Weakly o-minimal algebraic structures

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1 Introduction

Let $M$ be a linearly ordered structure and $A$ a subset $M$. The set $A$ is said to be convex if for all $a, b \in A$ and $c \in M$ with $a < c < b$ we have $c \in A$. A linearly ordered structure $M$ is said to be o-minimal if every definable subset of $M$ is a finite union of intervals (possibly with infinite endpoints). A linearly ordered structure $M$ is said to be weakly o-minimal if every definable subset of $M$ is a finite union of convex sets. A theory $T$ is said to be weakly o-minimal if every model of $T$ is weakly o-minimal. Henceforth, a linearly ordered structure is abbreviated as an ordered structure.

It is well-known the following fact.

**Fact 1** Let $M$ be an ordered structure. Then the following is equivalent:

1. $\text{Th}(M)$ is weakly o-minimal;

2. for each formula $\varphi(x, \overline{y})$ there exists some $n \in \omega$ such that for each tuple $\overline{a}$ from $M$ the set $\varphi(M, \overline{a})$ can be written as a union of at most $n$ many convex sets.

**Fact 2** Let $M$ be a weakly o-minimal structure. If $M$ is $\omega$-saturated, then $\text{Th}(M)$ is weakly o-minimal.

**Fact 3** [BP] Let $M$ be an expansion of an o-minimal structure by convex subsets. Then $\text{Th}(M)$ is weakly o-minimal.
2 Monoids and groups

In this section, we study weakly $o$-minimal monoids and groups. It is well-known the following fact.

**Fact 4** [MMS] Let $G$ be a weakly $o$-minimal group. Suppose that $H$ is a definable subgroup of $G$. Then, the following holds:

1. $G$ is abelian and divisible;
2. $H$ is convex.

Let $G$ be a weakly $o$-minimal group. Suppose that $H$ is a definable subgroup of $G$. Then, by Fact 4, $H$ is divisible.

** Lemma 5** Let $\mathcal{G} = (G, 0, +, <, \ldots)$ be a weakly $o$-minimal Archimedean group. Suppose that $H$ is a definable subgroup of $\mathcal{G}$. Then $H$ is either $\{0\}$ or $\mathcal{G}$.

**Proof.** Let $a \in G$. Without loss of generality, we may assume $a > 0$. Let $H \neq \{0\}$. Then, there exists some $b \in H$ such that $b > 0$. Since the group $\mathcal{G}$ is Archimedean, there exists some $n \in \omega$ such that $a < nb$. Hence, by Fact 4, we have $a \in H$. $\square$

From now on, we study monoids.

**Proposition 6** Let $\mathcal{N} = (N, 0, +, <, \ldots)$ be a weakly $o$-minimal monoid. Then $\mathcal{N}$ is commutative.

**Proof.** For all $a \in N$, let $C_N(a) := \{x \in N \mid x + a = a + x\}$.

**Claim** $C_N(a)$ is convex.

Clearly, $0 \in C_N(a)$ and, if $x, y \in C_N(a)$ then $x + y \in C_N(a)$. By weak o-minimality, $C_N(a)$ is the union of finitely many maximal convex subsets. Let $X$ be the greatest of these convex components with respect to the ordering induced by $<$. Let $x \in X$ with $x > 0$. Suppose that $y \in N$ with $0 < y < x$. We may show that $y \in C_N(a)$. By $x < y + x < 2x$ and $2x \in X$, we have $y + x \in X$. Hence $(y + x) + a = a + (y + x)$. By $x \in C_N(a)$, we have $(y + a) + x = (a + y) + x$. Hence, we have $y + a = a + y$. Thus, $y \in C_N(a)$, as desired.

Let $b, c \in N$ with $b < c$. Then the following is equivalent:
$b$ and $c$ are commutative;

- $b$ and $b + c$ are commutative;
- $b + c$ and $c$ are commutative.

Now $b, b + c \leq 0$ or $b + c, c \geq 0$. Hence we may assume $0 < b < c$. Then, as $C_N(c)$ is convex, we have $b \in C_N(c)$. Therefore $\mathcal{N}$ is commutative. \qed

Let $\mathcal{N} = (N, 0, +, <, \ldots)$ be an ordered monoid. Suppose that $I_N := \{x \in N \mid N \models \exists y(x + y = 0)\}$. Clearly, $I_N$ contains 0. We call an ordered monoid $(N, 0, +, <, \ldots)$ Archimedian if for all elements $a, b$ with $b > 0$ there exists some $n \in \omega$ such that $a < nb$, and for all elements $a, b$ with $b < 0$ there exists some $n \in \omega$ such that $nb < a$.

**Example 7** Let $\mathcal{M} = (\{0\} \cup \mathbb{Q}^{\geq 1}, 0, +, <, P)$, where $\mathbb{Q}^{\geq 1} = \{a \in \mathbb{Q} \mid a \geq 1\}$ and the unary predicate symbol $P$ is interpreted by the convex set $P^M = (\sqrt{2}, 3) \cap \mathbb{Q}$. Then, $\mathcal{M}$ is a weakly $0$-minimal Archimedean monoid and not divisible. Moreover $I_M = \{0\}$.

Hence, in generally a weakly $0$-minimal Archimedean monoid is not a group. However the following holds.

**Proposition 8** Let $\mathcal{N} = (N, 0, +, <, \ldots)$ be a weakly $0$-minimal Archimedean monoid. Suppose that $I_N \neq \{0\}$. Then $\mathcal{N}$ is a group.

**Proof.** Clearly $0 \in I_N$. Let $x, y \in I_N$. Then, there exist $x_1, y_1$ such that $x + x_1 = 0$ and $y + y_1 = 0$. Then $(x + y) + (y_1 + x_1) = 0$. Thus, $x + y \in I_N$. 

**Claim** $I_N$ is convex.

By weak $0$-minimality, $I_N$ is the union of finitely many maximal convex subsets. Let $C$ be the greatest of these convex components with respect to the ordering induced by $<$. Let $x \in C$ with $x > 0$. Suppose that $y \in N$ with $0 < y < x$. We may show that $y \in I_N$. By $x < y + x < 2x$ and $2x \in C$, we have $y + x \in C$. Hence, there exists some $z \in N$ such that $(y + x) + z = 0$. So $y + (x + z) = 0$. Thus, $y \in I_N$, as desired.

Let $g \in N$. By $I_N \neq \{0\}$, there exists some $a \in I_N$ such that $a \neq 0$. Without loss of generality, we may assume that $g > 0$ and $a > 0$. As $N$ is Archimedean, there exists some $n \in \omega$ such that $0 < g < na$. Since $I_N$ is convex, we have $g \in I_N$. Therefore $I_N = N$. \qed
Let $N$ be an ordered monoid and $A$ a subset $N$. The ordered monoid $N$ is said to be rich, if for all $a, b \in N$ if $0 \leq a \leq b$ or $b \leq a \leq 0$, then there exists some $c \in N$ such that $b = a + c$. The set $A$ admits right elimination, if for all $a \in A$ and all $b \in N$ if $b + a \in A$, then $b \in A$.

**Example 9** Let $\mathcal{M} = (\mathbb{Q}^{\geq 0}, 0, +, <, P)$, where $\mathbb{Q}^{\geq 0} = \{a \in \mathbb{Q} \mid a \geq 0\}$ and the unary predicate symbol $P$ is interpreted by the convex set $P^\mathcal{M} = (\sqrt{2}, 3) \cap \mathbb{Q}$. Then, $\mathcal{M}$ is a weakly o-minimal rich monoid and divisible.

**Proposition 10** Let $\mathcal{N} = (N, 0, +, <, \ldots)$ be a weakly o-minimal monoid. Then the following is equivalent:

1. $\mathcal{N}$ is divisible;
2. for all $n \in \omega$, $nN$ admits right elimination;
3. for all $n \in \omega$, $nN$ is convex.

**Proof.** (1 $\Rightarrow$ 2) It is clear.

(2 $\Rightarrow$ 3) Let $n \in \omega$. Let $x, y \in nN$. Then there exist $x_1, y_1 \in N$ such that $x = nx_1$ and $y = ny_1$. By Proposition 6, we have $x + y = nx_1 + ny_1 = n(x_1 + y_1)$ Hence, $x + y \in nN$. Now, by weak o-minimality, $nN$ is the union of finitely many maximal convex subsets. Let $C$ be the greatest of these convex components with respect to the ordering induced by $<$. Let $x \in C$ with $x > 0$. Suppose that $y \in N$ with $0 < y < x$. We may show that $y \in nN$. By $x < y + x < 2x$ and $2x \in C$, we have $y + x \in C$. As $nN$ admits right elimination, we have $y \in nN$, as desired.

(3 $\Rightarrow$ 1) Let $n$ be a nonzero natural number. For all positive $a \in N$, we have $0 < a < na$. As $nN$ is convex, we have $a \in nN$. Hence $\mathcal{N}$ is divisible. $\square$

**Proposition 11** Let $\mathcal{N} = (N, 0, +, <, \ldots)$ be a weakly o-minimal monoid. If $\mathcal{N}$ is rich, then $\mathcal{N}$ is divisible.

**Proof.** Let $n$ be a nonzero natural number. Now, by weak o-minimality, $nN$ is the union of finitely many maximal convex subsets. Let $C$ be the greatest of these convex components with respect to the ordering induced by $<$. Let $x \in C$ with $x > 0$. Suppose that $y \in N$ with $0 < y < x$. We show that $y \in nN$. By $x < y + x < 2x$ and $2x \in C$, we have $y + x \in C$. So there exist $z_1, z_2 \in N$ with $0 < z_1 < z_2$ such that $x = nz_1$ and $y + x = nz_2$. 

As $\mathcal{N}$ is rich, there exists some $a \in N$ such that $a + z_1 = z_2$. Hence, we have $y + nz_1 = na + nz_1$. Therefore we have $y = na \in nN$. It follows that $nN = N$.

**Proposition 12** [T] Let $N$ be an ordered monoid. Suppose that $\text{Th}(N)$ is weakly o-minimal. Then there exists an extending ordered group $G$ of $N$ such that $\text{Th}(G)$ is weakly o-minimal.

**Proof.** Let $N_1$ be an $\omega$-saturated elementary extension of $N$. Define the following relation on $N_1 \times N_1$:

$$(a, b) \sim (a', b') \iff a + b' = a' + b.$$  

Then $\sim$ is an equivalence relation on $N_1 \times N_1$. For each $(a, b) \in N_1 \times N_1$, let $[(a, b)]$ denote the $\sim$-class of $(a, b)$. Let $G := N_1 \times N_1 / \sim$. Then $G$ can be naturally expanded to an $\omega$-saturated ordered group. We may treat $N_1$ as a substructure of $G$ by identifying $a \in N_1$ and $[(a, 0)] \in G$. We may show that $G$ is weakly o-minimal. By way of a contradiction, assume that $G$ is not weakly o-minimal. Then there exists a definable subset $A \subseteq G$ and a monotone sequence $\{a_i \in G \mid i \in \omega\}$ such that for all $i \in \omega$, $a_i \in A$ if and only if $i$ is even. As $G$ is an eq-object of $N_1$, there exists a formula $\varphi(x, y)$ (parameters from $N_1$) such that $[(b, c)] \in A$ if and only if $N_1 \models \varphi(b, c)$. For all $i \in \omega$, let $a_i := [(b_i, c_i)]$. Then we have

$$N_1 \models \varphi(b_i, c_i) \iff i \text{ is even}.$$  

For all $n \in \omega$, let $d_i := \sum_{j=0}^{2n} c_i$ and $e := \Sigma_{j=0}^{2n} c_i$. Then we have

$$N_1 \models \varphi(b_i + d_i, e) \iff i \text{ is even}.$$  

Hence, the set $\varphi(N_1, e)$ can not be written as the union of $n$ convex sets, contradicting that $\text{Th}(N)$ is weakly o-minimal.

\[\square\]

### 3 Rings and fields

In this section, we study weakly o-minimal rings and fields.

A commutative ordered domain $R$ is said to be real closed if $R$ has intermediate value property, that is, for any polynomial $p(x)$ with coefficients in
$R$ and any $a, b \in R$ such that $a < b$ and $p(a) \cdot p(b) < 0$, there exists some $c \in R$ so that $a < c < b$ and $p(c) = 0$.

It is well-known the following fact.

**Fact 13** [MMS]

1. If a commutative ordered ring $R$ is weakly o-minimal, then $R$ is a real closed ring;

2. If an ordered field $F$ is weakly o-minimal, then $F$ is a real closed field.

In [PS1], it is shown that an o-minimal ring is a real closed field. However, in generally a weakly o-minimal ordered ring is not a field. We shall show that if a weakly o-minimal ordered ring $R$ which may not be associative is Archimedean, then $R$ is a real closed field.

**Lemma 14** If $\mathcal{R} = (R, 0, 1, +, \cdot, <, \ldots)$ is a weakly o-minimal ring, then $\mathcal{R}$ is commutative.

**Proof.** For all $a \in R$, let $C_R(a) := \{x \in R \mid xa = ax\}$. Then, $C_R(a)$ is a definable additive subgroup. Hence, by Fact 4, $C_R(a)$ is convex. Let $g, h \in R$. Without loss of generality, we may assume that $0 < g < h$. As $C_R(h)$ is convex, we have $g \in C_R(h)$. It follows that $\mathcal{R}$ is commutative. $\square$

We call an ordered ring $(R, 0, 1, +, \cdot, <, \ldots)$ standard if for all nonzero $a \in R$ there exists $b \in R$ such that $1 < ab$. Clearly, an Archimedian ordered ring is standard.

**Proposition 15** Let $\mathcal{R} = (R, 0, 1, +, \cdot, <, \ldots)$ be a weakly o-minimal ring. Then, the following is equivalent:

1. $\mathcal{R}$ is standard;

2. $\mathcal{R}$ is a field.

**Proof.** $(2 \Rightarrow 1)$ Let $a \in R$ with $a \neq 0$. Then, as $\mathcal{R}$ is field, there exists $a^{-1}$. Hence, $1 < a \cdot 2a^{-1} = 2$, as desired.

$(1 \Rightarrow 2)$ Let $a \in R$. Then, as $\mathcal{R}$ is standard, there exists some $b \in R$ such that $1 < ab$. Now $aR$ is a definable additive subgroup. Hence, as $aR$ is convex, we have $1 \in aR$. It follows that $\mathcal{R}$ is a field. $\square$
**Corollary 16** Let $\mathcal{R} = (R, 0, 1, +, \cdot, <, \ldots)$ be a weakly o-minimal Archimedean ring, where $\mathcal{R}$ may not be associative. Then, $\mathcal{R}$ is a real closed field.

**Proof.** By Fact 13, Lemma 14 and Proposition 15, we may show that $\mathcal{R}$ is associative. Let $a \in R$ with $a \neq 0$. Suppose that $D_R(a) := \{ x \in R \mid (xa)a = x(aa) \}$. Then, as $\mathcal{R}$ is commutative, $D_R(a)$ contains $a$ and is a definable additive subgroup. Hence, by Lemma 5, $D_R(a) = R$. Also, suppose that $E_R(a) := \{ x \in R \mid (za)x = z(ax) \text{ for each } z \}$. Then, by $D_R(a) = R$, $E_R(a)$ contains $a$ and is a definable additive subgroup. Thus, by Lemma 5, $E_R(a) = R$. It follows that $\mathcal{R}$ is associative. $\square$

**Proposition 17** Let $R$ be an ordered ring. Suppose that $\text{Th}(R)$ is weakly o-minimal. Then there exists an extending ordered field $F$ of $R$ such that $\text{Th}(F)$ is weakly o-minimal.

**Proof.** Let $R_1$ be an $\omega$-saturated elementary extension of $R$. Let $R_1^{>0} := \{ a \in R_1 \mid a > 0 \}$. Define the following relation on $R_1 \times R_1^{>0}$:

$$(a, b) \sim (a', b') \iff ab' = a'b.$$  

Then $\sim$ is an equivalence relation on $R_1 \times R_1^{>0}$. For each $(a, b) \in R_1 \times R_1^{>0}$, let $[(a, b)]$ denote the $\sim$-class of $(a, b)$. Let $F := R_1 \times R_1^{>0}/\sim$. Then $F$ can be naturally expanded to an $\omega$-saturated ordered field. We may treat $R_1$ as a substructure of $F$ by identifying $a \in R_1$ and $[(a, 1)] \in F$. We may show that $F$ is weakly o-minimal. By way of a contradiction, assume that $F$ is not weakly o-minimal. Then there exists a definable subset $A \subseteq F$ and a monotone sequence $\{ a_i \in F \mid i \in \omega \}$ such that for all $i \in \omega$, $a_i \in A$ if and only if $i$ is even. As $F$ is an eq-object of $R_1$, there exists a formula $\varphi(x, y)$ (parameters from $R_1$) such that $[(b, c)] \in A$ if and only if $R_1 \models \varphi(b, c)$. For all $i \in \omega$, let $a_i := [(b_i, c_i)]$. Then we have

$$R_1 \models \varphi(a_i, c_i) \iff i \text{ is even.}$$

For all $n \in \omega$, let $d_i := \Pi_{j=0}^{2n} i \neq d_i$ and $c := \Pi_{j=0}^{2n} c_i$. Then we have

$$R_1 \models \varphi(b_id_i, c) \iff i \text{ is even.}$$

Hence, the set $\varphi(R_1, c)$ can not be written as the union of $n$ convex sets, contradicting that $\text{Th}(R)$ is weakly o-minimal. $\square$
References


