数理解析研究所講究録 1450 巻 2005 年 75-82

On Predimensions of Finite Structures

神戸大学・工　桔梗宏孝 (Hirotaka Kikyo)
Faculty of Engineering, Kobe University
kikyo@kobe-u.ac.jp

September 7, 2005

Abstract

We give some families of finite structures with a predimension function, in which we can construct various structures with desired predimensions.

1 Introduction

N. Peatfield and B. Zilber investigated a generic structure of certain class of finite structures considered by Hrushovski and they showed that with some modification of the structure, they can put a topology called an analytic Zariski structure on it. One of their main tools is the following.

For the class $\mathbf{K}$ they considered, a structure M is elemetarily equivalent to a generic structure of $\mathbf{K}$ if and only if M satisfies Axioms 1 and 2:

Axiom 1. Any finite substructure A of M belongs to $\mathbf{K}$.

Axiom 2. For any $A, B \in \mathbf{K}$ such that $A \leq B$, if $f : A \to M$ is an $\mathcal{L}$-embedding then it can be extended to an $\mathcal{L}$-embedding $f' : B \to M$.

Axiom 2 represents a strong form of the amalgamation property. For any class $\mathbf{K}$ of finite structures, these axioms may have some meaning. This will be investigated in a joint work with K. Ikeda and A. Tsuboi [2]. This paper will be a part of this work.

2 Preliminaries

Throughout this paper, $\mathcal{L}$ is a finite relational language. If M is an $\mathcal{L}$-structure and R is a relation with n arguments then $R(M) = \{x \in M^n : M \models R(x)\}$.

Definition 2.1 Suppose $\mathcal{L} = \{R_1, R_2, \ldots, R_l\}$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ where α_i are positive real numbers at most 1.

For a finite $\mathcal{L}$-structure A, let

$$\delta(A) = |A| - |R_1(A)| - |R_2(A)| - \cdots - |R_l(A)|.$$

δ is called a *predimension function.*

Let B be an $\mathcal{L}$-structure and $A \subseteq B$. We write $A \leq B$ if for any subset X of B, whenever $A \subseteq X \subseteq B$ then $\delta(A) \leq \delta(X)$.

Let $\mathbf{K}_\alpha = \{A : \delta(X) \geq 0 \text{ for any } X \subseteq A\}$.

Definition 2.2 (Asymmetric amalgam) Let B and C be $\mathcal{L}$-structures, and let $A = B \cap C$ as a set. $B \oplus_A C$ is a structure with universe $B \cup C$ and for each relation R in $\mathcal{L}$, $R(B \oplus_A C) = R(B) \cup R(C) - (R(C) \cap A^n)$ where n is the number of arguments of R.

Note that $\delta(B \oplus_A C) = \delta(B) + \delta(C) - \delta(C|_A)$ where $C|_A$ is a substructure of C with the universe A.

Furthermore, if $B \cap C = \{a\}$ (a singleton), then we write $B \oplus_a C$ for $B \oplus_{\{a\}} C$, and if $B \cap C = \emptyset$, then we write $B \oplus C$ for $B \oplus_\emptyset C$.

Definition 2.3 Let A be an $\mathcal{L}$-structure, a, b elements in A, and suppose that for any relation $R \in \mathcal{L}$, there is no tuple x of elements in A such that $A \models R(x)$ and x contains a and b.

$A/(a = b)$ is an $\mathcal{L}$-structure obtained from A by identifying a and b. More precisely, it is defined as follows: The universe, also denoted $A/(a = b)$, is $A - \{b\}$. Let $f : A \to A/(a = b)$ be a projection map defined by $f(x) = x$ for $x \neq b$, and $f(b) = a$. For any relation $R \in \mathcal{L}$, $A/(a = b) \models R(c_1, \ldots, c_n)$ for $c_1, \ldots, c_n \in A/(a = b)$ if and only if $A \models R(c'_1, \ldots, c'_n)$ for some $c'_1, \ldots, c'_n \in A$ such that $f(c'_i) = c_i$ for $i = 1, \ldots, n$.

3 Arithmetic of Structures

In this section, we work in $\mathbf{K}_\alpha$ for some tuple α of positive real numbers at most 1. We show that there are a plenty of structures in $\mathbf{K}_\alpha$ to get structures in $\mathbf{K}_\alpha$ of desired δ-ranks.

Definition 3.1 Let s be a real number such that $0 \leq s \leq 2$. A triple (E, a, b) is an *s-component* if E is an $\mathcal{L}$-structure, $a, b \in E$, $a \neq b$, $\delta(\{a\}) = \delta(\{b\}) = 1$, and the following hold:

For any non-empty substructure X of E,

(1) $s \leq \delta(X)$ if $a, b \in X$,

(2) $1 \leq \delta(X)$ if $a \notin X$ or $b \notin X$, and

(3) $\delta(E) = s$.

We also say that E is an s-component with joints a and b, or just E is an s-component.

Lemma 3.2 *Let s, t, u be real numbers such that* $0 \leq s, t, u \leq 1$.

(1) *Suppose* (A, a, b) *is an* $(1+s)$*-component,* (B, b, c) *a* $(1+t)$*-component,* $s+t \leq 1$*, and* $C = A \oplus_b B$*. Then* (C, a, c) *is an* $(1+s+t)$*-component.*

(2) *Suppose* (A, a, b) *is an* $(1+s)$*-component,* (B, b, c) *a* $(1+t)$*-component,* (C, c, d) *a* $(1+u)$*-component,* $s+t \leq 1$*,* $D = A \oplus_b B \oplus_c C$*, and* $\overline{D} = D/(a = d)$*. Then* $(\overline{D}, c, a)$ *is an* $(s+t+u)$*-component.*

(3) *Suppose* $k \geq 3$*,* r_1*,* r_2*, ...,* r_{k-1}*,* r_k *are non-negative real numbers such that* $r_1 + r_2 + \cdots + r_{k-1} \leq 1$*, and an* r_i*-component exists for each* i*. Then there is an* $(r_1 + r_2 + \cdots + r_{k-1} + r_k)$*-component.*

Proof. (1) Suppose $X \subseteq C = A \oplus_b B$ and $X \neq \emptyset$.

We check the condition (1) of the definition of an $(1+s+t)$-component first. Assume that $a, c \in X$. Consider the case $b \in X$. Then $a, b \in X \cap A$, $b, c \in X \cap B$, and $X = (X \cap A) \oplus_b (X \cap B)$. Hence,

$$\delta(X) = \delta(X \cap A) + \delta(X \cap B) \geq (1+s) + (1+t) - 1 = 1 + (s+t).$$

Now, consider the case $b \notin X$. Then $b \notin X \cap A$, $b \notin X \cap B$, and $X = (X \cap A) \oplus (X \cap B)$. Hence,

$$\delta(X) = \delta(X \cap A) + \delta(X \cap B) \geq 1 + 1 \geq 1 + (s+t).$$

Therefore, $a, c \in X$ implies $\delta(X) \geq 1 + s + t$.

Now, we check the condition (2) of the definition of an $(1+s+t)$-component. Suppose X is a non-empty subset of C. Then $\delta(X) \geq \delta(X \cap A) + \delta(X \cap B) - 1 \geq 1 + 1 - 1 = 1$.

(2) Since there is no relation on a tuple containing a and d, going from D to $\overline{D}$, the number of relations does not change. Therefore, for any $X \subseteq D$, $\delta(X/(a = d)) = \delta(X) - 1$ if $a, d \in X$, and $\delta(X/(a = d)) = \delta(X)$ if $a, d \notin X$.

Suppose $c, a \in X \subset \overline{D}$. Let $Y \subset D$ be such that $X = Y/(a = d)$. We can assume that $a, d \in Y$.

Consider the case $b \in Y$. Since $a, b, c, d \in Y$, we have $\delta(Y \cap A) \geq 1 + s$, $\delta(Y \cap B) \geq 1 + t$, $\delta(Y \cap C) \geq 1 + u$, and $Y = (Y \cap A) \oplus_b (Y \cap B) \oplus_c (Y \cap C)$. Hence, $\delta(Y) \geq 1 + (s+t+u)$. Therefore, $\delta(X) = \delta(Y/(a = d)) = \delta(Y) - 1 \geq s + t + u$.

Now, consider the case $b \notin Y$. Then, $\delta(Y \cap A) \geq 1$, $\delta(Y \cap B) \geq 1$, $\delta(Y \cap C) \geq 1+u$, and $Y = (Y \cap A) \oplus (Y \cap B) \oplus_c (Y \cap C)$. We have $\delta(Y) \geq 1 + (1+u) \geq 1 + (s+t+u)$ by the assumption that $s+t \leq 1$. Therefore, $\delta(X) = \delta(Y/(a = d)) = \delta(Y) - 1 \geq s+t+u$. We have checked the condition (1) of the definition of $(s+t+u)$-component.

We turn to check the condition (2) of the definition of $(s+t+u)$-component. Suppose $a \notin X \subset \overline{D}$ and $X \neq \emptyset$. Then $X \subset D$ and $a, d \notin X$. Let $X_1 = X \cap (A \cup B)$ and $X_2 = X \cap C$. Since $s + t \leq 1$, $A \cup B = A \oplus_b B$ is a $(1+s+t)$-component by (1), and thus $\delta(X_1) \geq 1$ by $a \notin X$. $\delta(X_2) \geq 1$ since $d \notin X$. Therefore, $\delta(X) \geq \delta(X_1) + \delta(X_2) - 1 \geq 1$.

Suppose $c \notin X$ and $a \in X$. Let $Y \subseteq D$ be such that $X = Y/(a = d)$ and $a, d \in Y$. Since $c \notin Y$, $\delta(Y) = \delta(Y \cap (A \cup B)) + \delta(Y \cap C) \geq 2$. Therefore, $\delta(X) = \delta(Y) - 1 \geq 1$.
□

Lemma 3.3 *Suppose* $0 \le s < 1/2$ *and a* $(1+s)$*-component exists. If* s *is rational then a* 1*-component exists. If* s *is irrational, then for any* $\varepsilon > 0$ *there exists* ε' *such that* $\varepsilon > \varepsilon' > 0$ *and a* $(1+\varepsilon')$*-component exists. With Lemma 3.2, we can choose a* 1*-component or a* $(1+\varepsilon')$*-component arbitrarily large.*

Proof. Suppose $0 \le s < 1/2$ and a $(1+s)$-component exists. Consider the following non-negative decreasing sequence $\{s_i\}_{i<\omega}$: Let $s_0 = s$. If $s_i = 0$ then let $s_{i+1} = 0$. If $s_i > 0$ then choose the least positive integer m_i such that $m_i s_i \ge 1$ and let $s_{i+1} = m_i s_i - 1$.

By Lemma 3.2 (2), there is a $(1+s_i)$-component for each $i < \omega$.

Note that if $s_i > 0$ then $s_i > s_{i+1} \ge 0$. If $s_0 = s$ is rational then $s_i = 0$ eventually. Hence, 1-component exists.

If $s_0 = s$ is irrational, s_i converges to 0. This should be well-known, but we give a proof for convenience. Since $\{s_i\}_{i<\omega}$ is a decreasing sequence, $\{m_i\}_{i<\omega}$ is an increasing sequence. If $\{m_i\}_{i<\omega}$ is unbounded, then s_i converges to 0 because $s_i < 1/(m_i - 1)$. If $\{m_i\}_{i<\omega}$ is bounded, $m_i = m$ for some positive integer m eventually. So, we can assume that $s_{i+1} = ms_i - 1$ for all i. Since s_i converges to some $\beta > 0$, $\beta = m\beta - 1$. Therefore, $\beta = 1/(m-1)$. But $s_i > \beta = 1/(m-1)$ contradicts $(m-1)s_i = (m_i - 1)s_i < 1$. □

Lemma 3.4 *Let* (V, E) *be a finite graph where* V *is a set of vertices and* E *a set of edges, and let* $\delta(V, E) = |V| - \beta|E|$ *with* $0 < \beta \le 1$. *Then a graph which is a* $(1+s)$*-component exists for some real number* s *such that* $0 \le s < 1/2$ *with respect to this* δ.

Proof. We remark first that any two different points in each $(1+s)$-component below will be its joints.

Suppose $\alpha_i = 1/2$. A graph with 4 points and 6 edges (K_4) is a 1-component.

Suppose $1/2 < \alpha_i \le 1$. Let $s = 1 - \alpha_i$. Then a graph with 2 points and 1 edge is a $(1+s)$-component. These 2 points will be joints.

Suppose $0 < \alpha_i < 1/2$. Choose least natural number k such that $k - \alpha \binom{k}{2} < 1$.

Note that $k \le 4$ implies $k - \alpha \binom{k}{2} > 1$. Therefore, $k \ge 5$. Consider K_k, a complete graph with k points. Then for any non-empty proper subset X of K_k, $\delta(X) \ge 1$. Starting from K_k, remove 1 edge at a time. Then the δ-value of the graph increases by α_i at a time. Repeat this process until the δ-value of the graph exceeds 1. Let E be the graph with k points obtained by this process. We eventually get E because k points with no edge has the δ-value $k \ge 5$. If $\delta(E) = 1$, then E is a 1-component. If not, $1 < \delta(E) < 1 + \alpha_i < 1 + 1/2$ since if we put one more edge to E, the δ-value will be less than 1. □

If we have an s-component as a sufficiently large graph, then we can make an s-component as a structure with an n-ary relation for any n.

Lemma 3.5 *Let β be a real number such that $0 < \beta \leq 1$. Suppose (A, E) is a sufficiently large binary graph. Then there is $S \subseteq A^n$ such that $|A| - \beta|E| = |A| - \beta|S|$ and $|X| - \beta|E_X| \leq |X| - \beta|S_X|$ for any $X \subseteq A$. Here, E_X is the set of edges in E connecting vertices in X, and $S_X = X^n \cap S$. In particular, if (A, E) is an s-component then so is (A, S).*

Proof. Let $f : [A]^2 \to A^n$ be a one-to-one map such that a, b are members of $f(\{a, b\})$ for any distinct points in A ($[A]^2$ is the set of two point subsets of A). We can choose such f if $|A| > 2n$. Let $S = f(E)$ (Consider each edge as the set of two end points). Then $|A| - \beta|E| = |A| - \beta|S|$. Also, for any $X \subseteq A$, if $f(e) \in X$ then $e \in E_X$. Hence, $|S_X| \leq |E_X|$. Therefore, $|X| - \beta|E_X| \leq |X| - \beta|S_X|$. □

As a corollary, we get the following proposition.

Proposition 3.6 *If a rational number is a member of α then a 1-component exists in $\mathbf{K}_\alpha$.*

If an irrational number is a member of α, then for any $\varepsilon > 0$, there are ε' and ε'' such that $\varepsilon > \varepsilon' > \varepsilon'' > 0$ and a $(1 + \varepsilon')$-component and a $(1 - \varepsilon'')$-component exist in $\mathbf{K}_\alpha$.

Lemma 3.7 *Suppose that $r_1 + r_2 + \cdots + r_k - n = r$ where the r_i's are positive real numbers at most 1, $-1 \leq r \leq 1$, and n a natural number. Assume further that 1-component exists and so does $(1 + r_i)$-component for each $i = 1, \ldots, k$. Then $(1 + r)$-component exists.*

Proof. We prove the lemma by induction on n. The lemma holds for $n = 0$ by Lemma 3.2 (1). Suppose $n \geq 1$. Let i be the maximum suffix such that $r_1 + \cdots + r_i < 1$. Since a 1-component exists, we can assume that $i \geq 2$.

If $i = k$ then n must be 1. Therefore, there is a $(1 + r)$-component by Lemma 3.2 (3), and we are done.

Suppose $i < k$. Then $r_1 + \cdots + r_i + r_{i+1} \geq 1$. Let $q = r_1 + \cdots + r_i + r_{i+1} - 1 \geq 0$. By Lemma 3.2 (3), there is a $(1+q)$-component. We have $q + r_{i+2} + \cdots + r_k - (n-1) = r$. Therefore, r-component exists by the induction hypothesis. □

Proposition 3.8 *Suppose that a rational number is a member of α.*

(1) A 1-component exists in $\mathbf{K}_\alpha$.

(2) If a reduced fraction k/m is a member of α then a $(1 + 1/m)$-component exists in $\mathbf{K}_\alpha$.

(3) If a $(1 + 1/m)$-component and a $(1 + 1/m')$-component exist in $\mathbf{K}_\alpha$ then a $(1 + 1/\mathrm{lcm}(m, m'))$-component exists in $\mathbf{K}_\alpha$. Here, $\mathrm{lcm}(m, m')$ is the least common multiple of m and m'.

(4) *If a* $(1 + 1/m)$*-component exists in* $\mathbf{K}_\alpha$ *then a* $(1 - 1/m)$*-component exists in* $\mathbf{K}_\alpha$.

Proof. (1) is in Proposition 3.6.

(2) Suppose a reduced fraction $k/m = \alpha_i$ for some i. Then a graph with 2 points and 1 edge is a $(1+(m-k)/m)$-component for δ_{α_i} defined by $\delta_{\alpha_i}(A, E) = |A| - \alpha_i|E|$. Since there is a graph which is a 1-component for δ_{α_i}, we can make a $(1+(m-k)/m)$-component arbitrarily large. By Lemma 3.5, $(1 + (m - k)/m)$-component exists in $\mathbf{K}_\alpha$.

Let $k' = m - k$. k' and m are prime each other. Therefore, there are positive integers u, v such that $uk' - vm = 1$. Hence, $u(k'/m) - v = 1/m$. There is a $(1 + 1/m)$-component in $\mathbf{K}_\alpha$ by Lemma 3.7.

(3) Let d be the greatest common divisor of m and m'. Then there are positive integers $u < m'$ and $v < m$ such that $um - vm' = d$. Then $u(1/m') - v(1/m) = d/(mm')$ and thus $u(1/m') + (m - v)(1/m) - 1 = d/(mm')$. Therefore, there is a $(1 + d/(mm'))$-component in $\mathbf{K}_\alpha$ by Lemma 3.7.

(4) Also, by Lemma 3.7. □

4 A Property of $\mathbf{K}_\alpha$

In this section, we show a special property of $\mathbf{K}_\alpha$ with which we can show that the elementary theory of the generic structure of $\mathbf{K}_\alpha$ is axiomatized by Axioms 1 and 2 in the introduction. This will be investigated in [2].

Lemma 4.1 *Suppose* $0 \le s, t \le 1$, $C = B \oplus_a E$, *and* (E, a, b) *is a* $(1+t)$*-component. If* $\delta(A) + s \le \delta(X)$ *for any* X *such that* $A \subsetneq X \subseteq B$ *and* $a \in B - A$, *then* $\delta(A) + s \le \delta(X)$ *for any* X *such that* $A \subsetneq X \subseteq C$.

Proof. Suppose $A \subsetneq X \subseteq C$. If $a \notin X$, then $A \subseteq (X \cap B)$, $a \notin (X \cap E)$, and $X = (X \cap B) \oplus (X \cap E)$. If $(X \cap E) = \emptyset$, then $A \subsetneq X \subseteq B$, and thus $\delta(X) \ge \delta(A) + s$. If $(X \cap E) \neq \emptyset$, then $\delta(X \cap E) \ge 1$ and thus $\delta(X) = \delta(X \cap B) + \delta(X \cap E) \ge \delta(A) + 1 \ge \delta(A) + s$.

If $a \in X$, then $X = (X \cap B) \oplus_a (X \cap E)$, $A \subsetneq X \cap B$, and $X \cap E \neq \emptyset$. Hence,

$$\delta(X) = \delta(X \cap B) + \delta(X \cap E) - 1 \ge (\delta(A) + s) + (1 + t) - 1 \ge \delta(A) + s.$$

□

Lemma 4.2 *Suppose* $0 \le t \le s \le 1$, $C = B \oplus_a E$, *and* (E, a, b) *is a* $(1 - t)$*-component. If* $\delta(A) + s \le \delta(X)$ *for any* X *such that* $A \subsetneq X \subseteq B$ *and* $a \in B - A$, *then* $\delta(A) + (s - t) \le \delta(X)$ *for any* X *such that* $A \subsetneq X \subseteq C$.

Proof. Suppose $A \subsetneq X \subseteq C$. If $a \notin X$, then $A \subseteq (X \cap B)$, $a \notin (X \cap E)$, and $X = (X \cap B) \oplus (X \cap E)$. If $(X \cap E) = \emptyset$, then $A \subsetneq X \subseteq B$, and thus $\delta(X) \geq \delta(A) + s \geq \delta(A) + s - t$. If $(X \cap E) \neq \emptyset$, then $\delta(X \cap E) \geq 1$ and thus $\delta(X) = \delta(X \cap B) + \delta(X \cap E) \geq \delta(A) + 1 \geq \delta(A) + s - t$.

If $a \in X$, then $X = (X \cap B) \oplus_a (X \cap E)$, $A \subsetneq X \cap B$, and $X \cap E \neq \emptyset$. Hence,

$$\delta(X) = \delta(X \cap B) + \delta(X \cap E) - 1 \geq (\delta(A) + s) + (1 - t) - 1 \geq \delta(A) + s - t.$$

□

Theorem 4.3 *Consider $\mathbf{K}_\alpha$ where α consists of rational numbers only. Suppose A, B are $\mathcal{L}$-structures in $\mathbf{K}_\alpha$ and $A \leq B$. Then for any positive integer n, there is an $\mathcal{L}$-structure C in $\mathbf{K}_\alpha$ such that $B \subset C$, $B \leq_n C$, $A \leq C$ and $\delta(C) = \delta(A)$.*

Proof. We prove the theorem by the induction on $|B - A|$.

Suppose $|B - A| = 0$. In this case, $A = B$. Let $C = B = A$. Then the statement holds.

Suppose $|B - A| > 0$. Let B_0 be a substructure of B such that $A \subsetneq B_0 \subseteq B$ and $\delta(B_0) \leq \delta(X)$ for any set X such that $A \subsetneq X \subseteq B$.

Note that we have $A \leq B_0$ and $B_0 \leq B$.

Suppose $\delta(A) = \delta(B_0)$. We have $|B - B_0| < |B - A|$. By the induction hypothesis, for any integer $n > 0$, there is a structure $C \supset B$ such that $B \leq_n C$, $B_0 \leq C$ and $\delta(C) = \delta(B_0)$. Since $A \leq B_0$ and $\delta(A) = \delta(B_0)$, we have the statement.

Suppose $\delta(A) < \delta(B_0)$. If a is a point in $B - A$ then $\delta(\{a\}/A) \leq 1$. Therefore, $\delta(B_0/A) \leq 1$ by the choice of B_0. Let $s = \delta(B_0) - \delta(A) \leq 1$. Then $\delta(A) + s \leq \delta(X)$ for any substructure X of B_0 such that $A \subsetneq X$.

Let $\alpha = (n_1/m_1, n_2/m_2, \ldots, n_l/m_l)$ where each n_i/m_i is a reduced fraction. Let m be the least common multiple of m_1, m_2, …, m_l. Then $s = \delta(B_0) - \delta(A)$ is a multiple of $1/m$. Let $s = k/m$ where k is a positive integer.

For any positive integer n, let C' be an $\mathcal{L}$-structure such that

$$C' = B \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_n} E_n \oplus_{a_{n+1}} F_1 \oplus_{a_{n+2}} F_2 \cdots \oplus_{a_{n+k}} F_k,$$

where the a_i are pairwise distinct, $a_0 \in B_0 - A$, (E_i, a_i, a_{i+1}) is a 1-component for each $i = 0, 1, \ldots, n$, and $(F_j, a_{n+j}, a_{n+j+1})$ is a $(1 - 1/m)$-component for $j = 1, 2, \ldots, k$. Let C_0 be a substructure of C' such that

$$C_0 = B_0 \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_n} E_n \oplus_{a_{n+1}} F_1 \oplus_{a_{n+2}} F_2 \cdots \oplus_{a_{n+k}} F_k.$$

By Lemmas 4.1 and 4.2, we have $A \leq C_0$ and $\delta(A) = \delta(C_0)$. Therefore, $C_0 \in \mathbf{K}_\alpha$.

We show that $B_0 \leq_n C_0$. Suppose $X \subset C_0 - B_0$ and $|X| \leq n$. Then $X = \{a_0, a_1, \ldots, a_{n-1}\}$ or $a_i \notin X$ for some $i < n$. In either case, $X \subset (B_0 \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_j} E_j) \oplus D$ for some $j \leq n$ and $D \in \mathbf{K}_\alpha$. Since $B_0 \leq B_0 \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_j} E_j$ by Lemma 4.1, $\delta(B_0) \leq \delta(X)$.

Now we have $C' = B \oplus_{B_0} C_0$, $B_0 \leq C$, and $B_0 \leq_n C_0$. Hence, $C_0 \leq C'$ and $B \leq_n C'$. Since $|C' - C_0| = |B - B_0| < |B - A|$, we have $C \in \mathbf{K}_\alpha$ such that $C \supset C'$, $C' \leq_n C$, $C_0 \leq C$ and $\delta(C) = \delta(C_0)$. Therefore, $B \leq_n C$, $A \leq C$, and $\delta(C) = \delta(A)$. □

Theorem 4.4 *Suppose A, B are $\mathcal{L}$-structures in $\mathbf{K}_\alpha$ and $A \leq B$. Then for any real number $\varepsilon > 0$ and for any positive integer n, there is an $\mathcal{L}$-structure C in $\mathbf{K}_\alpha$ such that $B \subset C$, $B \leq_n C$, $A \leq C$ and $\delta(C) < \delta(A) + \varepsilon$.*

Proof. If α consists of ratinal numbers only, then the statement holds by Theorem 4.3.

Assume that α contains an irrational number. The proof for this case is similar to that of Theorem 4.3. So, we give only a sketch. Choose B_0 as in the proof of Theorem 4.3. Assume that $s = \delta(B_0) - \delta(A) > 0$. Let $\varepsilon > 0$ be an arbitrary (small) real number. Let

$$C' = B \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_n} E_n \oplus_{a_{n+1}} F_1 \oplus_{a_{n+2}} F_2 \cdots \oplus_{a_{n+k}} F_k$$

as in the proof of Theorem 4.3, except that (E_i, a_i, a_{i+1}) is a $(1+t)$-component with t sufficiently small for each $i = 0, 1, \ldots, n$ so that

$$\delta(B_0 \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_n} E_n) < \delta(B_0) + \varepsilon/4,$$

and $(F_j, a_{n+j}, a_{n+j+1})$ is a $(1-t')$-component with $0 < t' < \varepsilon/4$ for $j = 1, 2, \ldots, k$, where k is the largest integer such that $kt' < s$. Then $A \leq C_0$, and $\delta(C_0) < \delta(A) + \varepsilon/2$ for C_0 as in the proof of Theorem 4.3. Then we have $C_0 \leq C'$ and $B \leq_n C'$.

By the induction hypothesis, we can choose $C \in \mathbf{K}_\alpha$ such that $C' \leq_n C$, $C_0 \leq C$, and $\delta(C) < \delta(C_0) + \varepsilon/2$. Therefore, we have the theorem. □

References

[1] J.T. Baldwin, and N. Shi, Stable generic structures, Ann. Pure Appl. Logic **79** (1996) 1–35.

[2] K. Ikeda, H. Kikyo, and A. Tsuboi, On generic structures with a strong amalgamation property, in preparation.

[3] N. Peatfield and B. Zilber, Analytic Zariski structures and the Hrushovski construction, Ann. Pure Appl. Logic **132** (2005) 127–180.