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Generic 構造の安定性について II
(The Stability Spectrum of Generic Graphs)

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Abstract

Generic 構造でその理論が superstable であるが ω-stable でないものがあるか。という問題がある (Baldwin の問題)。この問題に関して次の部分的結果が得られた。定理: K が部分グラフに関して閉じているとき、K-generic グラフの理論は strictly stable かあるいは ω-stable になる。

1 Introduction

Our notation and definition are similar to those of [8] in this volume. We do not explain all of those details here. For further details, see [2, 3, 4, 5, 6, 9]. Our aim is to give a partial solution for the following problem.

Problem 1 (Baldwin [1]) Is there a generic structure that is superstable but not ω-stable?

This problem is still open. In fact, all of known stable generic structures is strictly stable or ω-stable. In this paper, we will consider the problem under the following assumption.

Assumption 2 K is a subclass of $K_\alpha$ that is closed under substructures. $M$ is a saturated K-generic graph. (It follows that K has the amalgamation property.)

For the stability of K-generic graphs, the following fact is well-known.

*Research partially supported by Grants-in-Aid for Scientific Research (no.16540123), Ministry of Education, Science and Culture.
Fact 3 ([3], [9]) Let \( T = \text{Th}(M) \). Then
(i) \( T \) is stable;
(ii) If \( \alpha \) is rational, then \( T \) is \( \omega \)-stable.

2 Triviality of \( K \)

For each \( n \in \omega \), \( I_n \) denotes a graph of size \( n \) with no relations.

Lemma 4 \( I_n \in K \) for each \( n \in \omega \).

Proof If \( m \leq n \) and \( I_n \in K \), then \( I_m \in K \) since \( K \) is closed under substructures. So we can assume \( n \geq 3 \). Then we can take \( k \in \omega \) such that \( \left\{ k \cdot \binom{k-2}{n-2}/\binom{k}{n} \right\} \leq \alpha \). Take any \( A \in K \) of size \( k \). Then it is enough to see that \( A \) contains a copy of \( I_n \). This can be shown as follows: If not, then \( r(A) \leq \left( \binom{k}{n}/\binom{k-2}{n-2} \right) \). So we have \( \delta(A) \leq k - \left\{ \left( \binom{k}{n}/\binom{k-2}{n-2} \right) \right\} \alpha < 0 \). This is a contradiction.

Lemma 5 If \( Ab \) is a finite graph with \( A \in K \) and \( r(A,b) = 0 \), then \( Ab \in K \).

Proof Let \( |A| = k \). Take \( n \in \omega \) with \( n > k/\alpha \). By lemma 4, we have \( I_n \in K \). So we can assume that \( I_n A (\in K) \) is an amalgamation of \( I_n \) and \( A \) over \( \emptyset \). It is enough to show that there is \( b' \in I_n - A \) with \( b' \cong_A b \). This can be shown as follows: If \( A \subset I_n \), then we easily get \( b' \in I_n - A \) with \( b' \cong_A b \). So we can assume \( A \subset I_n \). If not, then \( r(A - I_n, I_n) \geq n \). Then \( \delta(A/I_n) = \delta(A - I_n) - \alpha \cdot r(A - I_n, I_n) \leq k - \alpha \cdot n < 0 \). A contradiction.

Definition 6 \( K \) is said to be trivial, if there is some \( n \in \omega \) such that if \( A \in K \) is connected then \( |A| \leq n \).

Lemma 7 If \( K \) is trivial, then \( \text{Th}(M) \) is \( \omega \)-stable.

Proof Take any countable \( A \leq M \) and \( \bar{b} \in M \). To show that \( \text{Th}(M) \) is \( \omega \)-stable, it is enough to see that \( S(A) \) is countable. Since \( K \) is trivial, there is finite \( A_0 \leq A \) with \( d(\bar{b}/A) = d(\bar{b}/A_0) \). Let \( B = \text{cl}(\bar{b}A_0) \) and \( A_1 = B \cap A \). Note that \( A_1 \leq M \).

Claim: \( \text{tp}(\bar{b}/A_1) \vdash \text{tp}(\bar{b}/A) \).

Proof: Take any \( \bar{c} \in M \) such that \( \text{tp}(\bar{c}/A_1) = \text{tp}(\bar{b}/A_1) \) and \( d(\bar{c}/A_1) = d(\bar{c}/A) \). Let \( C = \text{cl}(\bar{c}A_1) \). Then we have \( B \cong_{A_1} C \). From proposition 13 it follows that \( B \) and \( A \) are free over \( A_1 \) and \( BA \leq M \), and that \( C \) and \( A \) are free over \( A_1 \) and...
$CA \leq M$. In particular we have $B \cong_A C$, and so $\text{tp}(B/A) = \text{tp}(C/A)$. Hence $\text{tp}(\hat{b}/A) = \text{tp}(\hat{c}/A)$. (End of Proof of Claim)

Since there is a countable saturated model, $\text{Th}(M)$ is small, and hence the number of the types over a given finite set is countable. It follows that $|S(A)| \leq \aleph_0 \cdot \aleph_0 = \aleph_0$.

For a finite graph $A$ and $e \in A$, we denote $\deg_A(e) = \max\{|B| : \forall b \in B, R(e,b)\}$.

**Lemma 8** Suppose that $K$ is non-trivial. Then for any $n \in \omega$ the following condition $(*)_n$ holds:

$(*)_n$ There is $A \in K$ and $a \in A$ with $\deg_A(a) \geq n$.

**Proof** For each $m \in \omega$, let $L_m$ denote a finite graph $a_0 a_1 \ldots a_m$ with the relations $R(a_0, a_1), R(a_1, a_2), \ldots, R(a_{m-1}, a_m)$. We divide into two cases.

Case 1: $L_m \not\in K$ for some $m \in \omega$.

Since $K$ is non-trivial, $(*)_n$ clearly holds for each $n \in \omega$. (mousukosi seikaku nii!)

Case 2: $L_m \in K$ for any $m \in \omega$.

We prove by induction. By induction hypothesis, we assume that $\deg_A(a) \geq n$ for some $A \in K$.

Subcase 2.1: $\alpha \leq \frac{1}{2}$.

Let $acd$ be a graph with the relations $R(a,c)$ and $R(c,d)$. Since $L_2 \in K$ and $\alpha < \frac{1}{2}$, we have $ad \leq acd \in K$. On the other hand, we can assume $r(A,d) = 0$. By lemma 5, we have $ad \leq Ad \in K$. So we can assume that $cdA(\in K)$ is an amalgamation of $acd$ and $Ad$ over $ad$. Note that $c \not\in A$ since $r(A,c) = 0$. Hence $\deg_{Ac}(a) \geq n + 1$.

Subcase 2.2: $\alpha > \frac{1}{2}$.

Let $m = \min\{k : (k - 1) - k\alpha > 0\}$. Note that $m \geq 3$ since $\alpha > \frac{1}{2}$. Then we have $L_m = a_0 a_1 \ldots a_m \in K$. We can assume $a = a_0$ and $r(A, a_m) = 0$. By lemma 5, we have $aa_m \leq Aa_m \in K$. By the definition of $m$, we have $aa_m \leq L_m \in K$. So we can assume that $AL_m(\in K)$ is an amalgamation of $Aa_m$ and $L_m$ over $aa_m$. Then we see $a_1 \not\in A$. (Proof: If not, then we have $aa_1 a_m \leq L_m$, and so $\delta(L_m/aa_1 a_m) = (m - 2) - (m - 1)\alpha > 0$. This contradicts the definition of $m$.) Therefore $\deg_{Aa_1}(a) \geq n + 1$.

For each $n \in \omega$, $S_n$ denote a finite graph $aa_1 a_2 \ldots a_n$ with the relations $R(a, a_i)$ for every $i = 1, 2, \ldots, n$.

**Lemma 9** Suppose that $K$ is non-trivial. Then $S_n \in K$ for each $n \in \omega$. 

Theorem of K-generic to If we take with we have can then have we on enough graph. Therefore graph such we have Take distinct that is have properties: hypothesis, then follows: that graph with have other shown lemma by any We finite following we there infinite there prove any induction. Then no can then will assume have the any contradiction. We assume there contradiction. We show that If be contradiction.) for required. By To (If then u-stable. We take not, cycles, cycles. is proved irrational, cycles; is no. Infinite cycles, we have that of K. Take 9, then we can easily pick b' as required. Then A ≠ S_n, we can take b' ∈ S_n - A with r(b', A) = r(b', a) = 1. (If not, then we have δ(A/S_n) = δ(A - S_n) - r(A - S_n, S_n) · α ≤ k - (n - k)α < 0. On the other hand, we have S_n ≤ AS_n since AS_n is an amalgamation. A contradiction.)

Proposition 11 Suppose that K is non-trivial. Then any finite graph with no cycles belongs to K.

Proof Let B be a finite graph with no cycles. We will prove by induction on |B|. Since B has no cycles, we can take b ∈ B such that there are no distinct c, d ∈ B with R(c, b) ∧ R(d, b). Let A = B - {b}. By induction hypothesis, we have A ∈ K. If r(b, A) = 0, then we have B = Ab ∈ K by lemma 10. Therefore we assume r(b, A) = 1. Since A has no cycles, we have a ≤ A ∈ K. Take n ∈ ω with n > |A|/α. By lemma 9, S_n ∈ K. We can assume that a ∈ S_n with R(a, c) for any c ∈ S_n - {a}. Hence we have a ≤ S_n ∈ K.

3 Proof of Theorem

The following proposition was proved in [7] to show that there is no K-generic pseudoplane that is stable but not ω-stable.

Proposition 12 ([7]) If α is irrational, then there is an infinite graph D with an element e and finite subgraphs B_1, B_2, · · · with the following properties:
(1) D = cl_D(eB_1B_2 · · ·) has no cycles;
(2) d_D(e/B_1) > d_D(e/B_2) > · · ·;
(3) B_1 ≤ B_2 ≤ · · · ≤ D.
In [8], we studied algebraic types of $K$-generic graphs, where $K$ is closed under subgraphs. As a corollary, we have the following proposition.

**Proposition 13 ([8])** Assume that $K$ is closed under subgraphs. Let $A, B, C$ be finite such that $B, C \leq M$ and $A = B \cap C$. Then the following are equivalent.
1. $d(B/A) = d(B/C)$;
2. $B$ and $C$ are free over $A$, and $BC \leq M$;
3. $tp(B/C)$ does not fork over $A$.

Using proposition 12 and 13, we obtain the following theorem.

**Theorem** Let $K$ be a subclass of $K_\alpha$ that is closed under subgraph and $M$ a saturated $K$-generic graph. Then $Th(M)$ is strictly stable or $\omega$-stable.

**Proof of Theorem** If $K$ is trivial or $\alpha$ is rational, then, by fact 3 and lemma 7, $Th(M)$ is $\omega$-stable. Thus we can assume that $K$ is non-trivial and $\alpha$ is irrational. By fact 3 again, $Th(M)$ is stable. So we have to show that $Th(M)$ is not superstable: Since $\alpha$ is irrational, by proposition 12, there is an infinite graph $D$ with an element $e$ and finite subgraphs $B_1, B_2, \ldots$ such that (i) $D = cl_D(eB_1B_2\cdots)$ has no cycles; (ii) $d_D(e/B_1) > d_D(e/B_2) > \cdots$; (iii) $B_1 \leq B_2 \leq \cdots \leq D$. Since $K$ is non-trivial and $D$ has no cycles, by proposition 11, any finite subset of $D$ belongs to $K$, and so we can assume $D \leq M$. It follows that $d_M(e/B_1) > d_M(e/B_2) > \cdots$. By proposition 13, we have $tp(e/B_1) \subset_f tp(e/B_2) \subset_f \cdots$. Hence $Th(M)$ is not superstable.

**Reference**


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