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CYCLE CLASS MAPS FOR ARITHMETIC SCHEMES

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X/k : a proper smooth variety over a field k of characteristic zero.
Let

$$\mathrm{CH}^r(X)=\Big(\bigoplus_{\substack{V\subset X\\ \text{irred. subvar.}}}\mathbb{Z}\Big)/\underset{\text{rat. equiv.}}{\sim}$$

be the group of cycles of codimension r in X modulo rational equivalence, called the Chow group of cycles of codimension r.

For $r=1$ we have

$$\mathrm{CH}^1(X)\simeq \mathrm{Pic}(X)$$

where $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles on X.

If X has a k-rational point, we have the exact sequence

$$0\to \mathrm{Pic}^0_{X/k}(k)\to \mathrm{Pic}(X)\to \mathrm{NS}(X)\to 0$$

where $\mathrm{NS}(X)$ is the Neron-Severi group of X and $\mathrm{Pic}^0_{X/k}$ is the Picard variety of X/k. It is known that:

(1) $\mathrm{NS}(X)$ is finitely generated (for an arbitrary k).

(2) $\mathrm{Pic}^0_{X/k}(k)$(=the group of the k-rational points of $\mathrm{Pic}^0_{X/k}$) is finitely generated if $[k:\mathbb{Q}]<\infty$. (the Mordell-Weil theorem).

Hence $\mathrm{CH}^1(X)$ is finitely generated if $[k:\mathbb{Q}]<\infty$.

Question: Is $\mathrm{CH}^r(X)$ is finitely generated if $[k:\mathbb{Q}]<\infty$?

Remark: The rank of $\mathrm{CH}^r(X)$ and the order of $\mathrm{CH}^r(X)_{\mathrm{tors}}$ are expected to be related to special values of L-function of X (Tate, Birch-Swinnerton-Dyer, Beilinson, Bloch-Kato,....).

Only little is known about the above question. Difficulty comes from the fact that $\mathrm{CH}^r(X)$ for $r\geq 2$ is in general "not representable" so that over $\mathbb{C}$ it is as large as $\mathbb{C}\otimes_{\mathbb{Z}}\mathbb{C}\cdots\otimes_{\mathbb{Z}}\mathbb{C}$ (Mumford theorem).

We now assume $[k:\mathbb{Q}]<\infty$ or $[k:\mathbb{Q}_\ell]<\infty$. We fix a prime p and are concerned with the finiteness of:

$$\mathrm{CH}^2(X)_{p\text{-tors}}\quad\text{and}\quad \mathrm{CH}^2(X)/p^n$$

where for an abelian group M, $M_{p\text{-tors}}$ denotes the p-primary torsion part. One way to approach to the fundamental question is to look at the cycle class map from Chow group to (continuous) étale cohomology of X:

$$\rho^r_{X,\mathbb{Z}/p^n\mathbb{Z}}:\mathrm{CH}^r(X)/p^n\to \mathrm{H}^{2r}_{\text{ét}}(X,\mathbb{Z}/p^n\mathbb{Z}(r))$$

$$\rho^r_{X,\mathbb{Z}_p}:\mathrm{CH}^r(X)\otimes\mathbb{Z}_p\to \mathrm{H}^{2r}_{\mathrm{cont}}(X,\mathbb{Z}_p(r))$$

where $\mathbb{Z}/p^n\mathbb{Z}(r) = \mu_{p^n}^{\otimes r}$ is the rth tensor power of the sheaf of p^nth roots of unity and $\mathbb{Z}_p(r) =$ "$\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}(r)$". Note that $\mathrm{H}^{2r}_{\text{ét}}(X, \mathbb{Z}/p^n\mathbb{Z}(r))$ is not in general finite if $[k:\mathbb{Q}] < \infty$. But one can show that $\mathrm{Im}(\rho^r_{X,\mathbb{Z}/p^n\mathbb{Z}})$ is fintie and $\mathrm{Im}(\rho^r_{X,\mathbb{Z}_p})$ is a finitely generated $\mathbb{Z}_p$-module. Hence the injectivity of the above maps would imply the desired finiteness.

For $r = 1$ one can show the injectivity of these maps by using the Kummer sequence

$$0 \to \mathbb{Z}/p^n\mathbb{Z}(1) \to \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \to 0$$

and the isomorphism

$$\mathrm{CH}^1(X) \simeq \mathrm{Pic}(X) \simeq H^1_{\text{ét}}(X, \mathbb{G}_m).$$

It is conjectured in case $[k:\mathbb{Q}] < \infty$ that the kernel of $\rho^r_{X,\mathbb{Z}_p}$ is torsion. On the other hand, using the theory of quadratic forms, Parimala and Suresh proved the following:

Theorem: There exists a smooth projective surface X over k with $\mathrm{H}^2(X, \mathcal{O}_X) = 0$ (in fact X is a rational surface) such that $\mathrm{Ker}(\rho^2_{X,\mathbb{Z}_2})$ is a nonzero finite group.

In this talk we present a new viewpoint on the injectivity problem of cycle class maps by investigating cycle maps for models of X over the ring of integers of k. We fix the following setup:

k: $[k:\mathbb{Q}] < \infty$ or $[k:\mathbb{Q}_\ell] < \infty$.

$\mathfrak{O}_k$: the integer ring of k and put $S := \mathrm{Spec}(\mathfrak{O}_k)$,

$\mathcal{X}$: a regular scheme which is proper flat of finite type over S.

$X = \mathcal{X} \times_S \mathrm{Spec}(k)$: the generic fiber of $\mathcal{X}$.

We fix a prime p and assume the following condition:

> *If p is not invertible on $\mathcal{X}$, then $\mathcal{X}$ has good or semistable reduction at each prime ideal of $\mathfrak{O}_k$ dividing (p).*

If p is not invertible on $\mathcal{X}$, etale cohomology of $\mathcal{X}$ with $\mu_{p^n}^{\otimes r}$-coefficient does not work well. Instead the p-adic étale Tate twist

$$\mathfrak{T}_n(r)_{\mathcal{X}} \in D^b(\mathcal{X}, \mathbb{Z}/p^n\mathbb{Z})$$

defined by K.Sato plays an important role. Here $D^b(\mathcal{X}, \mathbb{Z}/p^n\mathbb{Z})$ denotes the derived category of bounded complexes of étale sheaves of $\mathbb{Z}/p^n\mathbb{Z}$-modules on $\mathcal{X}$.

Remark

(1) Letting $\mathcal{X}[\frac{1}{p}] \subset \mathcal{X}$ be the open subschme obtained by removing the fibers over the points of characteristic p of S,
$$\mathfrak{T}_n(r)_{\mathcal{X}[\frac{1}{p}]} = \mu^{\otimes r}_{p^n,\mathcal{X}[\frac{1}{p}]}.$$
(2) Sato proved the finiteness of $\mathrm{H}^i_{\text{ét}}(\mathcal{X}, \mathfrak{T}_n(r)_{\mathcal{X}})$.

(3) It is expected that:
$$\mathfrak{T}_n(r)_{\mathcal{X}} = \mathbb{Z}(r)^{\text{ét}}_{\mathcal{X}} \otimes^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z},$$
where $\mathbb{Z}(r)^{\text{ét}}_{\mathcal{X}}$ denotes the conjectural étale motivic complex of Beilinson-Lichtenbaum for $\mathcal{X}$.

By the semi-purity property of $\mathfrak{T}_n(r)_{\mathcal{X}}$ shown by Sato, we can define the cycle map

$$\boxed{\rho^r_{\mathcal{X},\mathbb{Z}/p^n\mathbb{Z}} : \mathrm{CH}^r(\mathcal{X})/p^n \to \mathrm{H}^{2r}_{\text{ét}}(\mathcal{X}, \mathfrak{T}_n(r)_{\mathcal{X}})}$$

We are now concerned with the induced maps

$$\rho^r_{\mathcal{X},p\text{-tors}} : \mathrm{CH}^r(\mathcal{X})_{p\text{-tors}} \to \mathrm{H}^{2r}_{\text{ét}}(\mathcal{X}, \mathfrak{T}_{\mathbb{Z}_p}(r)_{\mathcal{X}})$$
$$\rho^r_{\mathcal{X},\mathbb{Z}_p} : \mathrm{CH}^r(\mathcal{X}) \otimes \mathbb{Z}_p \to \mathrm{H}^{2r}_{\text{ét}}(\mathcal{X}, \mathfrak{T}_{\mathbb{Z}_p}(r)_{\mathcal{X}}),$$

where

$$\mathrm{H}^*_{\text{ét}}(\mathcal{X}, \mathfrak{T}_{\mathbb{Z}_p}(r)_{\mathcal{X}}) = \varprojlim_{n\geq 1} \mathrm{H}^*_{\text{ét}}(\mathcal{X}, \mathfrak{T}_n(r)_{\mathcal{X}}).$$

Our main results on these maps concern the injectivity of these two maps in case $r = 2$. Roughly speaking, the injectivity of $\rho^2_{\mathcal{X},p\text{-tors}}$ and $\rho^2_{\mathcal{X},\mathbb{Z}_p}$ follows from a list of assumptions, each of which is a consequence of a well-known conjecture in arithmetic geometry. As a corollary we will get the following result: (Recall $X = \mathcal{X} \times_S \mathrm{Spec}(k)$)

Theorem 0.1. *Assume* $\mathrm{H}^2(X, \mathcal{O}_X) = 0$. *Then:*

(1) $\rho^2_{\mathcal{X},p\text{-tors}}$ *is injective.*
(2) *Suppose that* $[k : \mathbb{Q}_\ell] < \infty$ *with* $\ell \neq p$ *and* $\dim(X) = 2$. *Then* $\mathrm{Ker}(\rho^2_{\mathcal{X},\mathbb{Z}_p})$ *is uniquely p-divisible.*
(3) *Suppose that* $[k : \mathbb{Q}_p] < \infty$ *and* $\dim(X) = 2$ *with* $\kappa_X \leq 1$. *Then* $\rho^2_{\mathcal{X},\mathbb{Z}_p}$ *is injective.*
(4) *Suppose that* $[k : \mathbb{Q}] < \infty$ *and* $\dim(X) = 2$ *with* $\kappa_X \leq 1$. *Then* $\rho^2_{X,\mathbb{Z}_p}$ *is injective.*

Unramified cohomology:

Let $\mathcal{X}/\mathfrak{O}_k$ be as before and let K be its function field.

The *unramified cohomology of* K (here we write $\mathbb{Q}_p/\mathbb{Z}_p(n) = \mu_{p^\infty}^{\otimes n}$)

$$\mathrm{H}^{n+1}_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(n)) \subset \mathrm{H}^{n+1}_{\text{ét}}(\mathrm{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(n))$$

is defined to be the subgroup of those elements which are *unramified along every point of codimention one on* $\mathcal{X}$. More precisely it is the kernel of the boundary map

$$\mathrm{H}^{n+1}_{\text{ét}}(\mathrm{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(n)) \to \bigoplus_{y\in\mathcal{X}^1} \mathrm{H}^{n+2}_{y,\text{ét}}(\mathcal{X}, \mathfrak{T}_\infty(r)_{\mathcal{X}})$$

in the localization sequence, where $\mathfrak{T}_\infty(r)_{\mathcal{X}} = \varinjlim_{n\geq 1} \mathfrak{T}_n(r)_{\mathcal{X}}$ and $\mathcal{X}^1$ is the set of the points of codimension one in $\mathcal{X}$.

The following isomorphisms hold true:

$$\mathrm{H}^1_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(0)) \simeq \mathrm{H}^1_{\text{ét}}(\mathcal{X}, \mathbb{Q}_p/\mathbb{Z}_p) \simeq \mathrm{Hom}_{cont}(\pi_1^{ab}(\mathcal{X}), \mathbb{Q}_p/\mathbb{Z}_p),$$
$$\mathrm{H}^2_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(1)) \simeq \mathrm{Br}(\mathcal{X})_{p\text{-tors}},$$

where $\pi_1^{ab}(\mathcal{X})$ denotes the abelian fundamental group of $\mathcal{X}$ and $\mathrm{Br}(\mathcal{X})$ denotes the Grothendieck-Brauer group $\mathrm{H}^2_{\text{ét}}(\mathcal{X}, \mathbb{G}_{\text{m}})$.

In case $[k : \mathbb{Q}] < \infty$, $\mathrm{Br}(\mathcal{X})$ is isomorphic (up to finite groups) to the Tate-Shafarevich group of $\mathrm{Pic}^0_{X/k}$, the Picard variety of the generic fiber X of $\mathcal{X}$.

For $n = 0$, the quotient $\mathrm{H}^1_{\text{ét}}(\mathcal{X}, \mathbb{Q}/\mathbb{Z})/\mathrm{H}^1_{\text{ét}}(S, \mathbb{Q}/\mathbb{Z})$ is finite by a theorem of Katz-Lang and in case $[k : \mathbb{Q}] < \infty$, $\mathrm{H}^1_{\text{ét}}(\mathcal{X}, \mathbb{Q}/\mathbb{Z})$ is finite as well, because $\mathrm{H}^1_{\text{ét}}(S, \mathbb{Q}/\mathbb{Z})$ is finite.

In case $[k : \mathbb{Q}] < \infty$, $\mathrm{H}^2_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(1))$ is expected to be finite due to the finiteness conjecture of the Tate-Shafarevich group of the Picard variety of X.

In case $n = d := \dim(\mathcal{X})$, $\mathrm{H}^{d+1}_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(d))$ has been considered by K. Kato who conjectured $\mathrm{H}^{d+1}_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(d)) = 0$ if $p \neq 2$ or k has no embedding into $\mathbb{R}$ (The last conjecture is proved by Kato in case $d = 2$ and by Jannsen-Saito in case $d = 3$).

Motivated by the above facts we propose the following:

Conjecture 0.2. $\mathrm{H}^3_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ *is finite.*

The conjecture plays a central role in the proof of our main result. Indeed we have the following result.

Proposition 0.3. *Let*

$$\mathrm{H}^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2)) \subset \mathrm{H}^3_{\mathrm{ur}}(K,\mathbb{Q}_p/\mathbb{Z}_p(2))$$

be the intersection of $\mathrm{H}^3_{\mathrm{ur}}(K,\mathbb{Q}_p/\mathbb{Z}_p(2))$ *with*

$$\mathrm{Im}\Big(\mathrm{H}^3_{\text{ét}}(X,\mathbb{Q}_p/\mathbb{Z}_p(2)) \to \mathrm{H}^3_{\text{ét}}(\mathrm{Spec}(K),\mathbb{Q}_p/\mathbb{Z}_p(2))\Big).$$

(1) *If* $\mathrm{H}^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ *is finite, then* $\mathrm{Ker}(\rho^2_{\mathcal{X},p\text{-tors}})$ *coincides with the maximal divisible subgroup of* $\mathrm{CH}^2(\mathcal{X})_{p\text{-tors}}$.

(2) *If* $\mathrm{H}^3_{\mathrm{ur}}(K,\mathbb{Q}_p/\mathbb{Z}_p(2))$ *is finite then* $\mathrm{Ker}(\rho^2_{\mathcal{X},\mathbb{Z}_p})$ *coincides with the maximal divisible subgroup of* $\mathrm{CH}^2(\mathcal{X})\otimes\mathbb{Z}_p$.

The proposition is deduced from the exact sequence

$$\mathrm{H}^3_{\mathrm{ur}}(K,\mathbb{Z}/p^n\mathbb{Z}(2)) \to \mathrm{CH}^2(\mathcal{X})/p^n \xrightarrow{\rho^2_{\mathcal{X},\mathbb{Z}/p^n\mathbb{Z}}} \mathrm{H}^4_{\text{ét}}(\mathcal{X},\mathfrak{T}_n(r)_{\mathcal{X}})$$

which is constructed by using the semi-purity property of the Sato complex.

By the proposition the injectivity problem of our cycle class maps is reduced to the finiteness problem of the unramified cohomology $\mathrm{H}^3_{\mathrm{ur}}(K,\mathbb{Q}_p/\mathbb{Z}_p(2))$. We next relate it to other well-known conjectures in arithemtic geometry.

Bloch-Kato conjecture:

Let $\mathcal{X}/S = \mathrm{Spec}(\mathfrak{O}_k)$ and $X = \mathcal{X}\times_S \mathrm{Spec}(k)$ be as before. The conjecture concerns the p-adic regulator map from Bloch's higher Chow group to continuous Galois cohomology:

$$reg_X^{r,q} : \mathrm{CH}^r(X,q)\otimes\mathbb{Q}_p \to \mathrm{H}^1_{\mathrm{cont}}(G_k,\mathrm{H}^{2r-q-1}_{\text{ét}}(\overline{X},\mathbb{Q}_p(r))) \quad (r,q\geq 1)$$

where $G_k = \mathrm{Gal}(\overline{k}/k)$ and $\overline{X} = X\times_k \overline{k}$.

Conjecture (Bloch-Kato):

$$\mathrm{Im}(reg_X^{r,q}) = \mathrm{H}^1_g(G_k,\mathrm{H}^{2r-q-1}_{\text{ét}}(\overline{X},\mathbb{Q}_p(r)))$$

where the right hand side is the subspace defined by Bloch-Kato by using the p-adic Hodge theory. In case $[k:\mathbb{Q}_\ell]<\infty$,

$$\mathrm{H}^1_g(G_k,V) = \begin{cases} \mathrm{H}^1_{\mathrm{cont}}(G_k,V) & (p\neq\ell) \\ \mathrm{Ker}\Big(\mathrm{H}^1_{\mathrm{cont}}(G_k,V)\to\mathrm{H}^1_{\mathrm{cont}}(G_k,V\otimes B_{DR})\Big) & (p=\ell)\end{cases}$$

where $V = \mathrm{H}^*_{\text{ét}}(\overline{X},\mathbb{Q}_p(r))$.

The following special case is relevant to our problem.

$$reg_X = reg_X^{2,1} : \mathrm{CH}^2(X,1)\otimes\mathbb{Q}_p \to \mathrm{H}^1_{\mathrm{cont}}(G_k,\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Q}_p(2)))$$

where $\mathrm{CH}^2(X,1)$ coincides with the cohomology of the following complex

$$K_2(K) \xrightarrow{\delta_1} \bigoplus_{x\in X^1} k(x)^\times \xrightarrow{\delta_1} \bigoplus_{x\in X^2}\mathbb{Z},$$

(recall K is the function field of X), where

$$K_2(K) = (K^\times\otimes_{\mathbb{Z}}K^\times)/ < x\otimes y|\, x+y=1\ (x,y\in K^\times) >,$$

and X^r denotes the set of the points of codimension r on X and $k(x)$ is the residue field of $x \in X^r$. The map δ_1 is the so-called tame symbol and δ_2 is the map taking the divisors of functions.

We now state the Bloch-Kato conjecture in the relevant case as a condition:

$$\boxed{\textbf{(H1)}: \quad \mathrm{Im}(reg_X) = \mathrm{H}^1_g(G_k, \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p(2)))}$$

where

$$reg_X : \mathrm{CH}^2(X,1) \otimes \mathbb{Q}_p \to \mathrm{H}^1_{\mathrm{cont}}(G_k, \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p(2)))$$

(H1) is known to hold in the following cases:

(1) $H^2(X, \mathcal{O}_X) = 0$,
(2) $X = E \times E$ where E is a modular elliptic curve without CM over $\mathbb{Q}$ and $p \not|$(level of E), $p \geq 5$,
(3) X is an elliptic modular surface of level 4 over $\mathbb{Q}$ and $p \geq 5$,
(4) X is a Fermat quartic surface over $k = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-1})$,

The first case is easy and the other cases follow from the works of Mildenhall, Flach, Langer-Saito, Langer, Otsubo.

We now consider the regulator map with $\mathbb{Q}_p/\mathbb{Z}_p$-coefficient

$$reg_{X,\mathbb{Q}_p/\mathbb{Z}_p} : \mathrm{CH}^2(X,1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{H}^1(G_k, \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))$$

Consider the following variant of **H1**:

$$\boxed{\textbf{(H1*)}: \quad \mathrm{Im}(reg_{X,\mathbb{Q}_p/\mathbb{Z}_p}) = \mathrm{H}^1_g(G_k, \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p(2)))_{Div}}$$

where for an abelian group M, M_{Div} denotes its maximal divisible subgroup.

H1 always implies **H1*** and that the converse holds under some assumptions (for example in case $[k : \mathbb{Q}_\ell] < \infty$).

In what follows we assume $\mathrm{H}^3_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_p(2))^{G_k} = 0$, which holds if $[k : \mathbb{Q}] < \infty$ by the Weil conjecture (Deligne). If $[k : \mathbb{Q}_\ell] < \infty$, it is a consequence of the monodromy-weight conjecture so that it holds if $\dim(X) = 2$ or $\mathcal{X}$ is proper smooth over S. We also assume $p \geq 5$ by a technical reason coming from p-adic Hodge theory.

Theorem 0.4. *Let the aassumption be as above.*

(1) **H1*** *implies the following two finiteness conditions*:
F1: $\mathrm{CH}^2(X)_{p\text{-tors}}$ *is finite.*
F2: $\mathrm{H}^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ *is finite.*

(2) *Assume further*
T: *The reduced part of every fiber of $\mathcal{X}/\mathfrak{O}_k$ has simple normal crossings on $\mathcal{X}$ and the Tate conjecture for divisors holds for the irreducible components of those fibers.*
Then **F1** *and* **F2** *imply* **H1***.

As for the finiteness of $\mathrm{H}^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$, we need another condition:

H2: *Let*

$$AJ^2_X : \mathrm{CH}^2(X) \otimes \mathbb{Z}_p \to \mathrm{H}^1_{\mathrm{cont}}(k, \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_p(2)))$$

the p-adic Abel-Jacobi map for X. Then the quotient of $\mathrm{Ker}(AJ^2_X)$ *by its torsion subgroup is divisible.*

In case $[k : \mathbb{Q}] < \infty$, Beilinson conjectured that $\mathrm{Ker}(AJ^2_X)$ is torsion.
In case $\dim(X) = 2$, **H2** holds true in the following cases:

- $[k : \mathbb{Q}_\ell] < \infty$ with $\ell \neq p$ (Saito-Sujatha).

∘ $H^2(X, \mathcal{O}_X) = 0$ and $\kappa_X \leq 1$ (Bloch-Kas-Lieberman).

Theorem 0.5. *Let the assumption be as before. Then* **H1*** *and* **H2** *imply that* $H^3_{ur}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ *is finite.*

Summing up these, we get the following result which implies the first main result on the injectivity of cycle class map.

Corollary 0.6. *Assume* $H^2(X, \mathcal{O}_X) = 0$. *Then:*

(1) $H^3_{ur}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ *is finite.*
(2) $H^3_{ur}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ *is finite under one of the following:*
  (*i*) $[k : \mathbb{Q}_\ell] < \infty$ *with* $\ell \neq p$ *and* $\dim(X) = 2$,
  (*ii*) $[k : \mathbb{Q}_p] < \infty$ *and* $\dim(X) = 2$ *and* $\kappa_X \leq 1$.
  (*iii*) $[k : \mathbb{Q}] < \infty$ *and* $\dim(X) = 2$ *and* $\kappa_X \leq 1$.

Idea of Proof: We now explain the idea to show that **H1*** implies the finiteness of $CH^2(X)_{p\text{-tors}}$ and $H^3_{ur}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$. We only treat the case $[k : \mathbb{Q}_p] < \infty$. We consider the following groups

$$CH^2(X,1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \subset N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \subset U \subset H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

where $N^1H^3_{\acute{e}t}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is the kernel of the natural map

$$H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\iota} H^3(\mathrm{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(2)),$$

$$U = \iota^{-1}(H^3_{ur}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))),$$

The first inclusion comes from Bloch's exact sequence

$$0 \to CH^2(X,1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to CH^2(X)_{p\text{-tors}} \to 0$$

which is obtained by using the theorem of Mercuriev-Suslin on the surjectivity of the Galois symbol map for K_2. Thus it suffices to show

$$[U : CH^2(X,1) \otimes \mathbb{Q}_p/\mathbb{Z}_p] < \infty.$$

The assumption implies that $H^3(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_k}$ is finite and the Hochschild-Serre spectral sequence

$$E_2^{u,v} := H^u(G_k, H^v(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \Longrightarrow H^{u+v}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

induces the edge homomorphism

$$\nu : H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{Div} \to H^1(G_k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))$$

where for an abelian group M, M_{Div} denotes its maximal divisible subgroup. We note that the composition

$$CH^2(X,1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\nu} H^1(G_k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))$$

is the regulator map with $\mathbb{Q}_p/\mathbb{Z}_p$-coefficient. Thus **H1*** implies

$$\nu(CH^2(X,1) \otimes \mathbb{Q}_p/\mathbb{Z}_p) = H^1_g(G_k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{Div}.$$

Hence we are reduced to show the following:

Claim A: $\nu(U_{Div}) \subset H^1_g(G_k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))$.

Claim B: $U \cap \mathrm{Ker}(\nu)$ is finite.

To show Claim A, we first prove the inclusions

$$U \hookrightarrow W = H^3(\mathcal{X}, \tau_{\leq 2}Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2)) \hookrightarrow H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)),$$

where $j : X \hookrightarrow \mathcal{X}$ is the natural immersion. It is derived from the following purity theorem. Let $Y \subset \mathcal{X}$ be the special fiber of $\mathcal{X}/\mathfrak{O}_k$.

Theorem (Hagihara): *Let n, r and c be integers with $n \geq 0$ and $r, c \geq 1$. Then for any integer $q \leq n+c$ and any closed subscheme $Z \subset Y$ with $\mathrm{codim}_{\mathcal{X}}(Z) \geq c$, we have*

$$\mathrm{H}^q_Z(\mathcal{X}, \tau_{\leq n} Rj_*\mu_{p^r}^{\otimes n}) = 0 = \mathrm{H}^{q+1}_Z(\mathcal{X}, \tau_{\geq n+1} Rj_*\mu_{p^r}^{\otimes n}).$$

By the above inclusions the proof of Claim A is reduced to show

$$(*) \qquad \nu(W_{Div}) \subset \mathrm{H}^1_g(G_k, \mathrm{H}^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))).$$

The first step is to relate W with syntomic cohomology of $\mathcal{X}/\mathfrak{O}_k$.

Theorem (Kato-Kurihara-Tsuji) : *There is a canonical isomorphism*

$$\eta : s_n^{\log}(r)_{\mathcal{X}} \xrightarrow{\cong} i_* i^* \tau_{\leq r} Rj_* \mu_{p^n}^{\otimes r},$$

where the right hand side denotes the log-syntomic complex of Kato and $i : Y \to \mathcal{X}$ is the closed immersion of the closed fiber of $\mathcal{X}/S$.

Now put

$$\mathrm{H}^*(\mathcal{X}, s_{\mathbb{Q}_p}^{\log}(r)_{\mathcal{X}}) := \left\{ \varprojlim_{r \geq 1} \mathrm{H}^*(\mathcal{X}, s_n^{\log}(r)_{\mathcal{X}}) \right\} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

Assume $\mathrm{H}^{i+1}_{\text{ét}}(\overline{X}, \mathbb{Q}_p(r))^{G_k} = 0$. Let ξ be the composite map:

$$\mathrm{H}^{i+1}(\mathcal{X}, s_{\mathbb{Q}_p}^{\log}(n)_{\mathcal{X}}) \to \mathrm{H}^{i+1}_{\text{ét}}(X, \mathbb{Q}_p(r)) \to \mathrm{H}^1(G_k, \mathrm{H}^i_{\text{ét}}(\overline{X}, \mathbb{Q}_p(r)))$$

where the second map comes from the Hochschild-Serre spectral sequence. The desired assertion $(*)$ follows from the following result shown via theory of log-syntomic and log-crystalline cohomology.

Theorem (Langer and Nekovář) : We have

$$\mathrm{Im}(\xi) = \mathrm{H}^1_g(G_k, \mathrm{H}^i_{\text{ét}}(\overline{X}, \mathbb{Q}_p(r))).$$

Finally we explain the idea to show Claim B, namely the finiteness of $U \cap \mathrm{Ker}(\nu)$. By definition of $\mathrm{H}^3_{\text{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$, U is the kernel of the localization map

$$\delta : \mathrm{H}^3_{\text{ét}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to \bigoplus_{y \in \mathcal{X}^1} \mathrm{H}^4_y(\mathcal{X}, \mathfrak{T}_\infty(2)_{\mathcal{X}})$$

where $\mathfrak{T}_\infty(2)_{\mathcal{X}} = \varinjlim_n \mathfrak{T}_n(2)_{\mathcal{X}}$.

On the other hand, $\mathrm{Ker}(\nu) = F^2\mathrm{H}^3_{\text{ét}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ with F^2 denoting the filtration coming from the Hochschild-Serre spectral sequence. Hence we have a surjection

$$\mathrm{H}^2(G_k, \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \to \mathrm{Ker}(\nu).$$

Therefore we are reduced to show the finiteness of the kernel of the composite map

$$\mathrm{H}^2(G_k, \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \to \bigoplus_{y \in \mathcal{X}^1} \mathrm{H}^4_y(\mathcal{X}, \mathfrak{T}_\infty(2)_{\mathcal{X}}).$$

In order to show this, one is required to describe the above map explicitly in terms of geometry of the special fiber Y of $\mathcal{X}/\mathfrak{O}_k$. This is rather technical and complicate. Here we only point out one key ingredient.

Let $\overline{Y} = \mathcal{X} \times_{\mathfrak{O}_k} \overline{F}$ where F is the residue field of k and $\overline{F}$ is an algebraic closure of F. Let $W_n\omega^q_{Y,\log}$ be the logarithmic part of the de Rham-Witt differential $W_n\omega^q_Y$ associated to the semi-stable scheme $\mathcal{X}/\mathfrak{O}_k$ defined by Hyodo. Then one constructs a natural map

$$h : \mathrm{H}^2(G_k, \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \to \mathrm{H}^1(F, \mathrm{H}^0(\overline{Y}, W_\infty\omega^1_{Y,\log}))$$
$$(W_\infty\omega^1_{Y,\log} = \varinjlim_n W_n\omega^1_{Y,\log})$$

and show that it has finite kernel and cokernel by using the Fontaine-Jannsen conjecture (the comparison isomorphism between p-adic etale cohomology and log-crystalline cohomology of $\mathcal{X}/\mathfrak{O}_k$) proved by Hyodo-Kato and Tsuji.