x_1 y_1 z_1, x_2 y_2 z_2) \equiv \bigwedge_{i=1,2} (R(x_0, x_i) \land R(x_0, y_i) \land R(x_0, z_i)) \land \exists z, w, u(R(x_1, z) \land R(z, y_2) \land R(y_1, w) \land R(w, z_2) \land R(z_1, u) \land R(u, x_2))$.

Let $a_n, b_n, c_n$ be as in $E_n (n < \omega)$, and put $d_n = a_n b_n c_n$. Then $\Lambda I_{f_1} \models \varphi(d_i, d_j)$ for $i < j < \mu j$. By way of contradiction, suppose that there exist $N \models \text{Th}(M_{f_1})$ and $d_0', d_1', d_2' \in N$ such that $N \models \varphi(d_0', d_1') \land \varphi(d_1', d_2') \land \varphi(d_2', d_0')$. Let $d_i' = a_i'b_i'c_i'$.

Now we have $d_0' \not\in N$. But any substructure of $N$ is in $K_{f_1}$, a contradiction. 

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### 7. Appendix 1 (Free AP of $K_{f_1}$)

We show Lemma 6.2, when $|X_1| \leq 6$ and

$$\frac{\delta(X_1) - \delta(X_0)}{|X_1| - |X_0|} \leq \frac{\delta(X) - \delta(X_0)}{|X| - |X_0|} \leq \frac{\delta(X_2) - \delta(X_0)}{|X_2| - |X_0|}.$$ 

By assumption and $|X| - |X_1| = |X_2| - |X_0|$, $\delta(X_2/X_0) \geq \delta(X_1/X_0) \frac{|X| - |X_1|}{|X_1| - |X_0|}$ follows.

**Remark 7.1.**

1. $\delta(X) \geq \delta(X_1) + \delta(X_1/X_0) \frac{|X| - |X_1|}{|X_1| - |X_0|}$

2. $f'(x) (\leq \frac{1}{14})$ is decreasing for $x \geq 14$ by (F2).

3. $e(X_1 \setminus X_0, X_0) \leq |X_1 \setminus X_0|$ by $X_0 < X_1$. So we have

$$\delta(X_1/X_0) \geq |X_1 \setminus X_0| - e(X_1 \setminus X_0).$$

4. $X_0, X_1, X_2$ do not contain 3-cycles, since they belong to $K_{f_1}$.

**Proof.** (3): $\delta(X_1/X_0) = \delta(X_1 \setminus X_0) - e(X_1 \setminus X_0, X_0) \geq \delta(X_1 \setminus X_0) - |X_1 \setminus X_0| = |X_1 \setminus X_0| - e(X_1 \setminus X_0).$ 

Now we check $\delta(X) \geq f(2|X|)$ for each case on the size of $X_1 \setminus X_0, X_0$.

Recall (F1): $f(0) = 0, f(2) = 2, f(4) = 3, f(8) = 4 < f(10) < 4\frac{1}{2} < f(12) < 5 < f(14) < 5\frac{1}{3}$. 

The case that $|X_1 \setminus X_0| = 1$

- $|X_1 \setminus X_0| = 1, |X_0| = 0$

$\delta(X) \geq 2 + 2 \frac{|X| - 1}{1} = 2|X| \geq f(2|X|)$.
(By $\delta(X_1) = \delta(X_1/X_0) = 2$ and $2x \geq f(2x)$ for $x \geq 2$)

- $|X_1 \setminus X_0| = 1, |X_0| = 1$

$\delta(X) \geq (4 - 1) + (2 - 1) \frac{|X| - 2}{1} = 1 + |X| \geq f(2|X|)$.
(By $1 + x \geq f(2x)$ and $\delta(X_1) \geq 4 - 1, \delta(X_1/X_0) \geq 2 - 1$)

- $|X_1 \setminus X_0| = 1, |X_0| = 2$

$\delta(X) \geq (6 - 2) + 1 \frac{|X| - 3}{1} = 1 + |X| \geq f(2|X|)$.
(By $\delta(X_1) \geq 6 - 2, \delta(X_1/X_0) \geq 2 - 1$ and $1 + x \geq f(2x)$)

- $|X_1 \setminus X_0| = 1, |X_0| = 3$

$\delta(X) \geq (8 - 3) + 1 \frac{|X| - 4}{1} = 1 + |X| \geq f(2|X|)$.
(By $\delta(X_1) \geq 8 - 3, \delta(X_1/X_0) \geq 2 - 1$ and $1 + x \geq f(2x)$)

- $|X_1 \setminus X_0| = 1, |X_0| = 4$

$\delta(X) \geq (10 - 5) + 1 \frac{|X| - 5}{1} = 1 + |X| \geq f(2|X|)$.
(By $\delta(X_1) \geq 10 - 5, \delta(X_1/X_0) \geq 2 - 1$ and $x \geq f(2x)$ if $x \geq 6$)

- $|X_1 \setminus X_0| = 1, |X_0| = 5$

$\delta(X) \geq (12 - 6) + 1 \frac{|X| - 6}{1} = |X| \geq f(2|X|)$.
(By $\delta(X_1) \geq 12 - 6, \delta(X_1/X_0) \geq 2 - 1$ and $x \geq f(2x)$ if $x \geq 6$)

The case that $|X_1 \setminus X_0| = 2$

- $|X_1 \setminus X_0| = 2, |X_0| = 0$

$\delta(X) \geq 3 + 3 \frac{|X| - 1}{1} \geq f(2|X|)$.
(By $\delta(X_1) = \delta(X_1/X_0) \geq 3$ and $3x + 2 \geq f(2x)$.)

- $|X_1 \setminus X_0| = 2, |X_0| = 1$

$\delta(X) \geq 4 + 2 \frac{|X| - 3}{2} \geq f(2|X|)$.
(By $\delta(X_1) \geq 6 - 2, \delta(X_1/X_0) \geq 3 - 1$ and $x + 1 \geq f(2x)$.)

- $|X_1 \setminus X_0| = 2, |X_0| = 2$

$\delta(X) \geq 4 + 1 \frac{|X| - 4}{2} \geq f(2|X|)$.
(As $\delta(X_1) \geq 8 - 4, \delta(X_1/X_0) \geq 3 - 2$ and $4 + \frac{x - 4}{2} \geq f(2x)$ if $x \geq 5$.)
\[ |X_1 \setminus X_0| = 2, |X_0| = 3 \]
\[ \delta(X) \geq 5 + 1 \frac{|X| - 5}{2} \geq f(2|X|). \]
(As \( \delta(X_1) \geq 10 - 5, \delta(X_1/X_0) \geq 3 - 2 \) and \( 5 + \frac{x - 5}{2} \geq f(2x) \) if \( x \geq 6 \).

\[ |X_1 \setminus X_0| = 2, |X_0| = 4 \]
\[ \delta(X) \geq 5 + 1 \frac{|X| - 6}{2} \geq f(2|X|). \]
(As \( \delta(X_1) \geq 12 - 7, \delta(X_1/X_0) \geq 3 - 2 \) and \( 5 + \frac{x - 6}{2} \geq f(2x) \) if \( x \geq 7 \).

The case that \( |X_1 \setminus X_0| = 3 \)
\[ |X_1 \setminus X_0| = 3, |X_0| = 0 \]
\[ \delta(X) \geq 5 + 3 \frac{|X| - 4}{3} \geq f(2|X|). \]
(As \( \delta(X_1) = \delta(X_1/X_0) \geq 6 - 2 \) and \( 5 + 3 \frac{x - 4}{3} \geq f(2x) \) if \( x \geq 6 \).

\[ |X_1 \setminus X_0| = 3, |X_0| = 1 \]
\[ \delta(X) \geq 5 + 3 \frac{|X| - 4}{3} \geq f(2|X|). \]
(As \( \delta(X_1) \geq 8 - 3, \delta(X_1/X_0) \geq 4 - 1 \) and \( 5 + 3 \frac{x - 4}{3} \geq f(2x) \) if \( x \geq 6 \).

\[ |X_1 \setminus X_0| = 3, |X_0| = 2 \]
\[ \delta(X) \geq 5 + 2 \frac{|X| - 5}{3} \geq f(2|X|). \]
(As \( \delta(X_1) \geq 10 - 5, \delta(X_1/X_0) \geq 4 - 2 \) and \( 5 + 2 \frac{x - 5}{3} \geq f(2x) \) if \( x \geq 6 \).

\[ |X_1 \setminus X_0| = 3, |X_0| = 3 \]
\[ \tilde{\delta}(X) \geq 5 + 1 \frac{|X| - 6}{3} \geq f(2|X|). \]
(As \( \delta(X_1) \geq 12 - 7, \delta(X_1/X_0) \geq 4 - 3 \) and \( 5 + 1 \frac{x - 6}{3} \geq f(2x) \) if \( x \geq 7 \).

The case that \( |X_1 \setminus X_0| = 4 \)
\[ |X_1 \setminus X_0| = 4, |X_0| = 0 \]
\[ \delta(X) \geq 4 + 4 \frac{|X| - 4}{4} \geq f(2|X|). \]
(As \( \delta(X_1) = \delta(X_1/X_0) \geq 8 - 4 \) and \( 4 + 4 \frac{x - 4}{4} \geq f(2x) \) if \( x \geq 5 \).

\[ |X_1 \setminus X_0| = 4, |X_0| = 1 \]
\[\delta(X) \geq 5 + 3 \frac{|X| - 5}{4} \geq f(2|X|).\]

(As \(\delta(X_1) \geq 10 - 5, \delta(X_1/X_0) \geq 4 - 1\) and \(5 + 3 \frac{x - 5}{4} \geq f(2x)\) if \(x \geq 6\).)

- \(|X_1 \setminus X_0| = 4, |X_0| = 2\)

\[\delta(X) \geq 5 + 2 \frac{|X| - 6}{4} \geq f(2|X|).\]

(As \(\delta(X_1) \geq 12 - 7, \delta(X_1/X_0) \geq 4 - 2\) and \(5 + 3 \frac{x - 6}{2} \geq f(2x)\) if \(x \geq 7\).)

**The case that** \(|X_1 \setminus X_0| = 5\)

- \(|X_1 \setminus X_0| = 5, |X_0| = 0\)

\[\delta(X) \geq 5 + 5 \frac{|X| - 5}{5} = |X| \geq f(2|X|).\]

(As \(\delta(X_1) = \delta(X_1/X_0) \geq 10 - 5\) and \(x \geq f(2x)\) if \(x \geq 6\).)

- \(|X_1 \setminus X_0| = 5, |X_0| = 1\)

\[\delta(X) \geq 5 + 3 \frac{|X| - 6}{5} \geq f(2|X|).\]

(As \(\delta(X_1) \geq 12 - 7, \delta(X_1/X_0) \geq 5 - 2\) and \(5 + 3 \frac{x - 6}{5} \geq f(2x)\) if \(x \geq 7\).)

**The case that** \(|X_1 \setminus X_0| = 6\)

- \(|X_1 \setminus X_0| = 6, |X_0| = 0\)

\[\delta(X) \geq f(12) + f(12) \frac{|X| - 6}{6} = f(12)|X| \geq f(2|X|).\]

(As \(\delta(X_1) = \delta(X_1/X_0) \geq f(12)\) and \(f(12)|X| > 4|X| \geq f(2x)\) if \(x \geq 7\).) \(\Box\)

8. **Appendix 2 (The proof of Lemma 6.6)**

We show the following.

**Lemma 6.6** Suppose that \(A, B, C \in K_{f_1}, |A|, |B|, |C| \leq 4\). And suppose that \(A \cap B < A, B, A \cap C < A, C, B \cap C < B, C,\) and \(AB = A \otimes_{A \cap B} B, AC = A \otimes_{A \cap C} C, BC = B \otimes_{B \cap C} C\). Put \(X = A \cap B \cap C, Z = A \setminus (B \cup C), W = B \setminus (A \cup C), U = C \setminus (A \cup B)\).

If \(D = ABC \notin K_{f_1}\), then \(D\) is isomorphic to \(\begin{array}{c}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array}
\end{array}\), where \(a \in A \cap C, b \in A \cap B, c \in B \cap C, x \in X, z \in Z, w \in W, u \in U\).

**Proof.** We use the following easy fact: If \(X < Y, c \in Y \setminus X, a, b \in X\), then \(R(a, c) \land R(b, c)\) does not hold.
Clearly, $D = BCZ$.
We may assume that $Z, W, U \neq \emptyset$, since, for example, if $Z = \emptyset$, then $D = B \otimes_{B \cap C} C \in \mathbf{K}_{f_{1}}$ by free AP. As $|A|, |B|, |C| \leq 4$, we have $|A \cap C| \leq 3$.

$a, a'$ denote elements of $A \cap C$, $b, b'$ denote elements of $A \cap B$, $c, c'$ denote elements of $B \cap C$, $z, z'$ denote elements of $Z$, $w, w'$ denote elements of $W$, $u, u'$ denote elements of $U$ and $x, x'$ denote elements of $X$.

We check each case on the size of $|A \cap C|$.

The case that $|A \cap C| = 3$

We have $6 \leq |D| \leq 9$. As $|A| \leq 4$, $|Z| = 1$ and $A \cap B \setminus X = \emptyset$ follow. So, we have $\delta(Z/BC) \geq 1$. Thus $\delta(D) = \delta(BC) + \delta(Z/BC) \geq f(2|D| - 2) + 1 \geq f(2|D|)$.

The case that $|A \cap C| = 2$

- $|(A \cap C) \setminus X| = 2$ (i.e. $X = \emptyset$.)
Suppose that $|Z| = 2$. So, $6 \leq |D| \leq 10$.
As $A \cap B = \emptyset$, $\delta(Z/BC) \geq 3 - 2$ follows. So, $\delta(D) \geq \delta(BC) + 1 \geq f(2|D| - 4) + 1 \geq f(2|D|)$ by (F5), $f(8) + 1 = 5 \geq f(12), f(14) + 1 \geq 6 \geq f(18)$ and (F2).

Suppose that $|Z| = 1$, so $|A \cap B| \leq 1$.

If $A \cap B = \emptyset$, then $5 \leq |D| \leq 9$, $\delta(Z/BC) \geq 2 - 1$ follows. So, $\delta(D) \geq \delta(BC) + 1 \geq f(2|D| - 2) + 1 \geq f(2|D|)$.

If $|A \cap B| = 1$, then $6 \leq |D| \leq 9$.
If $|D| = 6$, then $D = aa'zbwv$. Then $\delta(D) = 12 - 5 = 7 \geq f(12).$
If $|D| = 7$, then $D = aa'zbwv, aa'zbwv'c, aa'zbwv'cu$, because $Z, W, U \neq \emptyset$. Then $\delta(D) \geq 14 - 7 = 7 \geq f(14).$
If $|D| = 8$, then $D = aa'zbwv'w'u'$ or $aa'zbwv'cu$, because $Z, W, U \neq \emptyset$. Then $\delta(D) \geq 16 - 9 = 7 \geq f(16).$
If $|D| = 9$, then $D = aa'zbwv'w'u'$, because $Z, W, U \neq \emptyset$. Then $\delta(D) \geq 18 - 9 = 9 \geq f(18).$

- $|(A \cap C) \setminus X| = |X| = 1$.
Suppose that $|A \cap B \setminus X| = 0$. Then $\delta(Z/BC) \geq 1$.
So, $\delta(D) \geq f(2|D| - 2|Z|)$ + 1.
If $|Z| = 1$, then $5 \leq |D| \leq 8$, so $f(2|D| - 2) + 1 \geq f(2|D|)$ holds.
If $|Z| = 2$, then $6 \leq |D| \leq 9$. $f(2|D| - 4) + 1 \geq f(2|D|)$ holds for $|D| = 6, 9$.
$f(8) + 1 = 5 \geq f(12)$ and $f(14) + 1 \geq 6 \geq f(18)$ holds for $|D| = 7, D = xazz'wcu,$
$xazz'wu'$ or $xazz'wu''u'$ and then $\delta(D) \geq 14 - 8 \geq f(14)$ holds. For $|D| = 8,$
$D = xazz'wu'cu$ or $xazz'wu'uu'$ and then $\delta(D) \geq 16 - 10 \geq f(16)$ holds.

Suppose that $|A \cap B \setminus X| = 1$. Then $6 \leq |D| \leq 8$.
If $|D| = 6$, then $D = xazbwu$ and $\delta(D) \geq 12 - 6 \geq f(12)$.
If $|D| = 7$, then $D = xazbwu'$, $xazbwu''u$ or $xazbwcu$. If the former two
cases hold, then $\delta(D) \geq 14 - 8 \geq f(14)$.

In the latter case, $D$ is $\xrightarrow{\text{or}}$ if and only if $\delta(D) = 14 - 9 < f(14)$.
If $|D| = 8$, then $D = xazbwu'uu'$ and $\delta(D) \geq 16 - 10 \geq f(16)$.

\begin{itemize}
  \item $|(A \cap C) \setminus X| = 0, |X| = 2$
\end{itemize}

We have $5 \leq |D| \leq 8$.
If $|D| = 5$, then $D = xx'zwu$ and $\delta(D) \geq 10 - 4 \geq f(10)$.
If $|D| = 6$, then $D = xx'zz'u, xx'zbwu, xx'zwu'u, xx'zwu'c$ or $xx'zwu''$ and
$\delta(D) \geq 12 - 7 \geq f(12)$.
If $|D| = 7$, then $D = xx'zz'w'u, xx'zbwu', xx'zwu'u', xx'zz'wu', xx'zz'wuc, xx'zz'wuu',
xx'zbwu'$ or $xx'zwu'u'$. If $D \neq xx'zz'wuc, xx'zbwu'$, then $\delta(D) \geq 14 - 8 \geq f(14)$. And we have
$\delta(D) = 14 - 9 < f(14)$ if and only if $D$ is $\xrightarrow{\text{or}}$ or $\xrightarrow{\text{or}}$. But this
never happens, because $B \cap C < B$ and $A \cap B < B$, so $w$ does not have two
edges to $B \cap C$, also to $A \cap B$.
If $|D| = 8$, then $D = xx'zz'w'u'u'$ and $\delta(D) = 16 - 10 \geq f(16)$.

The case that $|A \cap C| = 1$

\begin{itemize}
  \item $|(A \cap C) \setminus X| = 1$ ($|X| = 0$)
\end{itemize}

By symmetry, we may assume $|A \cap B|, |B \cap C| \leq 1$.

Suppose that $|A \cap B|, |B \cap C| = 1$. Then $6 \leq |D| \leq 9$.
If $|D| = 6$, $\delta(D) \geq 12 - 6 \geq f(12)$. If $|D| = 7$, $\delta(D) \geq 14 - 7 \geq f(14)$. If
$|D| = 8$, $\delta(D) \geq 16 - 8 \geq f(16)$. If $|D| = 9$, $\delta(D) \geq 18 - 9 \geq f(18)$. 


\begin{itemize}
  \item $|(A \cap C) \setminus X| = 0, |X| = 2$
\end{itemize}

We have $5 \leq |D| \leq 8$.
If $|D| = 5$, then $D = xx'zwu$ and $\delta(D) \geq 10 - 4 \geq f(10)$.
If $|D| = 6$, then $D = xx'zz'u, xx'zbwu, xx'zwu'u, xx'zwu'c$ or $xx'zwu''$ and
$\delta(D) \geq 12 - 7 \geq f(12)$.
If $|D| = 7$, then $D = xx'zz'w'u, xx'zbwu', xx'zwu'u', xx'zz'wu', xx'zz'wuc, xx'zz'wuu',
xx'zbwu'$ or $xx'zwu'u'$. If $D \neq xx'zz'wuc, xx'zbwu'$, then $\delta(D) \geq 14 - 8 \geq f(14)$. And we have
$\delta(D) = 14 - 9 < f(14)$ if and only if $D$ is $\xrightarrow{\text{or}}$ or $\xrightarrow{\text{or}}$. But this
never happens, because $B \cap C < B$ and $A \cap B < B$, so $w$ does not have two
edges to $B \cap C$, also to $A \cap B$.
If $|D| = 8$, then $D = xx'zz'w'u'u'$ and $\delta(D) = 16 - 10 \geq f(16)$.

The case that $|A \cap C| = 1$

\begin{itemize}
  \item $|(A \cap C) \setminus X| = 1$ ($|X| = 0$)
\end{itemize}

By symmetry, we may assume $|A \cap B|, |B \cap C| \leq 1$.

Suppose that $|A \cap B|, |B \cap C| = 1$. Then $6 \leq |D| \leq 9$.
If $|D| = 6$, $\delta(D) \geq 12 - 6 \geq f(12)$. If $|D| = 7$, $\delta(D) \geq 14 - 7 \geq f(14)$. If
$|D| = 8$, $\delta(D) \geq 16 - 8 \geq f(16)$. If $|D| = 9$, $\delta(D) \geq 18 - 9 \geq f(18)$.
Suppose that $|A \cap B| = 0$ or $|B \cap C| = 1$. By symmetry, we assume that $|A \cap B| = 0$. Then $AC \cap B = B \cap C$. By assumption on $A, B, C$, $B \cap C < C < AC$ and $AC = A \otimes_{AC} C \in K_{f_1}$ by free AP. As $B \cap C < AC, B$, and $D = AC \otimes_{AC} B$, we have $D \in K_{f_1}$ by free AP.

- $(A \cap C) \setminus X = 0$ and $|X| = 1$.
As we have shown that $|A \cap C| = 2, 3$, by symmetry, we may assume that $D = XZWU$. (i.e. $|(A \cap B) \setminus X| = 0$ and $|(B \cap C) \setminus X| = 0$) As $X < XZW = XZ \otimes_X XW \in K_{f_1}$ and $X < XU \in K_{f_1}$, we have $D = XZW \otimes_X XZ \in K_{f_1}$ by free AP.

The case that $|A \cap C| = 0$

As we have shown that $|A \cap C| = 1, 2, 3$, by symmetry, we may assume that $D = ZWU$. (i.e. $|A \cap B| = 0$ and $|B \cap C| = 0$.) By free AP, we see $D \in K_{f_1}$.  

References

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