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<th>On Predimensions of Finite Structures (Zariski Geometry and Arithmetic Geometry)</th>
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<td>Author(s)</td>
<td>Kikyo, Hirotaka</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2005), 1450: 75-82</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2005-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/47721">http://hdl.handle.net/2433/47721</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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京都大学
On Predimensions of Finite Structures

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September 7, 2005

Abstract

We give some families of finite structures with a predimension function, in which we can construct various structures with desired predimensions.

1 Introduction

N. Peatfield and B. Zilber investigated a generic structure of certain class of finite structures considered by Hrushovski and they showed that with some modification of the structure, they can put a topology called an analytic Zariski structure on it. One of their main tools is the following.

For the class \( \mathcal{K} \) they considered, a structure \( M \) is elementarily equivalent to a generic structure of \( \mathcal{K} \) if and only if \( M \) satisfies Axioms 1 and 2:

Axiom 1. Any finite substructure \( A \) of \( M \) belongs to \( \mathcal{K} \).

Axiom 2. For any \( A, B \in \mathcal{K} \) such that \( A \leq B \), if \( f: A \rightarrow M \) is an \( \mathcal{L} \)-embedding then it can be extended to an \( \mathcal{L} \)-embedding \( f': B \rightarrow M \).

Axiom 2 represents a strong form of the amalgamation property. For any class \( \mathcal{K} \) of finite structures, these axioms may have some meaning. This will be investigated in a joint work with K. Ikeda and A. Tsuboi [2]. This paper will be a part of this work.

2 Preliminaries

Throughout this paper, \( \mathcal{L} \) is a finite relational language. If \( M \) is an \( \mathcal{L} \)-structure and \( R \) is a relation with \( n \) arguments then \( R(M) = \{ x \in M^n : M \models R(x) \} \).

Definition 2.1 Suppose \( \mathcal{L} = \{ R_1, R_2, \ldots, R_l \} \), and \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \) where \( \alpha_i \) are positive real numbers at most 1.

For a finite \( \mathcal{L} \)-structure \( A \), let

\[
\delta(A) = |A| - |R_1(A)| - |R_2(A)| - \cdots - |R_l(A)|.
\]
\(\delta\) is called a \textit{predimension function}.

Let \(B\) be an \(\mathcal{L}\)-structure and \(A \subseteq B\). We write \(A \leq B\) if for any subset \(X\) of \(B\), whenever \(A \subseteq X \subseteq B\) then \(\delta(A) \leq \delta(X)\).

Let \(K_\alpha = \{A : \delta(X) \geq 0 \text{ for any } X \subseteq A\}\).

\textbf{Definition 2.2 (Asymmetric amalgam)} Let \(B\) and \(C\) be \(\mathcal{L}\)-structures, and let \(A = B \cap C\) as a set. \(B \oplus_A C\) is a structure with universe \(B \cup C\) and for each relation \(R\) in \(\mathcal{L}\), \(R(B \oplus_A C) = R(B) \cup R(C) - (R(C) \cap A^n)\) where \(n\) is the number of arguments of \(R\).

Note that \(\delta(B \oplus_A C) = \delta(B) + \delta(C) - \delta(C|_A)\) where \(C|_A\) is a substructure of \(C\) with the universe \(A\).

Furthermore, if \(B \cap C = \{a\}\) (a singleton), then we write \(B \oplus_a C\) for \(B \oplus\{a\} C\), and if \(B \cap C = \emptyset\), then we write \(B \oplus \emptyset C\).

\textbf{Definition 2.3} Let \(A\) be an \(\mathcal{L}\)-structure, \(a, b\) elements in \(A\), and suppose that for any relation \(R \in \mathcal{L}\), there is no tuple \(x\) of elements in \(A\) such that \(A \models R(x)\) and \(x\) contains \(a\) and \(b\).

\(A/(a = b)\) is an \(\mathcal{L}\)-structure obtained from \(A\) by identifying \(a\) and \(b\). More precisely, it is defined as follows: The universe, also denoted \(A/(a = b)\), is \(A - \{b\}\). Let \(f : A \to A/(a = b)\) be a projection map defined by \(f(x) = x\) for \(x \neq b\), and \(f(b) = a\).

For any relation \(R \in \mathcal{L}\), \(A/(a = b) \models R(c_1, \ldots, c_n)\) for \(c_1, \ldots, c_n \in A/(a = b)\) if and only if \(A \models R(c'_1, \ldots, c'_n)\) for some \(c'_1, \ldots, c'_n \in A\) such that \(f(c'_i) = c_i\) for \(i = 1, \ldots, n\).

\section{Arithmetic of Structures}

In this section, we work in \(K_\alpha\) for some tuple \(\alpha\) of positive real numbers at most 1. We show that there are a plenty of structures in \(K_\alpha\) to get structures in \(K_\alpha\) of desired \(\delta\)-ranks.

\textbf{Definition 3.1} Let \(s\) be a real number such that \(0 \leq s \leq 2\). A triple \((E, a, b)\) is an \textit{s-component} if \(E\) is an \(\mathcal{L}\)-structure, \(a, b \in E\), \(a \neq b\), \(\delta(\{a\}) = \delta(\{b\}) = 1\), and the following hold:

For any non-empty substructure \(X\) of \(E\),

1. \(s \leq \delta(X)\) if \(a, b \in X\),
2. \(1 \leq \delta(X)\) if \(a \notin X\) or \(b \notin X\), and
3. \(\delta(E) = s\).

We also say that \(E\) is an \(s\)-component with joints \(a\) and \(b\), or just \(E\) is an \(s\)-component.

\textbf{Lemma 3.2} Let \(s, t, u\) be real numbers such that \(0 \leq s, t, u \leq 1\).
(1) Suppose \((A, a, b)\) is an \((1+s)\)-component, \((B, b, c)\) a \((1+t)\)-component, \(s+t \leq 1\), and \(C = A \oplus_b B\). Then \((C, a, c)\) is an \((1+s+t)\)-component.

(2) Suppose \((A, a, b)\) is an \((1+s)\)-component, \((B, b, c)\) a \((1+t)\)-component, \((C, c, d)\) a \((1+u)\)-component, \(s+t \leq 1\), \(D = A \oplus_b B \oplus_c C\), and \(\overline{D} = D/(a = d)\). Then \((\overline{D}, c, a)\) is an \((s+t+u)\)-component.

(3) Suppose \(k \geq 3\), \(r_1, r_2, \ldots, r_{k-1}, r_k\) are non-negative real numbers such that \(r_1 + r_2 + \cdots + r_{k-1} \leq 1\), and an \(r_i\)-component exists for each \(i\). Then there is an \((r_1 + r_2 + \cdots + r_{k-1} + r_k)\)-component.

**Proof.** (1) Suppose \(X \subseteq C = A \oplus_b B\) and \(X \neq \emptyset\).

We check the condition (1) of the definition of an \((1+s+t)\)-component first. Assume that \(a, c \in X\). Consider the case \(b \in X\). Then \(a, b \in X \cap A\), \(b, c \in X \cap B\), and \(X = (X \cap A) \oplus_b (X \cap B)\). Hence,

\[
\delta(X) = \delta(X \cap A) + \delta(X \cap B) \geq (1+s) + (1+t) - 1 = 1 + (s+t).
\]

Now, consider the case \(b \notin X\). Then \(b \notin X \cap A\), \(b \notin X \cap B\), and \(X = (X \cap A) \oplus (X \cap B)\). Hence,

\[
\delta(X) = \delta(X \cap A) + \delta(X \cap B) \geq 1 + 1 \geq 1 + (s+t).
\]

Therefore, \(a, c \in X\) implies \(\delta(X) \geq 1 + s + t\).

Now, we check the condition (2) of the definition of an \((1+s+t)\)-component. Suppose \(X\) is a non-empty subset of \(C\). Then \(\delta(X) \geq \delta(X \cap A) + \delta(X \cap B) - 1 \geq 1 + 1 - 1 = 1\).

(2) Since there is no relation on a tuple containing \(a\) and \(d\), going from \(D\) to \(\overline{D}\), the number of relations does not change. Therefore, for any \(X \subseteq D\), \(\delta(X/(a = d)) = \delta(X) - 1\) if \(a, d \in X\), and \(\delta(X/(a = d)) = \delta(X)\) if \(a, d \notin X\).

Suppose \(c, a \in X \subseteq \overline{D}\). Let \(Y \subseteq D\) be such that \(X = Y/(a = d)\). We can assume that \(a, d \in Y\).

Consider the case \(b \in Y\). Since \(a, b, c, d \in Y\), we have \(\delta(Y \cap A) \geq 1 + s\), \(\delta(Y \cap B) \geq 1 + t\), \(\delta(Y \cap C) \geq 1 + u\), and \(Y = (Y \cap A) \oplus_b (Y \cap B) \oplus_c (Y \cap C)\). Hence, \(\delta(Y) \geq 1 + (s+t+u)\). Therefore, \(\delta(X) = \delta(Y/(a = d)) = \delta(Y) - 1 \geq s + t + u\).

Now, consider the case \(b \notin Y\). Then, \(\delta(Y \cap A) \geq 1\), \(\delta(Y \cap B) \geq 1\), \(\delta(Y \cap C) \geq 1 + u\), and \(Y = (Y \cap A) \oplus (Y \cap B) \oplus_c (Y \cap C)\). We have \(\delta(Y) \geq 1 + (1+u) \geq 1 + (s+t+u)\) by the assumption that \(s+t \leq 1\). Therefore, \(\delta(X) = \delta(Y/(a = d)) = \delta(Y) - 1 \geq s + t + u\).

We have checked the condition (1) of the definition of \((s+t+u)\)-component.

We turn to check the condition (2) of the definition of \((s+t+u)\)-component. Suppose \(a \notin X \subseteq \overline{D}\) and \(X \neq \emptyset\). Then \(X \subseteq D\) and \(a, d \notin X\). Let \(X_1 = X \cap (A \cup B)\) and \(X_2 = X \cap C\). Since \(s+t \leq 1\), \(A \cup B = A \oplus_b B\) is a \((1+s+t)\)-component by (1), and thus \(\delta(X_1) \geq 1\) by \(a \notin X\). \(\delta(X_2) \geq 1\) since \(d \notin X\). Therefore, \(\delta(X) \geq \delta(X_1) + \delta(X_2) - 1 \geq 1\).

Suppose \(c \notin X\) and \(a \in X\). Let \(Y \subseteq D\) be such that \(X = Y/(a = d)\) and \(a, d \in Y\).

Since \(c \notin Y\), \(\delta(Y) = \delta(Y \cap (A \cup B)) + \delta(Y \cap C) \geq 2\). Therefore, \(\delta(X) = \delta(Y) - 1 \geq 1\).

\(\square\)
Lemma 3.3 Suppose $0 \leq s < 1/2$ and a $(1 + s)$-component exists. If $s$ is rational then a 1-component exists. If $s$ is irrational, then for any $\varepsilon > 0$ there exists $\varepsilon'$ such that $\varepsilon > \varepsilon' > 0$ and a $(1 + \varepsilon')$-component exists. With Lemma 3.2, we can choose a 1-component or a $(1 + \varepsilon')$-component arbitrarily large.

Proof. Suppose $0 \leq s < 1/2$ and a $(1 + s)$-component exists. Consider the following non-negative decreasing sequence $\{s_i\}_{i<\omega}$: Let $s_0 = s$. If $s_i = 0$ then let $s_{i+1} = 0$. If $s_i > 0$ then choose the least positive integer $m_i$ such that $m_i s_i \geq 1$ and let $s_{i+1} = m_i s_i - 1$.

By Lemma 3.2 (2), there is a $(1 + s_i)$-component for each $i < \omega$.

Note that if $s_i > 0$ then $s_i > s_{i+1} \geq 0$. If $s_0 = s$ is rational then $s_i = 0$ eventually. Hence, 1-component exists.

If $s_0 = s$ is irrational, $s_i$ converges to 0. This should be well-known, but we give a proof for convenience. Since $\{s_i\}_{i<\omega}$ is a decreasing sequence, $\{m_i\}_{i<\omega}$ is an increasing sequence. If $\{m_i\}_{i<\omega}$ is unbounded, then $s_i$ converges to 0 because $s_i < 1/(m_i-1)$. If $\{m_i\}_{i<\omega}$ is bounded, $m_i = m$ for some positive integer $m$ eventually. So, we can assume that $s_{i+1} = m s_i - 1$ for all $i$. Since $s_i$ converges to some $\beta > 0$, $\beta = m \beta - 1$. Therefore, $\beta = 1/(m - 1)$. But $s_i > \beta = 1/(m - 1)$ contradicts $(m - 1)s_i = (m_i - 1)s_i < 1$.

Lemma 3.4 Let $(V, E)$ be a finite graph where $V$ is a set of vertices and $E$ a set of edges, and let $\delta(V, E) = |V| - |E|$ with $0 < \beta \leq 1$. Then a graph which is a $(1 + s)$-component exists for some real number $s$ such that $0 \leq s < 1/2$ with respect to this $\delta$.

Proof. We remark first that any two different points in each $(1 + s)$-component below will be its joints.

Suppose $\alpha_i = 1/2$. A graph with 4 points and 6 edges $(K_4)$ is a 1-component.

Suppose $1/2 < \alpha_i \leq 1$. Let $s = 1 - \alpha_i$. Then a graph with 2 points and 1 edge is a $(1 + s)$-component. These 2 points will be joints.

Suppose $0 < \alpha_i < 1/2$. Choose least natural number $k$ such that $k - \alpha \left(\begin{array}{c}k \\ 2 \end{array}\right) < 1$.

Note that $k \leq 4$ implies $k - \alpha \left(\begin{array}{c}k \\ 2 \end{array}\right) > 1$. Therefore, $k \geq 5$. Consider $K_k$, a complete graph with $k$ points. Then for any non-empty proper subset $X$ of $K_k$, $\delta(X) \geq 1$. Starting from $K_k$, remove 1 edge at a time. Then the $\delta$-value of the graph increases by $\alpha_i$ at a time. Repeat this process until the $\delta$-value of the graph exceeds 1. Let $E$ be the graph with $k$ points obtained by this process. We eventually get $E$ because $k$ points with no edge has the $\delta$-value $k \geq 5$. If $\delta(E) = 1$, then $E$ is a 1-component. If not, $1 < \delta(E) < 1 + \alpha_i < 1 + 1/2$ since if we put one more edge to $E$, the $\delta$-value will be less than 1.

If we have an $s$-component as a sufficiently large graph, then we can make an $s$-component as a structure with an $n$-ary relation for any $n$.  

\[ \text{Lemma 3.3} \] Suppose $0 \leq s < 1/2$ and a $(1 + s)$-component exists. If $s$ is rational then a 1-component exists. If $s$ is irrational, then for any $\varepsilon > 0$ there exists $\varepsilon'$ such that $\varepsilon > \varepsilon' > 0$ and a $(1 + \varepsilon')$-component exists. With Lemma 3.2, we can choose a 1-component or a $(1 + \varepsilon')$-component arbitrarily large.

\[ \text{Proof.} \] Suppose $0 \leq s < 1/2$ and a $(1 + s)$-component exists. Consider the following non-negative decreasing sequence $\{s_i\}_{i<\omega}$: Let $s_0 = s$. If $s_i = 0$ then let $s_{i+1} = 0$. If $s_i > 0$ then choose the least positive integer $m_i$ such that $m_i s_i \geq 1$ and let $s_{i+1} = m_i s_i - 1$.

By Lemma 3.2 (2), there is a $(1 + s_i)$-component for each $i < \omega$.

Note that if $s_i > 0$ then $s_i > s_{i+1} \geq 0$. If $s_0 = s$ is rational then $s_i = 0$ eventually. Hence, 1-component exists.

If $s_0 = s$ is irrational, $s_i$ converges to 0. This should be well-known, but we give a proof for convenience. Since $\{s_i\}_{i<\omega}$ is a decreasing sequence, $\{m_i\}_{i<\omega}$ is an increasing sequence. If $\{m_i\}_{i<\omega}$ is unbounded, then $s_i$ converges to 0 because $s_i < 1/(m_i-1)$. If $\{m_i\}_{i<\omega}$ is bounded, $m_i = m$ for some positive integer $m$ eventually. So, we can assume that $s_{i+1} = m s_i - 1$ for all $i$. Since $s_i$ converges to some $\beta > 0$, $\beta = m \beta - 1$. Therefore, $\beta = 1/(m - 1)$. But $s_i > \beta = 1/(m - 1)$ contradicts $(m - 1)s_i = (m_i - 1)s_i < 1$. \[ \square \]

\[ \text{Lemma 3.4} \] Let $(V, E)$ be a finite graph where $V$ is a set of vertices and $E$ a set of edges, and let $\delta(V, E) = |V| - |E|$ with $0 < \beta \leq 1$. Then a graph which is a $(1 + s)$-component exists for some real number $s$ such that $0 \leq s < 1/2$ with respect to this $\delta$.

\[ \text{Proof.} \] We remark first that any two different points in each $(1 + s)$-component below will be its joints.

Suppose $\alpha_i = 1/2$. A graph with 4 points and 6 edges $(K_4)$ is a 1-component.

Suppose $1/2 < \alpha_i \leq 1$. Let $s = 1 - \alpha_i$. Then a graph with 2 points and 1 edge is a $(1 + s)$-component. These 2 points will be joints.

Suppose $0 < \alpha_i < 1/2$. Choose least natural number $k$ such that $k - \alpha \left(\begin{array}{c}k \\ 2 \end{array}\right) < 1$.

Note that $k \leq 4$ implies $k - \alpha \left(\begin{array}{c}k \\ 2 \end{array}\right) > 1$. Therefore, $k \geq 5$. Consider $K_k$, a complete graph with $k$ points. Then for any non-empty proper subset $X$ of $K_k$, $\delta(X) \geq 1$. Starting from $K_k$, remove 1 edge at a time. Then the $\delta$-value of the graph increases by $\alpha_i$ at a time. Repeat this process until the $\delta$-value of the graph exceeds 1. Let $E$ be the graph with $k$ points obtained by this process. We eventually get $E$ because $k$ points with no edge has the $\delta$-value $k \geq 5$. If $\delta(E) = 1$, then $E$ is a 1-component. If not, $1 < \delta(E) < 1 + \alpha_i < 1 + 1/2$ since if we put one more edge to $E$, the $\delta$-value will be less than 1. \[ \square \]
Lemma 3.5 Let $\beta$ be a real number such that $0 < \beta \leq 1$. Suppose $(A, E)$ is a sufficiently large binary graph. Then there is $S \subseteq A^n$ such that $|A| - \beta|E| = |A| - \beta|S|$ and $|X| - \beta|E_X| \leq |X| - \beta|S_X|$ for any $X \subseteq A$. Here, $E_X$ is the set of edges in $E$ connecting vertices in $X$, and $S_X = X^n \cap S$. In particular, if $(A, E)$ is an $s$-component then so is $(A, S)$.

Proof. Let $f : [A]^2 \rightarrow A^n$ be a one-to-one map such that $a, b$ are members of $f(\{a, b\})$ for any distinct points in $A$ ($[A]^2$ is the set of two point subsets of $A$). We can choose such $f$ if $|A| > 2n$. Let $S = f(E)$ (Consider each edge as the set of two end points). Then $|A| - \beta|E| = |A| - \beta|S|$. Also, for any $X \subseteq A$, if $f(e) \in X$ then $e \in E_X$. Hence, $|S_X| \leq |E_X|$. Therefore, $|X| - \beta|E_X| \leq |X| - \beta|S_X|$.

As a corollary, we get the following proposition.

Proposition 3.6 If a rational number is a member of $\alpha$ then a 1-component exists in $K_\alpha$.

If an irrational number is a member of $\alpha$, then for any $\epsilon > 0$, there are $\epsilon'$ and $\epsilon''$ such that $\epsilon > \epsilon' > \epsilon'' > 0$ and a $(1 + \epsilon')$-component and a $(1 + \epsilon'')$-component exist in $K_\alpha$.

Lemma 3.7 Suppose that $r_1 + r_2 + \cdots + r_k - n = r$ where the $r_i$'s are positive real numbers at most 1, $-1 \leq r \leq 1$, and $n$ a natural number. Assume further that 1-component exists and so does $(1 + r_i)$-component for each $i = 1, \ldots, k$. Then $(1 + r)$-component exists.

Proof. We prove the lemma by induction on $n$. The lemma holds for $n = 0$ by Lemma 3.2 (1). Suppose $n \geq 1$. Let $i$ be the maximum suffix such that $r_1 + \cdots + r_i < 1$. Since a 1-component exists, we can assume that $i \geq 2$.

If $i = k$ then $n$ must be 1. Therefore, there is a $(1 + r)$-component by Lemma 3.2 (3), and we are done.

Suppose $i < k$. Then $r_1 + \cdots + r_i + r_{i+1} \geq 1$. Let $q = r_1 + \cdots + r_i + r_{i+1} - 1 \geq 0$. By Lemma 3.2 (3), there is a $(1 + q)$-component. We have $q + r_{i+2} + \cdots + r_k - (n-1) = r$. Therefore, $r$-component exists by the induction hypothesis.

Proposition 3.8 Suppose that a rational number is a member of $\alpha$.

(1) A 1-component exists in $K_\alpha$.

(2) If a reduced fraction $k/m$ is a member of $\alpha$ then a $(1 + 1/m)$-component exists in $K_\alpha$.

(3) If a $(1 + 1/m)$-component and a $(1 + 1/m')$-component exist in $K_\alpha$ then a $(1 + 1/\text{lcm}(m, m'))$-component exists in $K_\alpha$. Here, $\text{lcm}(m, m')$ is the least common multiple of $m$ and $m'$. 
(4) If a $(1 + 1/m)$-component exists in $K_\alpha$ then a $(1 - 1/m)$-component exists in $K_\alpha$.

**Proof.** (1) is in Proposition 3.6.

(2) Suppose a reduced fraction $k/m = \alpha_i$ for some $i$. Then a graph with 2 points and 1 edge is a $(1 + (m - k)/m)$-component for $\delta_{\alpha_i}$ defined by $\delta_{\alpha_i}(A, E) = |A| - \alpha_i|E|$. Since there is a graph which is a 1-component for $\delta_{\alpha_i}$, we can make a $(1 + (m - k)/m)$-component arbitrarily large. By Lemma 3.5, $(1 + (m - k)/m)$-component exists in $K_\alpha$.

Let $k' = m - k$. $k'$ and $m$ are prime each other. Therefore, there are positive integers $u, v$ such that $uk' - vm = 1$. Hence, $u(k'/m) - v = 1/m$. There is a $(1 + 1/m)$-component in $K_\alpha$ by Lemma 3.7.

(3) Let $d$ be the greatest common divisor of $m$ and $m'$. Then there are positive integers $u < m'$ and $v < m$ such that $um - vm' = d$. Then $(1/m') = v(1/m) = d/(mm')$ and thus $u(1/m') + (m - v)/(1/m) = (m - v)/m - 1 = d/(mm')$. Therefore, there is a $(1 + d/(mm'))$-component in $K_\alpha$ by Lemma 3.7.

(4) Also, by Lemma 3.7.

\[\square\]

4 A Property of $K_\alpha$

In this section, we show a special property of $K_\alpha$ with which we can show that the elementary theory of the generic structure of $K_\alpha$ is axiomatized by Axioms 1 and 2 in the introduction. This will be investigated in [2].

**Lemma 4.1** Suppose $0 \leq s, t \leq 1$, $C = B \oplus_a E$, and $(E, a, b)$ is a $(1 + t)$-component. If $\delta(A) + s \leq \delta(X)$ for any $X$ such that $A \subseteq X \subseteq B$ and $a \in B - A$, then $\delta(A) + s \leq \delta(X)$ for any $X$ such that $A \subseteq X \subseteq C$.

**Proof.** Suppose $A \subseteq X \subseteq C$. If $a \notin X$, then $A \subseteq (X \cap B)$, $a \notin (X \cap E)$, and $X = (X \cap B) \oplus (X \cap E)$. If $(X \cap E) = \emptyset$, then $A \subseteq X \subseteq B$, and thus $\delta(X) \geq \delta(A) + s$.

If $(X \cap E) \neq \emptyset$, then $\delta(X \cap E) \geq 1$ and thus $\delta(X) = \delta(X \cap B) + \delta(X \cap E) \geq \delta(A) + 1 \geq \delta(A) + s$.

If $a \in X$, then $X = (X \cap B) \oplus_a (X \cap E)$, $A \subseteq X \subseteq B$, and $X \cap E \neq \emptyset$. Hence,

$$\delta(X) = \delta(X \cap B) + \delta(X \cap E) - 1 \geq (\delta(A) + s) + (1 + t) - 1 \geq \delta(A) + s.$$

\[\square\]

**Lemma 4.2** Suppose $0 \leq t \leq s \leq 1$, $C = B \oplus_a E$, and $(E, a, b)$ is a $(1 - t)$-component. If $\delta(A) + s \leq \delta(X)$ for any $X$ such that $A \subseteq X \subseteq B$ and $a \in B - A$, then $\delta(A) + (s - t) \leq \delta(X)$ for any $X$ such that $A \subseteq X \subseteq C$.
Proof. Suppose \( A \subseteq X \subseteq C \). If \( a \not\in X \), then \( A \subseteq (X \cap B) \), \( a \not\in (X \cap E) \), and \( X = (X \cap B) \oplus (X \cap E) \). If \( X \cap E = \emptyset \), then \( A \not\subseteq X \subseteq B \), and thus \( \delta(X) \geq \delta(A) + s \geq \delta(A) + s - t \). If \( X \cap E \neq \emptyset \), then \( \delta(X \cap E) \geq 1 \) and thus \( \delta(X) = \delta(X \cap B) + \delta(X \cap E) \geq \delta(A) + 1 \geq \delta(A) + s - t \).

If \( a \in X \), then \( X = (X \cap B) \oplus_f (X \cap E) \), \( A \not\subseteq X \cap B \), and \( X \cap E \neq \emptyset \). Hence,

\[
\delta(X) = \delta(X \cap B) + \delta(X \cap E) - 1 \geq (\delta(A) + s) + (1 - t) - 1 \geq \delta(A) + s - t.
\]

\( \Box \)

Theorem 4.3 Consider \( K_\alpha \) where \( \alpha \) consists of rational numbers only. Suppose \( A \), \( B \) are \( \mathcal{L} \)-structures in \( K_\alpha \) and \( A \subseteq B \). Then for any positive integer \( n \), there is an \( \mathcal{L} \)-structure \( C \) in \( K_\alpha \) such that \( B \subseteq C \), \( B \leq_n C \), \( A \subseteq C \) and \( \delta(C) = \delta(A) \).

Proof. We prove the theorem by the induction on \(|B - A|\).

Suppose \(|B - A| = 0\). In this case, \( A = B \). Let \( C = B = A \). Then the statement holds.

Suppose \(|B - A| > 0\). Let \( B_0 \) be a substructure of \( B \) such that \( A \not\subseteq B_0 \subseteq B \) and \( \delta(B_0) \leq \delta(X) \) for any set \( X \) such that \( A \not\subseteq X \subseteq B \).

Note that we have \( A \leq B_0 \) and \( B_0 \leq B \).

Suppose \( \delta(A) = \delta(B_0) \). We have \(|B - B_0| < |B - A| \). By the induction hypothesis, for any integer \( n > 0 \), there is a structure \( C \supseteq B \) such that \( B \leq_n C \), \( B_0 \leq C \) and \( \delta(C) = \delta(B_0) \). Since \( A \subseteq B_0 \) and \( \delta(A) = \delta(B_0) \), we have the statement.

Suppose \( \delta(A) < \delta(B_0) \). If \( a \) is a point in \( B - A \) then \( \delta(\{a\}/A) \leq 1 \). Therefore, \( \delta(B_0/A) \leq 1 \) by the choice of \( B_0 \). Let \( s = \delta(B_0) - \delta(A) \leq 1 \). Then \( \delta(A) + s \leq \delta(X) \) for any substructure \( X \) of \( B_0 \) such that \( A \not\subseteq X \).

Let \( \alpha = (n_1/m_1, n_2/m_2, \ldots, n_i/m_i) \) where each \( n_i/m_i \) is a reduced fraction. Let \( m \) be the least common multiple of \( m_1, m_2, \ldots, m_i \). Then \( s = \delta(B_0) - \delta(A) \) is a multiple of \( 1/m \). Let \( k = m \) where \( m \) is a positive integer.

For any positive integer \( n \), let \( C' \) be an \( \mathcal{L} \)-structure such that

\[
C' = B \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_n} E_n \oplus_{a_{n+1}} F_1 \oplus_{a_{n+2}} F_2 \cdots \oplus_{a_{n+k}} F_k,
\]

where the \( a_i \) are pairwise distinct, \( a_0 \in B_0 - A \), \( (E_i, a_i, a_{i+1}) \) is a 1-component for each \( i = 0, 1, \ldots, n \), and \( (F_j, a_{n+j}, a_{n+j+1}) \) is a \((1 - 1/m)\)-component for \( j = 1, 2, \ldots, k \). Let \( C_0 \) be a substructure of \( C' \) such that

\[
C_0 = B_0 \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_n} E_n \oplus_{a_{n+1}} F_1 \oplus_{a_{n+2}} F_2 \cdots \oplus_{a_{n+k}} F_k.
\]

By Lemmas 4.1 and 4.2, we have \( A \subseteq C_0 \) and \( \delta(A) = \delta(C_0) \). Therefore, \( C_0 \in K_\alpha \).

We show that \( B_0 \leq_n C_0 \). Suppose \( X \subseteq C_0 - B_0 \) and \(|X| \leq n \). Then \( X = \{a_0, a_1, \ldots, a_{n-1}\} \) or \( a_i \not\in X \) for some \( i < n \). In either case, \( X \subseteq (B_0 \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_j} E_j) \oplus D \) for some \( j \leq n \) and \( D \in K_\alpha \). Since \( B_0 \leq B_0 \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_j} E_j \) by Lemma 4.1, \( \delta(B_0) \leq \delta(X) \).
Now we have $C' = B \oplus_{B_0} C_0$, $B_0 \leq C$, and $B_0 \leq_n C_0$. Hence, $C_0 \leq C'$ and $B \leq_n C'$. Since $|C' - C_0| = |B - B_0| < |B - A|$, we have $C \in K_\alpha$ such that $C \supset C'$, $C' \leq_n C$, $C_0 \leq C$ and $\delta(C) = \delta(C_0)$. Therefore, $B \leq_n C$, $A \leq C$, and $\delta(C) = \delta(A)$.

Theorem 4.4 Suppose $A$, $B$ are $L$-structures in $K_\alpha$ and $A \leq B$. Then for any real number $\varepsilon > 0$ and for any positive integer $n$, there is an $L$-structure $C$ in $K_\alpha$ such that $B \subset C$, $B \leq_n C$, $A \leq C$ and $\delta(C) < \delta(A) + \varepsilon$.

Proof. If $\alpha$ consists of rational numbers only, then the statement holds by Theorem 4.3.

Assume that $\alpha$ contains irrational numbers. The proof for this case is similar to that of Theorem 4.3. So, we give only a sketch. Choose $B_0$ as in the proof of Theorem 4.3. Assume that $s = \delta(B_0) - \delta(A) > 0$. Let $\varepsilon > 0$ be an arbitrary (small) real number. Let

$$C' = B \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_n} E_n \oplus_{a_{n+1}} F_1 \oplus_{a_{n+2}} F_2 \cdots \oplus_{a_{n+k}} F_k$$

as in the proof of Theorem 4.3, except that $(E_i, a_i, a_{i+1})$ is a $(1+t)$-component with $t$ sufficiently small for each $i = 0, 1, \ldots, n$ so that

$$\delta(B_0 \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_n} E_n) < \delta(B_0) + \varepsilon/4,$$

and $(F_j, a_{n+j}, a_{n+j+1})$ is a $(1-t')$-component with $0 < t' < \varepsilon/4$ for $j = 1, 2, \ldots, k$, where $k$ is the largest integer such that $kt' < s$. Then $A \leq C_0$, and $\delta(C_0) < \delta(A) + \varepsilon/2$ for $C_0$ as in the proof of Theorem 4.3. Then we have $C_0 \leq C'$ and $B \leq_n C'$.

By the induction hypothesis, we can choose $C \in K_\alpha$ such that $C' \leq_n C$, $C_0 \leq C$, and $\delta(C) < \delta(C_0) + \varepsilon/2$. Therefore, we have the theorem.

References

