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On Predimensions of Finite Structures

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Abstract

We give some families of finite structures with a predimension function, in which we can construct various structures with desired predimensions.

1 Introduction

N. Peatfield and B. Zilber investigated a generic structure of certain class of finite structures considered by Hrushovski and they showed that with some modification of the structure, they can put a topology called an analytic Zariski structure on it. One of their main tools is the following.

For the class $\mathbf{K}$ they considered, a structure $M$ is elementarily equivalent to a generic structure of $\mathbf{K}$ if and only if $M$ satisfies Axioms 1 and 2:

Axiom 1. Any finite substructure $A$ of $M$ belongs to $\mathbf{K}$.

Axiom 2. For any $A, B \in \mathbf{K}$ such that $A \leq B$, if $f : A \rightarrow M$ is an $\mathcal{L}$-embedding then it can be extended to an $\mathcal{L}$-embedding $f' : B \rightarrow M$.

Axiom 2 represents a strong form of the amalgamation property. For any class $\mathbf{K}$ of finite structures, these axioms may have some meaning. This will be investigated in a joint work with K. Ikeda and A. Tsuboi [2]. This paper will be a part of this work.

2 Preliminaries

Throughout this paper, $\mathcal{L}$ is a finite relational language. If $M$ is an $\mathcal{L}$-structure and $R$ is a relation with $n$ arguments then $R(M) = \{x \in M^n : M \models R(x)\}$.

Definition 2.1 Suppose $\mathcal{L} = \{R_1, R_2, \ldots, R_l\}$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ where $\alpha_i$ are positive real numbers at most 1.

For a finite $\mathcal{L}$-structure $A$, let

$$\delta(A) = \lvert A \rvert - \lvert R_1(A) \rvert - \lvert R_2(A) \rvert - \cdots - \lvert R_l(A) \rvert.$$
$\delta$ is called a \textit{pre-dimension function}.

Let $B$ be an $\mathcal{L}$-structure and $A \subseteq B$. We write $A \leq B$ if for any subset $X$ of $B$, whenever $A \subseteq X \subseteq B$ then $\delta(A) \leq \delta(X)$.

Let $\mathbf{K}_\alpha = \{ A : \delta(X) \geq 0 \text{ for any } X \subseteq A \}$.

\textbf{Definition 2.2 (Asymmetric amalgam)} Let $B$ and $C$ be $\mathcal{L}$-structures, and let $A = B \cap C$ as a set. $B \oplus_{A} C$ is a structure with universe $B \cup C$ and for each relation $R$ in $\mathcal{L}$, $R(B \oplus_{A} C) = R(B) \cup R(C) - (R(C) \cap A^n)$ where $n$ is the number of arguments of $R$.

Note that $\delta(B \oplus_{A} C) = \delta(B) + \delta(C) - \delta(C|_A)$ where $C|_A$ is a substructure of $C$ with the universe $A$.

Furthermore, if $B \cap C = \{a\}$ (a singleton), then we write $B \oplus_a C$ for $B \oplus_{\{a\}} C$, and if $B \cap C = \emptyset$, then we write $B \oplus C$ for $B \oplus_{\emptyset} C$.

\textbf{Definition 2.3} Let $A$ be an $\mathcal{L}$-structure, $a, b$ elements in $A$, and suppose that for any relation $R \in \mathcal{L}$, there is no tuple $x$ of elements in $A$ such that $A \models R(x)$ and $x$ contains $a$ and $b$.

$A/(a = b)$ is an $\mathcal{L}$-structure obtained from $A$ by identifying $a$ and $b$. More precisely, it is defined as follows: The universe, also denoted $A/(a = b)$, is $A - \{b\}$. Let $f : A \rightarrow A/(a = b)$ be a projection map defined by $f(x) = x$ for $x \neq b$, and $f(b) = a$.

For any relation $R \in \mathcal{L}$, $A/(a = b) \models R(c_1, \ldots, c_n)$ for $c_1, \ldots, c_n \in A/(a = b)$ if and only if $A \models R(c_1', \ldots, c_n')$ for some $c_1', \ldots, c_n' \in A$ such that $f(c_i') = c_i$ for $i = 1, \ldots, n$.

\section{Arithmetic of Structures}

In this section, we work in $\mathbf{K}_\alpha$ for some tuple $\alpha$ of positive real numbers at most 1. We show that there are plenty of structures in $\mathbf{K}_\alpha$ to get structures in $\mathbf{K}_\alpha$ of desired $\delta$-ranks.

\textbf{Definition 3.1} Let $s$ be a real number such that $0 \leq s \leq 2$. A triple $(E, a, b)$ is an $s$-component if $E$ is an $\mathcal{L}$-structure, $a, b \in E$, $a \neq b$, $\delta(\{a\}) = \delta(\{b\}) = 1$, and the following hold:

For any non-empty substructure $X$ of $E$,

(1) $s \leq \delta(X)$ if $a, b \in X$,

(2) $1 \leq \delta(X)$ if $a \notin X$ or $b \notin X$, and

(3) $\delta(E) = s$.

We also say that $E$ is an $s$-component with joints $a$ and $b$, or just $E$ is an $s$-component.

\textbf{Lemma 3.2} Let $s, t, u$ be real numbers such that $0 \leq s, t, u \leq 1$. 
(1) Suppose \((A, a, b)\) is an \((1+s)\)-component, \((B, b, c)\) a \((1+t)\)-component, \(s+t \leq 1\), and \(C = A \oplus_b B\). Then \((C, a, c)\) is an \((1+s+t)\)-component.

(2) Suppose \((A, a, b)\) is an \((1+s)\)-component, \((B, b, c)\) a \((1+t)\)-component, \((C, c, d)\) a \((1+u)\)-component, \(s+t \leq 1\), \(D = A \oplus_b B \oplus_c C\), and \(\overline{D} = D/(a = d)\). Then \((\overline{D}, c, a)\) is an \((s+t+u)\)-component.

(3) Suppose \(k \geq 3\), \(r_1, r_2, \ldots, r_{k-1}, r_k\) are non-negative real numbers such that \(r_1 + r_2 + \cdots + r_{k-1} \leq 1\), and an \(r_i\)-component exists for each \(i\). Then there is an \((r_1 + r_2 + \cdots + r_{k-1} + r_k)\)-component.

**Proof.** (1) Suppose \(X \subseteq C = A \oplus_b B\) and \(X \neq \emptyset\).

We check the condition (1) of the definition of an \((1+s+t)\)-component first. Assume that \(a, c \in X\). Consider the case \(b \in X\). Then \(a, b \in X \cap A, b, c \in X \cap B\), and \(X = (X \cap A) \oplus_b (X \cap B)\). Hence,

\[
\delta(X) = \delta(X \cap A) + \delta(X \cap B) \geq (1+s) + (1+t) - 1 = 1 + (s+t).
\]

Now, consider the case \(b \not\in X\). Then \(b \not\in X \cap A, b \not\in X \cap B\), and \(X = (X \cap A) \oplus_b (X \cap B)\).

Hence,

\[
\delta(X) = \delta(X \cap A) + \delta(X \cap B) \geq 1 + 1 \geq 1 + (s+t).
\]

Therefore, \(a, c \in X\) implies \(\delta(X) \geq 1 + s + t\).

Now, we check the condition (2) of the definition of an \((1+s+t)\)-component. Suppose \(X\) is a non-empty subset of \(C\). Then \(\delta(X) \geq \delta(X \cap A) + \delta(X \cap B) - 1 \geq 1 + 1 - 1 = 1\).

(2) Since there is no relation on a tuple containing \(a\) and \(d\), going from \(D\) to \(\overline{D}\), the number of relations does not change. Therefore, for any \(X \subseteq D\), \(\delta(X/(a = d)) = \delta(X) - 1\) if \(a, d \in X\), and \(\delta(X/(a = d)) = \delta(X)\) if \(a, d \not\in X\).

Suppose \(c, a \in X \subseteq \overline{D}\). Let \(Y \subseteq D\) be such that \(X = Y/(a = d)\). We can assume that \(a, d \in Y\).

Consider the case \(b \in Y\). Since \(a, b, c, d \in Y\), we have \(\delta(Y \cap A) \geq 1 + s\), \(\delta(Y \cap B) \geq 1 + t\), \(\delta(Y \cap C) \geq 1 + u\), and \(Y = (Y \cap A) \oplus_b (Y \cap B) \oplus_c (Y \cap C)\). Hence, \(\delta(Y) \geq 1 + (s + t + u)\). Therefore, \(\delta(X) = \delta(Y/(a = d)) = \delta(Y) - 1 \geq 1 + 1 - 1 = 1\).

Now, consider the case \(b \not\in Y\). Then, \(\delta(Y \cap A) \geq 1\), \(\delta(Y \cap B) \geq 1\), \(\delta(Y \cap C) \geq 1 + s + t + u\), and \(Y = (Y \cap A) \oplus_b (Y \cap B) \oplus_c (Y \cap C)\). We have \(\delta(Y) \geq 1 + (1 + u) \geq 1 + (s + t + u)\) by the assumption that \(s + t \leq 1\). Therefore, \(\delta(X) = \delta(Y/(a = d)) = \delta(Y) - 1 \geq 1 + 1 - 1 = 1\).

We have checked the condition (1) of the definition of \((s+t+u)\)-component.

We turn to check the condition (2) of the definition of \((s+t+u)\)-component.

Suppose \(a \not\in X \subseteq \overline{D}\) and \(X \neq \emptyset\). Then \(X \subseteq D\) and \(a, d \not\in X\). Let \(X_1 = X \cap (A \cup B)\) and \(X_2 = X \cap C\). Since \(s + t \leq 1\), \(A \cup B = A \oplus_b B\) is a \((1+s+t)\)-component by (1), and thus \(\delta(X_1) \geq 1\) by \(a \not\in X\). \(\delta(X_2) \geq 1\) since \(d \not\in X\). Therefore, \(\delta(X) \geq \delta(X_1) + \delta(X_2) - 1 \geq 1\).

Suppose \(c \not\in X\) and \(a \in X\). Let \(Y \subseteq D\) be such that \(X = Y/(a = d)\) and \(a, d \in Y\). Since \(c \not\in Y\), \(\delta(Y) = \delta(Y \cap (A \cup B)) + \delta(Y \cap C) \geq 2\). Therefore, \(\delta(X) = \delta(Y) - 1 \geq 1\). \(\square\)
Lemma 3.3 Suppose $0 \leq s < 1/2$ and a $(1 + s)$-component exists. If $s$ is rational then a 1-component exists. If $s$ is irrational, then for any $\varepsilon > 0$ there exists $\varepsilon'$ such that $\varepsilon > \varepsilon' > 0$ and a $(1 + \varepsilon')$-component exists. With Lemma 3.2, we can choose a 1-component or a $(1 + \varepsilon')$-component arbitrarily large.

Proof. Suppose $0 \leq s < 1/2$ and a $(1 + s)$-component exists. Consider the following non-negative decreasing sequence $\{s_i\}_{i<\omega}$: Let $s_0 = s$. If $s_i = 0$ then let $s_{i+1} = 0$. If $s_i > 0$ then choose the least positive integer $m_i$ such that $m_i s_i \geq 1$ and let $s_{i+1} = m_i s_i - 1$.

By Lemma 3.2 (2), there is a $(1 + s_i)$-component for each $i < \omega$.

Note that if $s_i > 0$ then $s_i > s_{i+1} \geq 0$. If $s_0 = s$ is rational then $s_i = 0$ eventually. Hence, 1-component exists.

If $s_0 = s$ is irrational, $s_i$ converges to 0. This should be well-known, but we give a proof for convenience. Since $\{s_i\}_{i<\omega}$ is a decreasing sequence, $\{m_i\}_{i<\omega}$ is an increasing sequence. If $\{m_i\}_{i<\omega}$ is unbounded, then $s_i$ converges to 0 because $s_i < 1/(m_i-1)$. If $\{m_i\}_{i<\omega}$ is bounded, $m_i = m$ for some positive integer $m$ eventually. So, we can assume that $s_{i+1} = ms_i - 1$ for all $i$. Since $s_i$ converges to some $\beta > 0$, $\beta = m\beta - 1$. Therefore, $\beta = 1/(m-1)$. But $s_i > \beta = 1/(m-1)$ contradicts $(m-1)s_i = (m-1)s_0 < 1$.

Lemma 3.4 Let $(V, E)$ be a finite graph where $V$ is a set of vertices and $E$ a set of edges, and let $\delta(V, E) = |V| - \beta|E|$ with $0 < \beta \leq 1$. Then a graph which is a $(1 + s)$-component exists for some real number $s$ such that $0 \leq s < 1/2$ with respect to this $\delta$.

Proof. We remark first that any two different points in each $(1 + s)$-component below will be its joints.

Suppose $\alpha_i = 1/2$. A graph with 4 points and 6 edges ($K_4$) is a 1-component.

Suppose $1/2 < \alpha_i < 1$. Let $s = 1 - \alpha_i$. Then a graph with 2 points and 1 edge is a $(1 + s)$-component. These 2 points will be joints.

Suppose $0 < \alpha_i < 1/2$. Choose least natural number $k$ such that $k - \alpha \left(\frac{k}{2}\right) < 1$.

Note that $k \leq 4$ implies $k - \alpha \left(\frac{k}{2}\right) > 1$. Therefore, $k \geq 5$. Consider $K_k$, a complete graph with $k$ points. Then for any non-empty proper subset $X$ of $K_k$, $\delta(X) \geq 1$.

Starting from $K_k$, remove 1 edge at a time. Then the $\delta$-value of the graph increases by $\alpha_i$ at a time. Repeat this process until the $\delta$-value of the graph exceeds 1. Let $E$ be the graph with $k$ points obtained by this process. We eventually get $E$ because $k$ points with no edge has the $\delta$-value $k \geq 5$. If $\delta(E) = 1$, then $E$ is a 1-component. If not, $1 < \delta(E) < 1 + \alpha_i < 1 + 1/2$ since if we put one more edge to $E$, the $\delta$-value will be less than 1. \qed

If we have an $s$-component as a sufficiently large graph, then we can make an $s$-component as a structure with an $n$-ary relation for any $n$. 


Lemma 3.5 Let $\beta$ be a real number such that $0 < \beta \leq 1$. Suppose $(A, E)$ is a sufficiently large binary graph. Then there is $S \subseteq A^n$ such that $|A| - \beta|E| = |A| - \beta|S|$ and $|X| - \beta|E_X| \leq |X| - \beta|S_X|$ for any $X \subseteq A$. Here, $E_X$ is the set of edges in $E$ connecting vertices in $X$, and $S_X = X^n \cap S$. In particular, if $(A, E)$ is an $s$-component then so is $(A, S)$.

Proof. Let $f : [A]^2 \to A^n$ be a one-to-one map such that $a, b$ are members of $f(\{a, b\})$ for any distinct points in $A$ ([A]^2 is the set of two point subsets of $A$). We can choose such $f$ if $|A| > 2n$. Let $S = f(E)$ (Consider each edge as the set of two end points). Then $|A| - \beta|E| = |A| - \beta|S|$. Also, for any $X \subseteq A$, if $f(e) \in X$ then $e \in E_X$. Hence, $|S_X| \leq |E_X|$. Therefore, $|X| - \beta|E_X| \leq |X| - \beta|S_X|$.

As a corollary, we get the following proposition.

Proposition 3.6 If a rational number is a member of $\alpha$ then a 1-component exists in $K_\alpha$.

If an irrational number is a member of $\alpha$, then for any $\varepsilon > 0$, there are $\varepsilon'$ and $\varepsilon''$ such that $\varepsilon > \varepsilon' > \varepsilon'' > 0$ and a $(1 + \varepsilon')$-component and a $(1 - \varepsilon'')$-component exist in $K_\alpha$.

Lemma 3.7 Suppose that $r_1 + r_2 + \cdots + r_k - n = r$ where the $r_i$'s are positive real numbers at most 1, $-1 \leq r \leq 1$, and $n$ a natural number. Assume further that a 1-component exists and so does $(1 + r_i)$-component for each $i = 1, \ldots, k$. Then $(1 + r)$-component exists.

Proof. We prove the lemma by induction on $n$. The lemma holds for $n = 0$ by Lemma 3.2 (1). Suppose $n \geq 1$. Let $i$ be the maximum suffix such that $r_1 + \cdots + r_i < 1$. Since a 1-component exists, we can assume that $i \geq 2$.

If $i = k$ then $n$ must be 1. Therefore, there is a $(1 + r)$-component by Lemma 3.2 (3), and we are done.

Suppose $i < k$. Then $r_1 + \cdots + r_i + r_{i+1} \geq 1$. Let $q = r_1 + \cdots + r_i + r_{i+1} - 1 \geq 0$. By Lemma 3.2 (3), there is a $(1 + q)$-component. We have $q + r_{i+2} + \cdots + r_k - (n-1) = r$. Therefore, $r$-component exists by the induction hypothesis.

Proposition 3.8 Suppose that a rational number is a member of $\alpha$.


2. If a reduced fraction $k/m$ is a member of $\alpha$ then a $(1 + 1/m)$-component exists in $K_\alpha$.

3. If a $(1 + 1/m)$-component and a $(1 + 1/m')$-component exist in $K_\alpha$ then a $(1 + 1/\text{lcm}(m, m'))$-component exists in $K_\alpha$. Here, $\text{lcm}(m, m')$ is the least common multiple of $m$ and $m'$. 


(4) If a \((1 + 1/m)\)-component exists in \(K_\alpha\) then a \((1 - 1/m)\)-component exists in \(K_\alpha\).

**Proof.** (1) is in Proposition 3.6.

(2) Suppose a reduced fraction \(k/m = \alpha_i\) for some \(i\). Then a graph with 2 points and 1 edge is a \((1 + (m - k)/m)\)-component for \(\delta_{\alpha_i}\) defined by \(\delta_{\alpha_i}(A, E) = |A| - \alpha_i|E|\). Since there is a graph which is a 1-component for \(\delta_{\alpha_i}\), we can make a \((1 + (m - k)/m)\)-component arbitrarily large. By Lemma 3.5, \((1 + (m - k)/m)\)-component exists in \(K_\alpha\).

Let \(k' = m - k\). \(k'\) and \(m\) are prime each other. Therefore, there are positive integers \(u, v\) such that \(uk' - vm = 1\). Hence, \(u(k'/m) - v = 1/m\). There is a \((1 + 1/m)\)-component in \(K_\alpha\) by Lemma 3.7.

(3) Let \(d\) be the greatest common divisor of \(m\) and \(m'\). Then there are positive integers \(u < m'\) and \(v < m\) such that \(um - vm' = d\). Then \(u(1/m') - v(1/m) = d/(mm')\) and thus \(u(1/m') + (m - v)(1/m) - 1 = d/(mm')\). Therefore, there is a \((1 + d/(mm'))\)-component in \(K_\alpha\) by Lemma 3.7.

(4) Also, by Lemma 3.7.

\[\square\]

4 A Property of \(K_\alpha\)

In this section, we show a special property of \(K_\alpha\) with which we can show that the elementary theory of the generic structure of \(K_\alpha\) is axiomatized by Axioms 1 and 2 in the introduction. This will be investigated in [2].

**Lemma 4.1** Suppose \(0 \leq s, t \leq 1\), \(C = B \oplus_a E\), and \((E, a, b)\) is a \((1 + t)\)-component. If \(\delta(A) + s \leq \delta(X)\) for any \(X\) such that \(A \subseteq X \subseteq B\) and \(a \in B - A\), then \(\delta(A) + s \leq \delta(X)\) for any \(X\) such that \(A \not\subseteq X \subseteq C\).

**Proof.** Suppose \(A \not\subseteq X \subseteq C\). If \(a \not\in X\), then \(A \subseteq (X \cap B), a \not\in (X \cap E)\), and \(X = (X \cap B) \oplus (X \cap E)\). If \((X \cap E) = \emptyset\), then \(A \not\subseteq X \subseteq B\), and thus \(\delta(X) \geq \delta(A) + s\). If \((X \cap E) \neq \emptyset\), then \(\delta(X \cap E) \geq 1\) and thus \(\delta(X) = \delta(X \cap B) + \delta(X \cap E) \geq \delta(A) + 1 \geq \delta(A) + s\).

If \(a \in X\), then \(X = (X \cap B) \oplus_a (X \cap E), A \subseteq X \cap B\), and \(X \cap E \neq \emptyset\). Hence,

\[
\delta(X) = \delta(X \cap B) + \delta(X \cap E) - 1 \geq (\delta(A) + s) + (1 + t) - 1 \geq \delta(A) + s.
\]

\[\square\]

**Lemma 4.2** Suppose \(0 \leq t \leq s \leq 1\), \(C = B \oplus_a E\), and \((E, a, b)\) is a \((1 - t)\)-component. If \(\delta(A) + s \leq \delta(X)\) for any \(X\) such that \(A \not\subseteq X \subseteq B\) and \(a \in B - A\), then \(\delta(A) + (s - t) \leq \delta(X)\) for any \(X\) such that \(A \not\subseteq X \subseteq C\).

\[\square\]
Proof. Suppose $A \subseteq X \subseteq C$. If $a \notin X$, then $A \subseteq (X \cap B)$, $a \notin (X \cap E)$, and $X = (X \cap B) \oplus (X \cap E)$. If $(X \cap E) = \emptyset$, then $A \subseteq X \subseteq B$, and thus $\delta(X) \geq \delta(A) + s \geq \delta(A) + s - t$. If $(X \cap E) \neq \emptyset$, then $\delta(X \cap E) \geq 1$ and thus $\delta(X) = \delta(X \cap B) + \delta(X \cap E) \geq \delta(A) + 1 \geq \delta(A) + s - t$.

If $a \in X$, then $X = (X \cap B) \oplus (X \cap E)$, $A \subseteq X \subseteq B$, and $X \cap E \neq \emptyset$. Hence,

$$
\delta(X) = \delta(X \cap B) + \delta(X \cap E) - 1 \geq (\delta(A) + s) + (1 - t) - 1 \geq \delta(A) + s - t.
$$

\[\square\]

Theorem 4.3 Consider $\mathbb{K}_\alpha$ where $\alpha$ consists of rational numbers only. Suppose $A$, $B$ are $\mathcal{L}$-structures in $\mathbb{K}_\alpha$ and $A \subseteq B$. Then for any positive integer $n$, there is an $\mathcal{L}$-structure $C$ in $\mathbb{K}_\alpha$ such that $B \subseteq C$, $B \leq_n C$, $A \leq C$ and $\delta(C) = \delta(A)$.

Proof. We prove the theorem by the induction on $|B - A|$.

Suppose $|B - A| = 0$. In this case, $A = B$. Let $C = B = A$. Then the statement holds.

Suppose $|B - A| > 0$. Let $B_0$ be a substructure of $B$ such that $A \subseteq B_0 \subseteq B$ and $\delta(B_0) \leq \delta(X)$ for any set $X$ such that $A \subseteq X \subseteq B$.

Note that we have $A \leq B_0$ and $B_0 \leq B$.

Suppose $\delta(A) = \delta(B_0)$. We have $|B - B_0| < |B - A|$. By the induction hypothesis, for any integer $n > 0$, there is a structure $C \supseteq B$ such that $B \leq_n C$, $B_0 \leq C$ and $\delta(C) = \delta(B_0)$. Since $A \leq B_0$ and $\delta(A) = \delta(B_0)$, we have the statement.

Suppose $\delta(A) < \delta(B_0)$. If $a$ is a point in $B - A$ then $\delta\{a\} / A \leq 1$. Therefore, $\delta(B_0 / A) \leq 1$ by the choice of $B_0$. Let $s = \delta(B_0) - \delta(A) \leq 1$. Then $\delta(A) + s \leq \delta(X)$ for any substructure $X$ of $B_0$ such that $A \subseteq X$.

Let $\alpha = (n_1 / m_1, n_2 / m_2, \ldots, n_l / m_l)$ where each $n_i / m_i$ is a reduced fraction. Let $m$ be the least common multiple of $m_1, m_2, \ldots, m_l$. Then $s = \delta(B_0) - \delta(A)$ is a multiple of $1 / m$. Let $s = k / m$ where $k$ is a positive integer.

For any positive integer $n$, let $C'$ be an $\mathcal{L}$-structure such that

$$
C' = B \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_n} E_n \oplus_{a_{n+1}} F_1 \oplus_{a_{n+2}} F_2 \cdots \oplus_{a_{n+k}} F_k,$$

where the $a_i$ are pairwise distinct, $a_0 \in B_0 - A$, $(E_i, a_i, a_{i+1})$ is a 1-component for each $i = 0, 1, \ldots, n$, and $(F_j, a_{n+j}, a_{n+j+1})$ is a $(1 - 1 / m)$-component for $j = 1, 2, \ldots, k$. Let $C_0$ be a substructure of $C'$ such that

$$
C_0 = B_0 \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_n} E_n \oplus_{a_{n+1}} F_1 \oplus_{a_{n+2}} F_2 \cdots \oplus_{a_{n+k}} F_k.
$$

By Lemmas 4.1 and 4.2, we have $A \leq C_0$ and $\delta(A) = \delta(C_0)$. Therefore, $C_0 \in \mathbb{K}_\alpha$.

We show that $B_0 \leq_n C_0$. Suppose $X \subset C_0 - B_0$ and $|X| \leq n$. Then $X = \{a_0, a_1, \ldots, a_{n-1}\}$ or $a_i \notin X$ for some $i < n$. In either case, $X \subset (B_0 \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_j} E_j) \oplus D$ for some $j \leq n$ and $D \in \mathbb{K}_\alpha$. Since $B_0 \leq B_0 \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_j} E_j$, by Lemma 4.1, $\delta(B_0) \leq \delta(X)$. 

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Now we have $C' = B \oplus_{B_0} C_0$, $B_0 \leq C$, and $B_0 \leq_n C_0$. Hence, $C_0 \leq C'$ and $B \leq_n C'$. Since $|C' - C_0| = |B - B_0| < |B - A|$, we have $C \in K_\alpha$ such that $C \supset C'$, $C' \leq_n C$, $C_0 \leq C$ and $\delta(C) = \delta(C_0)$. Therefore, $B \leq_n C$, $A \leq C$, and $\delta(C) = \delta(A)$.

\[ \square \]

**Theorem 4.4** Suppose $A$, $B$ are $\mathcal{L}$-structures in $K_\alpha$ and $A \leq B$. Then for any real number $\epsilon > 0$ and for any positive integer $n$, there is an $\mathcal{L}$-structure $C$ in $K_\alpha$ such that $B \subset C$, $B \leq_n C$, $A \leq C$ and $\delta(C) < \delta(A) + \epsilon$.

**Proof.** If $\alpha$ consists of rational numbers only, then the statement holds by Theorem 4.3.

Assume that $\alpha$ contains an irrational number. The proof for this case is similar to that of Theorem 4.3. So, we give only a sketch. Choose $B_0$ as in the proof of Theorem 4.3. Assume that $s = \delta(B_0) - \delta(A) > 0$. Let $\epsilon > 0$ be an arbitrary (small) real number. Let

$$C' = B \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_n} E_n \oplus_{a_{n+1}} F_1 \oplus_{a_{n+2}} F_2 \cdots \oplus_{a_{n+k}} F_k$$

as in the proof of Theorem 4.3, except that $(E_i, a_i, a_{i+1})$ is a $(1 + t)$-component with $t$ sufficiently small for each $i = 0, 1, \ldots, n$ so that

$$\delta(B_0 \oplus_{a_0} E_0 \oplus_{a_1} E_1 \cdots \oplus_{a_n} E_n) < \delta(B_0) + \epsilon/4,$$

and $(F_j, a_{n+j}, a_{n+j+1})$ is a $(1 - t')$-component with $0 < t' < \epsilon/4$ for $j = 1, 2, \ldots, k$, where $k$ is the largest integer such that $kt' < s$. Then $A \leq C_0$, and $\delta(C_0) < \delta(A) + \epsilon/2$ for $C_0$ as in the proof of Theorem 4.3. Then we have $C_0 \leq C'$ and $B \leq_n C'$.

By the induction hypothesis, we can choose $C \in K_\alpha$ such that $C' \leq_n C$, $C_0 \leq C$, and $\delta(C) < \delta(C_0) + \epsilon/2$. Therefore, we have the theorem.

\[ \square \]

**References**

