弱い0-最小化代数的構造（ゼリック幾何と数論幾何）

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Weakly o-minimal algebraic structures

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1 Introduction

Let $M$ be a linearly ordered structure and $A$ a subset $M$. The set $A$ is said to be convex if for all $a, b \in A$ and $c \in M$ with $a < c < b$ we have $c \in A$. A linearly ordered structure $M$ is said to be o-minimal if every definable subset of $M$ is a finite union of intervals (possibly with infinite endpoints). A linearly ordered structure $M$ is said to be weakly o-minimal if every definable subset of $M$ is a finite union of convex sets. A theory $T$ is said to be weakly o-minimal if every model of $T$ is weakly o-minimal. Henceforth, a linearly ordered structure is abbreviated as an ordered structure.

It is well-known the following fact.

**Fact 1** Let $M$ be an ordered structure. Then the following is equivalent:

1. $\text{Th}(M)$ is weakly o-minimal;
2. for each formula $\varphi(x, \overline{y})$ there exists some $n \in \omega$ such that for each tuple $\overline{a}$ from $M$ the set $\varphi(M, \overline{a})$ can be written as a union of at most $n$ many convex sets.

**Fact 2** Let $M$ be a weakly o-minimal structure. If $M$ is $\omega$-saturated, then $\text{Th}(M)$ is weakly o-minimal.

**Fact 3** [BP] Let $M$ be an expansion of an o-minimal structure by convex subsets. Then $\text{Th}(M)$ is weakly o-minimal.
2 Monoids and groups

In this section, we study weakly o-minimal monoids and groups. It is well-known the following fact.

**Fact 4** [MMS] Let $G$ be a weakly o-minimal group. Suppose that $H$ is a definable subgroup of $G$. Then, the following holds:

1. $G$ is abelian and divisible;
2. $H$ is convex.

Let $G$ be a weakly o-minimal group. Suppose that $H$ is a definable subgroup of $G$. Then, by Fact 4, $H$ is divisible.

We call an ordered group $(G, 0, +, <, \ldots)$ Archimedean if for all elements $a, b$ with $b > 0$ there exists some $n \in \omega$ such that $a < nb$.

**Lemma 5** Let $G = (G, 0, +, <, \ldots)$ be a weakly o-minimal Archimedean group. Suppose that $H$ is a definable subgroup of $G$. Then $H$ is either $\{0\}$ or $G$.

**Proof.** Let $a \in G$. Without loss of generality, we may assume $a > 0$. Let $H \neq \{0\}$. Then, there exists some $b \in H$ such that $b > 0$. Since the group $G$ is Archimedean, there exists some $n \in \omega$ such that $a < nb$. Hence, by Fact 4, we have $a \in H$. \qed

From now on, we study monoids.

**Proposition 6** Let $N = (N, 0, +, <, \ldots)$ be a weakly o-minimal monoid. Then $N$ is commutative.

**Proof.** For all $a \in N$, let $C_N(a) := \{x \in N \mid x + a = a + x\}$.

**Claim** $C_N(a)$ is convex.

Clearly, $0 \in C_N(a)$ and, if $x, y \in C_N(a)$ then $x + y \in C_N(a)$. By weak o-minimality, $C_N(a)$ is the union of finitely many maximal convex subsets. Let $X$ be the greatest of these convex components with respect to the ordering induced by $<$. Let $x \in X$ with $x > 0$. Suppose that $y \in N$ with $0 < y < x$. We may show that $y \in C_N(a)$. By $x - y + x < 2x$ and $2x \in X$, we have $y + x \in X$. Hence $(y + x) + a = a + (y + x)$. By $x \in C_N(a)$, we have $(y + a) + x = (a + y) + x$. Hence, we have $y + a = a + y$. Thus, $y \in C_N(a)$, as desired.

Let $b, c \in N$ with $b < c$. Then the following is equivalent:
• $b$ and $c$ are commutative;
• $b$ and $b + c$ are commutative;
• $b + c$ and $c$ are commutative.

Now $b, b + c \leq 0$ or $b + c, c \geq 0$. Hence we may assume $0 < b < c$. Then, as $C_N(c)$ is convex, we have $b \in C_N(c)$. Therefore $N$ is commutative. \[\square\]

Let $\mathcal{N} = (N, 0, +, <, \ldots)$ be an ordered monoid. Suppose that $I_N := \{x \in N \mid N \models \exists y(x + y = 0)\}$. Clearly, $I_N$ contains 0. We call an ordered monoid $(N, 0, +, <, \ldots)$ Archimedean if for all elements $a, b$ with $b > 0$ there exists some $n \in \omega$ such that $a < nb$, and for all elements $a, b$ with $b < 0$ there exists some $n \in \omega$ such that $nb < a$.

**Example 7** Let $\mathcal{M} = (\{0\} \cup \mathbb{Q}_{\leq 1}, 0, +, <, P)$, where $\mathbb{Q}_{\leq 1} = \{a \in \mathbb{Q} \mid a \geq 1\}$ and the unary predicate symbol $P$ is interpreted by the convex set $P_{\mathcal{M}} = (\sqrt{2}, 3) \cap \mathbb{Q}$. Then, $\mathcal{M}$ is a weakly o-minimal Archimedean monoid and not divisible. Moreover $I_{\mathcal{M}} = \{0\}$.

Hence, in generally a weakly o-minimal Archimedean monoid is not a group. However the following holds.

**Proposition 8** Let $\mathcal{N} = (N, 0, +, <, \ldots)$ be a weakly o-minimal Archimedean monoid. Suppose that $I_N \neq \{0\}$. Then $\mathcal{N}$ is a group.

**Proof.** Clearly $0 \in I_N$. Let $x, y \in I_N$. Then, there exist $x_1, y_1$ such that $x + x_1 = 0$ and $y + y_1 = 0$. Then $(x + y) + (y_1 + x_1) = 0$. Thus, $x + y \in I_N$. Let $g \in I_N$.

**Claim** $I_N$ is convex.

By weak o-minimality, $I_N$ is the union of finitely many maximal convex subsets. Let $C$ be the greatest of these convex components with respect to the ordering induced by $<$. Let $x \in C$ with $x > 0$. Suppose that $y \in N$ with $0 < y < x$. We may show that $y \in I_N$. By $x < y + x < 2x$ and $2x \in C$, we have $y + x \in C$. Hence, there exists some $z \in N$ such that $(y + x) + z = 0$. Thus, $y \in I_N$, as desired.

Let $g \in N$. By $I_N \neq \{0\}$, there exists some $a \in I_N$ such that $a \neq 0$. Without loss of generality, we may assume that $g > 0$ and $a > 0$. As $N$ is Archimedean, there exists some $n \in \omega$ such that $0 < g < na$. Since $I_N$ is convex, we have $g \in I_N$. Therefore $I_N = N$. \[\square\]
Let $N$ be an ordered monoid and $A$ a subset $N$. The ordered monoid $N$ is said to be rich, if for all $a, b \in N$ if $0 \leq a \leq b$ or $b \leq a \leq 0$, then there exists some $c \in N$ such that $b = a + c$. The set $A$ admits right elimination, if for all $a \in A$ and all $b \in N$ if $b + a \in A$, then $b \in A$.

**Example 9** Let $\mathcal{M} = (\mathbb{Q}^{\geq 0}, 0, +, <, P)$, where $\mathbb{Q}^{\geq 0} = \{a \in \mathbb{Q} \mid a \geq 0\}$ and the unary predicate symbol $P$ is interpreted by the convex set $P^{\mathcal{M}} = (\sqrt{2}, 3) \cap \mathbb{Q}$. Then, $\mathcal{M}$ is a weakly $o$-minimal rich monoid and divisible.

**Proposition 10** Let $\mathcal{N} = (N, 0, +, <, \ldots)$ be a weakly $o$-minimal monoid. Then the following is equivalent:

1. $\mathcal{N}$ is divisible;
2. for all $n \in \omega$, $nN$ admits right elimination;
3. for all $n \in \omega$, $nN$ is convex.

**Proof.** (1 $\Rightarrow$ 2) It is clear.

(2 $\Rightarrow$ 3) Let $n \in \omega$. Let $x, y \in nN$. Then there exist $x_{1}, y_{1} \in N$ such that $x = nx_{1}$ and $y = ny_{1}$. By Proposition 6, we have $x + y = nx_{1} + ny_{1} = n(x_{1} + y_{1})$ Hence, $x + y \in nN$. Now, by weak $o$-minimality, $nN$ is the union of finitely many maximal convex subsets. Let $C$ be the greatest of these convex components with respect to the ordering induced by <. Let $x \in C$ with $x > 0$. Suppose that $y \in N$ with $0 < y < x$. We may show that $y \in nN$. By $x < y + x < 2x$ and $2x \in C$, we have $y + x \in C$. As $nN$ admits right elimination, we have $y \in nN$, as desired.

(3 $\Rightarrow$ 1) Let $n$ be a nonzero natural number. For all positive $a \in N$, we have $0 < a < na$. As $nN$ is convex, we have $a \in nN$. Hence $\mathcal{N}$ is divisible. 

**Proposition 11** Let $\mathcal{N} = (N, 0, +, <, \ldots)$ be a weakly $o$-minimal monoid. If $\mathcal{N}$ is rich, then $\mathcal{N}$ is divisible.

**Proof.** Let $n$ be a nonzero natural number. Now, by weak $o$-minimality, $nN$ is the union of finitely many maximal convex subsets. Let $C$ be the greatest of these convex components with respect to the ordering induced by <. Let $x \in C$ with $x > 0$. Suppose that $y \in N$ with $0 < y < x$. We show that $y \in nN$. By $x < y + x < 2x$ and $2x \in C$, we have $y + x \in C$. So there exist $z_{1}, z_{2} \in N$ with $0 < z_{1} < z_{2}$ such that $x = nz_{1}$ and $y + x = nz_{2}$. 


As $\mathcal{N}$ is rich, there exists some $a \in N$ such that $a + z_1 = z_2$. Hence, we have $y + nz_1 = na + nz_1$. Therefore we have $y = na \in nN$. It follows that $nN = N$.

\[\square\]

**Proposition 12** [T] Let $N$ be an ordered monoid. Suppose that $\text{Th}(N)$ is weakly o-minimal. Then there exists an extending ordered group $G$ of $N$ such that $\text{Th}(G)$ is weakly o-minimal.

**Proof.** Let $N_1$ be an $\omega$-saturated elementary extension of $N$. Define the following relation on $N_1 \times N_1$:

$$(a,b) \sim (a',b') \iff a + b' = a' + b.$$  

Then $\sim$ is an equivalence relation on $N_1 \times N_1$. For each $(a,b) \in N_1 \times N_1$, let $[(a,b)]$ denote the $\sim$-class of $(a,b)$. Let $G := N_1 \times N_1 / \sim$. Then $G$ can be naturally expanded to an $\omega$-saturated ordered group. We may treat $N_1$ as a substructure of $G$ by identifying $a \in N_1$ and $[(a,0)] \in G$. We may show that $G$ is weakly o-minimal. By way of a contradiction, assume that $G$ is not weakly o-minimal. Then there exists a definable subset $A \subseteq G$ and a monotone sequence $\{a_i \in G \mid i \in \omega\}$ such that for all $i \in \omega$, $a_i \in A$ if and only if $i$ is even. As $G$ is an eq-object of $N_1$, there exists a formula $\varphi(x,y)$ (parameters from $N_1$) such that $[(b,c)] \in A$ if and only if $N_1 \models \varphi(b,c)$. For all $i \in \omega$, let $a_i := [(b_i, c_i)]$. Then we have

$$N_1 \models \varphi(b_i, c_i) \iff i \text{ is even.}$$

For all $n \in \omega$, let $d_i := \Sigma_{j=0,j \neq i}^{2n} c_i$ and $e := \Sigma_{j=0}^{2n} c_i$. Then we have

$$N_1 \models \varphi(b_i + d_i, e) \iff i \text{ is even.}$$

Hence, the set $\varphi(N_1,e)$ can not be written as the union of $n$ convex sets, contradicting that $\text{Th}(N)$ is weakly o-minimal.

\[\square\]

3 Rings and fields

In this section, we study weakly o-minimal rings and fields.

A commutative ordered domain $R$ is said to be real closed if $R$ has intermediate value property, that is, for any polynomial $p(x)$ with coefficients in
\( R \) and any \( a, b \in R \) such that \( a < b \) and \( p(a) \cdot p(b) < 0 \), there exists some \( c \in R \) so that \( a < c < b \) and \( p(c) = 0 \).

It is well-known the following fact.

**Fact 13** [MMS]

1. If a commutative ordered ring \( R \) is weakly o-minimal, then \( R \) is a real closed ring.

2. If an ordered field \( F \) is weakly o-minimal, then \( F \) is a real closed field.

In [PS1], it is shown that an o-minimal ring is a real closed field. However, in generally a weakly o-minimal ordered ring is not a field. We shall show that if a weakly o-minimal ordered ring \( R \) which may not be associative is Archimedean, then \( R \) is a real closed field.

**Lemma 14** If \( \mathcal{R} = (R, 0, 1, +, \cdot, <, \ldots) \) is a weakly o-minimal ring, then \( \mathcal{R} \) is commutative.

**Proof.** For all \( a \in R \), let \( C_R(a) := \{ x \in R \mid xa = ax \} \). Then, \( C_R(a) \) is a definable additive subgroup. Hence, by Fact 4, \( C_R(a) \) is convex. Let \( g, h \in R \). Without loss of generality, we may assume that \( 0 < g < h \). As \( C_R(h) \) is convex, we have \( g \in C_R(h) \). It follows that \( \mathcal{R} \) is commutative. \( \square \)

We call an ordered ring \( (R, 0, 1, +, \cdot, <, \ldots) \) standard if for all nonzero \( a \in R \) there exists \( b \in R \) such that \( 1 < ab \). Clearly, an Archimedian ordered ring is standard.

**Proposition 15** Let \( \mathcal{R} = (R, 0, 1, +, \cdot, <, \ldots) \) be a weakly o-minimal ring. Then, the following is equivalent:

1. \( \mathcal{R} \) is standard;

2. \( \mathcal{R} \) is a field.

**Proof.** \((2 \Rightarrow 1)\) Let \( a \in R \) with \( a \neq 0 \). Then, as \( \mathcal{R} \) is field, there exists \( a^{-1} \). Hence, \( 1 < a \cdot 2a^{-1} = 2 \), as desired.

\((1 \Rightarrow 2)\) Let \( a \in R \). Then, as \( \mathcal{R} \) is standard, there exists some \( b \in R \) such that \( 1 < ab \). Now \( aR \) is a definable additive subgroup. Hence, as \( aR \) is convex, we have \( 1 \in aR \). It follows that \( \mathcal{R} \) is a field. \( \square \)
Corollary 16 Let $\mathcal{R} = (R, 0, 1, +, \cdot, <, \ldots)$ be a weakly o-minimal Archimedean ring, where $\mathcal{R}$ may not be associative. Then, $\mathcal{R}$ is a real closed field.

Proof. By Fact 13, Lemma 14 and Proposition 15, we may show that $\mathcal{R}$ is associative. Let $a \in R$ with $a \neq 0$. Suppose that $D_R(a) := \{x \in R \mid (xa)a = x(aa)\}$. Then, as $\mathcal{R}$ is commutative, $D_R(a)$ contains $a$ and is a definable additive subgroup. Hence, by Lemma 5, $D_R(a) = R$. Also, suppose that $E_R(a) := \{x \in R \mid (xa)x = z(ax)\}$ for each $z$. Then, by $D_R(a) = R$, $E_R(a)$ contains $a$ and is a definable additive subgroup. Thus, by Lemma 5, $E_R(a) = R$. It follows that $\mathcal{R}$ is associative.

Proposition 17 Let $R$ be an ordered ring. Suppose that $\text{Th}(R)$ is weakly o-minimal. Then there exists an extending ordered field $F$ of $R$ such that $\text{Th}(F)$ is weakly o-minimal.

Proof. Let $R_1$ be an $\omega$-saturated elementary extension of $R$. Let $R_1^{\geq 0} := \{a \in R_1 \mid a > 0\}$. Define the following relation on $R_1 \times R_1^{\geq 0}$:

$$(a, b) \sim (a', b') \iff ab' = a'b.$$ 

Then $\sim$ is an equivalence relation on $R_1 \times R_1^{\geq 0}$. For each $(a, b) \in R_1 \times R_1^{\geq 0}$, let $[(a, b)]$ denote the $\sim$-class of $(a, b)$. Let $F := R_1 \times R_1^{\geq 0}/\sim$. Then $F$ can be naturally expanded to an $\omega$-saturated ordered field. We may treat $R_1$ as a substructure of $F$ by identifying $a \in R_1$ and $[(a, 1)] \in F$. We may show that $F$ is weakly o-minimal. By way of a contradiction, assume that $F$ is not weakly o-minimal. Then there exists a definable subset $A \subseteq F$ and a monotone sequence $\{a_i \in F \mid i \in \omega\}$ such that for all $i \in \omega$, $a_i \in A$ if and only if $i$ is even. As $F$ is an eq-object of $R_1$, there exists a formula $\varphi(x, y)$ (parameters from $R_1$) such that $[(b, c)] \in A$ if and only if $R_1 \models \varphi(b, c)$. For all $i \in \omega$, let $a_i := [(b_i, c_i)]$. Then we have

$$R_1 \models \varphi(b_i, c_i) \iff i \text{ is even.}$$

For all $n \in \omega$, let $d_i := \Pi_{j=0, j \neq i}^{2n}c_j$ and $e := \Pi_{j=0}^{2n}c_j$. Then we have

$$R_1 \models \varphi(b_id_i, e) \iff i \text{ is even.}$$

Hence, the set $\varphi(R_1, e)$ can not be written as the union of $n$ convex sets, contradicting that $\text{Th}(R)$ is weakly o-minimal. \qed
References


