<table>
<thead>
<tr>
<th>Title</th>
<th>On $p$-adic families of Hilbert cusp forms of finite slope (Algebraic number theory and related topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Yamagami, Atsushi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2005), 1451: 19-36</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2005-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/47737">http://hdl.handle.net/2433/47737</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
On $p$-adic families of Hilbert cusp forms of finite slope

京大理　山上 敦士 (Atsushi Yamagami)
Department of Mathematics, Kyoto University

0. Introduction

Let $p$ be an odd prime number. We fix an algebraic closure $\overline{\mathbb{Q}}$ of the field $\mathbb{Q}$ of rational numbers in the field $\mathbb{C}$ of complex numbers and an embedding $i_p : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$, where $\mathbb{Q}_p$ is an algebraic closure of the field $\mathbb{Q}_p$ of $p$-adic numbers. We denote by $i_\infty$ the fixed embedding $\mathbb{Q} \hookrightarrow \mathbb{C}$. Then we take the $p$-adic completion $\mathbb{C}_p$ of $\mathbb{Q}_p$ and fix an isomorphism $\mathbb{C}_p \cong \mathbb{C}$ of fields which is compatible with the embeddings $i_p$ and $i_\infty$. We denote by $\text{ord}_p$ the normalized $p$-adic valuation in $\mathbb{C}_p$ so that $\text{ord}_p(p) = 1$ and by $| \cdot |$ the absolute value given by $\text{ord}_p$. In this section, we would like to see the author's motivation, which is a story over $\mathbb{Q}$, for working on $p$-adic families of Hilbert cusp forms of finite slope.

Let $N$ be a positive integer prime to $p$ and $k \geq 2$ an integer. We take a normalized cuspidal Hecke eigenform $f$ of level $Np$ and weight $k$ whose Fourier expansion is given by $f(q) = \sum_{n \geq 1} a_n(f)q^n$ with $a_1(f) = 1$. Then we know that the Fourier coefficient $a_n$ is the $T(n)$-eigenvalue of $f$ for each $n \geq 1$, where $T(n)$ is the Hecke operator at $n$. In particular, all $a_n(f)$'s belong to $\overline{\mathbb{Q}}$. We then put $\alpha := \text{ord}_p(i_p(a_p(f)))$ and call it the $T(p)$-slope of $f$, which is a non-negative rational number in this case. Then it is known that if $f$ satisfies some technical assumptions, then there exists a family $\{f_{k'}\}_{k' \in \mathcal{K}}$ of normalized cuspidal Hecke eigenforms $f_{k'}$ of weight $k'$ and level $Np$ having fixed $T(p)$-slope $\alpha$ parametrized by an arithmetic progression $\mathcal{K}$ of radius $p^m$ starting from $k$ with some non-negative integer $m$. This fact has been proved in the case where $\alpha = 0$, i.e., ordinary case, by Hida [8] and [9], and his result has been generalized to the case where $\alpha$ is any non-negative rational number by Coleman [5] and [6].

The author [16, Main Theorem] used such families of finite $T(p)$-slopes to prove Gouvêa's conjecture in the unobstructed case, which asserts that all deformations of the mod $p$ Galois representation associated with $f$ to complete Noetherian local rings are associated with Katz's generalized $p$-adic modular forms of tame level $N$ (for the details of this conjecture, see [16]). The author would like to generalize this result to the case over totally real fields.

The author is a JSPS Postdoctoral Fellow in Department of Mathematics, Kyoto University.
Now let us recall Coleman’s arguments in [6] to obtain $p$-adic families $\{f_{k'}\}_{k' \in K}$ of eigenforms having fixed $T(p)$-slope $\alpha$ as above. He constucted in [6, Section B4] the Banach module $S^{\dagger}(N)$ consisting of families of overconvergent cusp forms which is specialized to the Banach space $S_{k}^{\dagger}(N)$ of overconvergent cusp forms of weight $k$. One of the key points is that the Hecke operator $T(p)$ acts on these spaces completely continuously. The space $S_{k}^{\mathrm{cl}}(Np)$ of classical cusp forms of weight $k$ and level $Np$ is included in $S^{\dagger}(N)$. For any non-negative rational number $\alpha$, we denote by $S_{k}^{\dagger}(N)^{\alpha}$ (resp. $S_{k}^{\mathrm{cl}}(Np)^{\alpha}$) the subspace of $S_{k}^{\dagger}(N)$ (resp. $S_{k}^{\mathrm{cl}}(Np)$) generated by all generalized $T(p)$-eigenspaces for all $T(p)$-eigenvalues whose $p$-adic valuation are $\alpha$. Coleman [5, Theorem 8.1] proved that if $k > \alpha + 1$, then

$$S_{k}^{\dagger}(N)^{\alpha} = S_{k}^{\mathrm{cl}}(Np)^{\alpha},$$

i.e., the classciality of overconvergent cusp forms of small $T(p)$-slope, and that if $k \equiv k' \pmod{p^{m(\alpha)}}$ with some non-negative integer $m(\alpha)$ depending on $\alpha$, then we have

$$\dim_{\mathbb{C}_{p}} S_{k}^{\dagger}(N)^{\alpha} = \dim_{\mathbb{C}_{p}} S_{k'}^{\dagger}(N)^{\alpha},$$

i.e, the local constancy of $\dim_{\mathbb{C}_{p}} S_{k}^{\dagger}(N)^{\alpha}$ with respect to weights $k$ (cf. [6, Theorem B3.4]). Then as an application of these facts, under some technical conditions, he constructed $p$-adic families $\{f_{k'}\}_{k' \in K}$ as above by means of the duality theorems between then classical Hecke algebras and the spaces of classical cusp forms and the theory of newforms and oldforms (see [6, Corollary B5.7.1]).

The aim of this article is to generalize Coleman’s arguments above to the case over totally real fields. Namely, we shall define in Section 1.1 the spaces $S_{(n,v)}^{\mathrm{cl}}(G; \Gamma_{1}(N); \mathbb{C}_{p})$ of classical Hilbert cusp forms which are interpolated by the Banach module $S(G; \Gamma_{1}(N))$ of “$p$-adic Hilbert cusp forms” defined in Section 1.2. Then in Section 2.1 we shall define the Hecke operator $T(\pi)$ which acts on them completely continuously, and prove in Section 2.2 the classicality of $p$-adic Hilbert cusp forms of small $T(\pi)$-slope and in Section 2.3 the local constancy of dimensions of submodules having fixed $T(\pi)$-slope $\alpha$. The method which we shall use is based on works of Buzzard [3] on “eigenvariety machine,” and of Chenevier [4] dealing with automorphic forms on any twisted form of $\mathrm{GL}_{n}$ over $\mathbb{Q}$ which is compact at infinity modulo center.

**Acknowledgement.** The author is grateful to Professor Morishita for giving him an opportunity to give a talk in the conference “Algebraic Number Theory and Related Topics” at RIMS in Kyoto.
1. Classical and $p$-adic automorphic forms

In this section, we define spaces of classical automorphic forms and $p$-adic ones on the algebraic groups defined by the unit groups of totally definite quaternion algebras over totally real fields. In this article, we assume that $p$ is an odd prime number for simplicity, although the case of $p = 2$ can be also done as well.

1.1. Classical automorphic forms

Let $F$ be a totally real field of degree $g$ and $O$ its ring of integers. Let $p_1, \ldots, p_r$ be all prime ideals of $F$ above $p$. Then the set $I$ of all embeddings $\sigma : F \hookrightarrow \mathbb{Q}$ has the partition $I = \bigcup_{i=1}^{r} I_i$, where $I_i$ is the subset of $I$ consisting of embeddings $\sigma$ such that the completion of $i_p(F^\sigma)$ in $\mathbb{C}_p$ coincides with the $p_i^\sigma$-adic completion $F_{p_i}^\sigma$ of $F^\sigma$.

In this article, we shall formulate “modular forms” as “automorphic forms” on adelic groups on quaternion algebras defined over $F$. Let $B$ be a totally definite quaternion algebra over $F$. We fix a maximal order $R$ of $B$ and a finite Galois extension $K_0$ over $\mathbb{Q}$ containing $F$ for which there is an isomorphism

$$B \otimes_{\mathbb{Q}} K_0 \cong M_2(K_0)^I$$

such that we have $R \otimes_{\mathbb{Z}} O_0 \cong M_2(O_0)^I$, where $M_2(A)$ with some ring $A$ stands for the ring of $2 \times 2$ matrices with coefficients in $A$ and $\mathbb{Z}$ and $O_0$ are the rings of integers in $\mathbb{Q}$ and $K_0$, respectively. Then we may assume that for a prime ideal $\mathfrak{i}$ at which $B$ is unramified, this isomorphism induces an isomorphism

$$B \otimes_{F} F_{\mathfrak{i}} \cong M_2(F_{\mathfrak{i}})$$

such that we have $R \otimes_{O} O_{\mathfrak{i}} \cong M_2(O_{\mathfrak{i}})$, where $O_{\mathfrak{i}}$ is the $\mathfrak{i}$-adic completion of $O$. We fix this isomorphism in this article. Let $G$ be the algebraic group defined over $\mathbb{Q}$ given by

$$G(A) := (B \otimes_{\mathbb{Q}} A)^{\times}$$

for $\mathbb{Q}$-algebras $A$. Let $A$ be the adele ring of $\mathbb{Q}$ and $A_f$ its finite part. We denote by $K$ the $p$-adic completion of $i_p(K_0)$ in $\mathbb{C}_p$ whose ring of integers is denoted by $O$. For $\gamma \in G(A_f)$, under the natural identification

$$F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{i=1}^{r} F_{p_i}$$

we then take the $\sigma$-projection $\gamma_\sigma \in \text{GL}_2(K)$ of the $p$-part $\gamma_p = (\gamma_i)_{i=1}^{r} \in G(\mathbb{Q}_p) = \prod_{i=1}^{r} (B \otimes_{F} F_{p_i})^{\times}$ of $\gamma$ as the image in $\text{GL}_2(K)$ of $\gamma_i$ under the projection $\sigma$ with the subscript $i$ determined by the condition that $\sigma \in I_i$ for each $\sigma \in I$. 
Let $N$ be an integral ideal of $F$ at which $B$ is unramified. We put
\[ \hat{R} := R \otimes \mathbb{Z}, \]
where $\hat{Z} := \prod_{l \text{ prime}} Z_l$ with the rings $Z_l$ of $l$-adic integers.
We then define an open compact subgroup
\[ \Gamma_1(N) := \{ x \in \hat{R}^\times \mid x_N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a-1, c, d-1 \in NO_N \} \]
of $\hat{R}^\times$, where $x_N$ is the $N$-part of $x$ and $O_N := \prod_{l \mid N} O_l$. By the approximation theorem, there exist $t_1, \ldots, t_h \in G(A)$ for some positive integer $h$ such that $(t_i)_N = 1$ and $(t_i)_\infty = 1$ for each $i = 1, \ldots, h$ and
\[
(1) \quad G(A) = \bigcap_{i=1}^h G(\mathbb{Q}) t_i \Gamma_1(N) G(\mathbb{R})_+,
\]
where $G(\mathbb{R})_+$ is the connected component of $G(\mathbb{R})$ with the identity.
We fix the decomposition (1) in this article and put $\Gamma_i := (t_i^{-1} G(\mathbb{Q}) t_i) \cap \Gamma_1(N) G(\mathbb{R})_+$ for each $i = 1, \ldots, h$, which is a discrete subgroup of $G(\mathbb{R})_+$ (cf. [10, Section 2]). Since we assume that $B$ is totally definite, we see that the quotient subgroup $\Gamma_i / \Gamma_i \cap (F \otimes \mathbb{Q})^\times$ of $G(\mathbb{R})_+ / G(\mathbb{R})_+ \cap (F \otimes \mathbb{Q})^\times$ is finite for each $i = 1, \ldots, h$.

Let $\mathbb{Z}[I]$ be the free $\mathbb{Z}$-module generated by $I$. We define an equivalence relation $\sim$ in $\mathbb{Z}[I]$ as follows: for $a, b \in \mathbb{Z}[I]$, $a \sim b$ if and only if $a - b \in \mathbb{Z} t_0$, where $t_0 := \sum_{\sigma \in I} \sigma$. We then put
\[ W^\cl := \{ (n, v) \in \mathbb{Z}[I] \times \mathbb{Z}[I] \mid n + 2v \sim 0, n > 0 \}, \]
where we mean by $n > 0$ that $n$ is positive, i.e., all coefficients $n_\sigma$ of $n$ are positive integers. We call $W^\cl$ the set of classical weights. For $(n, v) \in W^\cl$ and any $\mathcal{O}$-algebra $A$, we denote by $L(n, v; A)$ the left $GL_2(\mathcal{O})^I$-module consisting of polynomials $P$ of $2g$-parameters $(X_\sigma, Y_\sigma)_{\sigma \in I}$ with coefficients in $A$ which are homogeneous of degree $n_\sigma$ for each variable $(X_\sigma, Y_\sigma)$, on which $\gamma = (\gamma_\sigma)_{\sigma \in I} \in GL_2(\mathcal{O})^I$ acts by
\[
(2) \quad \gamma \cdot P := \det(\gamma)^v P((X_\sigma, Y_\sigma)^t \gamma_\sigma^t)_{\sigma \in I}.
\]
Here we define $\det(\gamma)^v := \prod_{\sigma \in I} \det(\gamma_\sigma)^v$ and for a $2 \times 2$ matrix $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we put $x^e := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

**Definition 1.1.** For $(n, v) \in W^\cl$ and an $\mathcal{O}$-algebra $A$, we put
\[ S^\cl_{(n, v)}(G; \Gamma_1(N); A) := \{ f : G(\mathbb{Q}) \backslash G(A_f) \rightarrow L(n, v; A) : \text{ function} \mid f(xu) = u^{-1} \cdot f(x) \text{ for } u \in \Gamma_1(N), x \in G(A_f) \}, \]
which we call the space of classical automorphic forms of level $\Gamma_1(N)$ and weight $(n, v)$ on $G$ (defined over $A$).
Remark 1.1. In the case where we regard $A = \mathbb{C}$ as an $\mathcal{O}$-algebra via the fixed isomorphism $\mathbb{C}_p \cong \mathbb{C}$ and $B$ is unramified at all finite places of $F$ (hence $g$ must be even by Hasse principle (cf. [15, XIII, Sections 3 and 6])), it is known that $S^{\text{cl}}(n,v)(G; \Gamma_1(N; \mathbb{C})$ are isomorphic to the spaces of classical holomorphic Hilbert cusp forms of weight $(n_{\sigma} + 2)_{\sigma \in I}$ and level $N$ by a result of Jacquet-Langlands and Shimizu (cf. [10, Theorem 2.1]).

1.2. $p$-Adic automorphic forms

We fix a classical weight $(n, v) \in W^{\text{cl}}$. Let $N$ be an integral ideal of $F$ which is not prime to $p$ and unramified in $B$. We now take arbitrarily $s(\leq r)$ prime ideals above $p$ which divide $N$. We may denote them by $p_1, \ldots, p_s$. We then put $I' := \sqcup_{i=1}^{s} I_i \subseteq I$ and denote the cardinality of $I'$ by $g'(\leq g)$. We fix a prime element $\pi_i$ of the $p_i$-adic completion $F_{p_i}$ of $F$ at $p_i$ for each $i = 1, \ldots, s$. We then denote by $\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$ the element of $G(A_F)$ whose $p_i$-part is the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & \pi_i \end{pmatrix}$ for each $i = 1, \ldots, s$ and other parts are trivial. In the following, for an element $\gamma \in \Gamma_1(N)$, we write its $\sigma$-projection as

$$\gamma_\sigma = \begin{pmatrix} 1 + \pi^\sigma_i a_{\sigma} & b_{\sigma} \\ \pi^\sigma_i c_{\sigma} & 1 + \pi^\sigma_i d_{\sigma} \end{pmatrix}$$

with some $a_{\sigma}, b_{\sigma}, c_{\sigma}, d_{\sigma} \in \mathcal{O}$ for each $\sigma \in I$ with $i$ such that $\sigma \in I_i$. Then we have

(3) $$(X_\sigma, Y_\sigma)^t \gamma_\sigma^t = ((1 + \pi^\sigma_i d_{\sigma}) X_\sigma - b_{\sigma} Y_\sigma, Y_\sigma + \pi^\sigma_i (a_{\sigma} Y_\sigma - c_{\sigma} X_\sigma))$$

for all $\sigma \in I'$ with $i$ such that $\sigma \in I_i$, and

(4) $$(X_\sigma, Y_\sigma)^t \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}_\sigma^t = \begin{cases} (\pi^\sigma_i X_\sigma, Y_\sigma) & (\sigma \in I_i \subset I'), \\ (X_\sigma, Y_\sigma) & (\sigma \in I \setminus I'). \end{cases}$$

For any elements $\gamma = \gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$ with $\gamma_1, \gamma_2 \in \Gamma_1(N)$ of the double coset $\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \Gamma_1(N)$, using actions (3) and (4), we define a $K$-endomorphism $[\gamma](n,v)$ on $L(n,v; K)$ with normalization of the $\det^v$-part by

(5) $$[\gamma](n,v) \cdot P := \prod_{\tau \in I \setminus I'} \det(\gamma_\tau)^{v_{\tau}} \prod_{\sigma \in I'} \det(\gamma_{1\sigma} \gamma_{2\sigma})^{v_{\sigma}}$$

$$\times P(((X_\tau, Y_\tau)^t \gamma_\tau^t)_{\tau \in I}).$$
Let $K\langle x_\sigma | \sigma \in I' \rangle$ be the strictly convergent power series ring of $g'$-variables $(x_\sigma)_{\sigma \in I'}$ with coefficients in $K$, which is the subring of the formal power series ring $K[x_\sigma | \sigma \in I']$ consisting of power series $P(x) = \sum_{(i_\tau)_{\tau \in I'} \in \mathbb{Z}_{\geq 0}^{I'}} a_{(i_\tau)_{\tau \in I'}} \prod_{\sigma \in I'} x_\sigma^{i_\sigma}$ such that $|a_{(i_\tau)_{\tau \in I'}}| \to 0$ as $\sum_{\sigma \in I'} i_\sigma \to \infty$. This is an orthonormalizable $K$-Banach algebra with sup norm $| \cdot |$ with respect to coefficients in $K$ (for the notion in the $p$-adic Banach theory, see [6, Chapter A]). We can take the set $\{\prod_{\sigma \in I'} x_\sigma^{i_\sigma} | i_\sigma \geq 0, \sigma \in I'\}$ as an orthonormal basis of $K\langle x_\sigma | \sigma \in I' \rangle$. We define actions on the variables $(x_\sigma)_{\sigma \in I'}$ of the $\sigma$-projections of $\gamma \in \Gamma_1(N)$ and $\left(\begin{array}{ll} 1 & 0 \\ 0 & \pi \end{array}\right)$ for $\sigma \in I'$ as follows:

\begin{equation}
(6) \gamma_\sigma \cdot x_\sigma := \frac{-b_\sigma + (1 + \pi_i^\sigma d_\sigma)x_\sigma}{1 + \pi_i^\sigma(a_\sigma - c_\sigma x_\sigma)} \quad \text{and} \quad \left(\begin{array}{ll} 1 & 0 \\ 0 & \pi \end{array}\right)_\sigma \cdot x_\sigma := \pi_i^\sigma x_\sigma
\end{equation}

with $i$ such that $\sigma \in I_i$. Note that the denominator $1 + \pi_i^\sigma(a_\sigma - c_\sigma x_\sigma)$ in the action (6) is a unit in $\mathcal{O}\langle x_\sigma \rangle$. Then by [6, Lemma A1.6], we see that elements in the double coset $\Gamma_1(N) \left(\begin{array}{ll} 1 & 0 \\ 0 & \pi \end{array}\right) \Gamma_1(N)$ give completely continuous $K$-endomorphisms on $K\langle x_\sigma | \sigma \in I' \rangle$ whose operator norms are at most 1. Here the operator norm $|L|$ of a continuous endomorphism $L$ on a Banach module $M$ is defined by

$$|L| := \sup_{0 \neq m \in M} \frac{|L(m)|}{|m|}.$$

Now we define a Banach module $S$ over the strictly convergent power series ring $K\langle \xi_\sigma | \sigma \in I' \rangle$ of $g'$-variables $(\xi_\sigma)_{\sigma \in I'}$ as follows: $S$ is the set of polynomials $P$ of $2(g - g')$-parameters $(X_\tau, Y_\tau)_{\tau \in I \setminus I'}$ with coefficients in $K\langle \xi_\sigma, x_\sigma | \sigma \in I' \rangle$ which are homogeneous of degree $n_\tau$ for each variable $(X_\tau, Y_\tau)$. We can take the set

$$\{(\prod_{\tau \in I \setminus I'} X_\tau^{a_\tau} Y_\tau^{b_\tau}) \prod_{\sigma \in I'} x_\sigma^{m_\sigma} | a_\tau + b_\tau = n_\tau \text{ with } a_\tau, b_\tau \geq 0, m_\sigma \geq 0\}$$

as an orthonormal basis of $S$ over $K\langle \xi_\sigma | \sigma \in I' \rangle$. Let $e(p_i)$ be the ramification index of the prime ideal $p_i$ in $F/\mathbb{Q}$. In order to define an action of $\Gamma_1(N)$ on $S$, we assume the condition that

$$\text{(ram)} \quad e(p_i) < p - 1 \quad \text{for each } i = 1, \ldots, s$$

is satisfied in the following. We see that $j_\sigma(\gamma_\sigma)$ for elements $\gamma$ of $\Gamma_1(N)$ and $\Gamma_1(N) \left(\begin{array}{ll} 1 & 0 \\ 0 & \pi \end{array}\right) \Gamma_1(N)$, and $\det(\gamma_\sigma)$ for $\gamma \in \Gamma_1(N)$ are of the form $1 + \pi_i^\sigma a$ with some $a \in \mathcal{O}$ for each $\sigma \in I'$ with $i$ such that $\sigma \in I_i$. Then
we can define their powers with any element $s$ in $C_p$ (resp. $C_p\langle \xi_\sigma \rangle$) such that $|s| \leq 1$ by a convergent power series as

$$(7) \quad (1 + \pi^*_\sigma a)^s := 1 + \sum_{k \geq 1} \frac{s(s-1) \cdots (s-k+1)}{k!} (\pi^*_\sigma)^k a^k$$

in $O_{C_p}$ (resp. $O_{C_p}\langle \xi_\sigma \rangle$) because of the assumption (ram) (cf. [4, Lemme 3.6.1]). Here we denote by $O_{C_p}$ the ring of $p$-adic integers in $C_p$, i.e., the subring of $C_p$ consisting of elements $s$ such that $|s| \leq 1$. We then define an action $[\gamma]$ of $\gamma \in \Gamma_1(N)$ on $S$ as

$$(8) \quad [\gamma] \cdot P := \prod_{\tau \in I \setminus I'} \det(\gamma_{\tau})^{v_\tau} (\prod_{\sigma \in I'} j_{\sigma}(\gamma_{\sigma})^{\xi_\sigma} \det(\gamma_{\sigma})^{\frac{\mu(n,v)-\xi_\sigma}{2}})$$

$\times P(((X_{\tau}, Y_{\tau})^{t}\gamma_{\tau})_{\tau \in I \setminus I'}; (\xi_{\sigma}, \gamma_{\sigma} \cdot x_{\sigma})_{\sigma \in I'})$.

As for $\gamma = \gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \gamma_2 \in \Gamma_1(N)$, we define a $K(\xi_\sigma | \sigma \in I')$-endomorphism on $S$ as

$$(9) \quad [\gamma] \cdot P := \prod_{\tau \in I \setminus I'} \det(\gamma_{\tau})^{v_\tau} (\prod_{\sigma \in I'} j_{\sigma}(\gamma_{\sigma})^{\xi_\sigma} \det(\gamma_{1\sigma} \gamma_{2\sigma})^{\frac{\mu(n,v)-\xi_\sigma}{2}})$$

$\times P(((X_{\tau}, Y_{\tau})^{t}\gamma_{\tau})_{\tau \in I \setminus I'}; (\xi_{\sigma}, \gamma_{\sigma} \cdot x_{\sigma})_{\sigma \in I'})$, which is completely continuous with operator norm $\leq 1$.

**Definition 1.2.** We denote by $W(n,v)$ the $g'$-dimensional closed affinoid ball over $K$ of radius 1 around $(n_\sigma)_{\sigma \in I'}$. Then the set $W(n,v)(C_p)$ of its $C_p$-valued points coincides with $O_{C_p}^{I'}$ and $K(\xi_\sigma | \sigma \in I')$ is the affinoid algebra associated to $W(n,v)$. (For the details of affinoid algebras and affinoid varieties, see [1, Part B and Chapter 7] and [6, Section A5].) We call it the space of the $I'$-parts of $p$-adic weights associated to $(n,v)$. We then associate $(t_\sigma := \frac{\mu(n,v)-s_\sigma}{2})_{\sigma \in I'}$ to any point $(s_\sigma)_{\sigma \in I'} \in W(n,v)(C_p)$, and put the $p$-adic weight $(s,t)$ as

$$s := \sum_{\sigma \in I'} s_\sigma \sigma + \sum_{\tau \in I \setminus I'} n_\tau \tau$$

and

$$t := \frac{\mu(n,v)t_0 - s}{2} = \sum_{\sigma \in I'} t_\sigma \sigma + \sum_{\tau \in I \setminus I'} v_\tau \tau.$$ 

Further, we denote by $W_{cl}(n,v)$ the subset of $W(n,v)(C_p)$ consisting of elements $(n'_\sigma)_{\sigma \in I'}$ whose components are positive integers of the same parity as $\mu(n,v)$ for all $\sigma \in I'$. We call it the set of the $I'$-parts of classical weights associated to $(n,v)$. For $(n'_\sigma)_{\sigma \in I'} \in W_{cl}(n,v)$, we put $(v'_\sigma := \frac{\mu(n,v)-n'_\sigma}{2})_{\sigma \in I'}$ and define $(n',v')$ as well as $(s,t)$. By the definition
of \( W_{\{n,v\}}^\text{cl} \), we see that \( v'_\sigma \) are also integers for all \( \sigma \in I' \) and that
\[
n' + 2v' = \mu(n,v)t_0.
\]

For \( (s_\sigma)_{\sigma \in I'} \in W_{\{n,v\}}(\mathbb{C}_p) \), we denote by \( K_{(s,t)} \) the \( p \)-adic completion in \( \mathbb{C}_p \) of the fraction field of \( K(\xi_\sigma | \sigma \in I')/(\xi_\sigma - s_\sigma | \sigma \in I') \). We denote by \( S_{(s,t)} \) the specialized orthonormalizable \( K_{(s,t)} \)-Banach space \( S \otimes_{K(\xi_\sigma | \sigma \in I')} K_{(s,t)} \). Then we denote by \( [\gamma]_{(s,t)} \) the specialized \( K_{(s,t)} \)-endomorphism \( [\gamma] \otimes K_{(s,t)} \) on \( S_{(s,t)} \) for elements \( \gamma \) of \( \Gamma_1(N) \) and \( \Gamma_1(N) \left[ \begin{array}{cc} 1 & 0 \\ 0 & \pi \end{array} \right] \Gamma_1(N) \).

**Definition 1.3.** (1) Assume the condition (ram). We define the space of \( p \)-adic automorphic forms of level \( \Gamma_1(N) \) on \( G \) (with coefficients in \( K \)) as
\[
S(G; \Gamma_1(N)) := \{ f : G(\mathbb{Q}) \backslash G(A_f) \to S : \text{function} | f(xu) = [u^{-1}] \cdot f(x), u \in \Gamma_1(N), x \in G(A_f) \}.
\]

We then have a \( K \)-isomorphism
\[
(10) \quad S(G; \Gamma_1(N)) \sim \bigoplus_{i=1}^{h} S_{\Gamma_i}, \quad f \mapsto (f(t_1), \ldots, f(t_h)),
\]

where \( t_1, \ldots, t_h \in G(A) \) are the fixed representatives of the decomposition (1). Here each \( S_{\Gamma_i} \) is the submodule of the orthonormalizable \( K(\xi_\sigma | \sigma \in I') \)-module \( S \) consisting of elements fixed under the action of \( \Gamma_i = (t_i^{-1}G(\mathbb{Q})t_i) \cap \Gamma_1(N)G(\mathbb{R})_+ \). Since \( \Gamma_i \) acts on \( S \) via the finite quotient group \( \Gamma_i/\Gamma_i \cap (F \otimes_{\mathbb{Q}} \mathbb{R})^\times \) because of the assumption \( n + 2v \sim 0 \), we then see that \( S_{\Gamma_i} \) satisfies the property (Pr) of [3, Section 2] for each \( i = 1, \ldots, h \). We now define a norm in \( S(G; \Gamma_1(N)) \) via this isomorphism as
\[
|f| := \sup_{1 \leq i \leq h} |f(t_i)|.
\]

Therefore, \( S(G; \Gamma_1(N)) \) can be regarded as a \( K(\xi_\sigma | \sigma \in I') \)-Banach module with the norm \( | \cdot | \) which satisfies the property (Pr) of [3, Section 2].

(2) Let \( (s_\sigma)_{\sigma \in I'} \in W_{\{n,v\}}(\mathbb{C}_p) \). Assume the condition (ram) in the case where \( (s_\sigma)_{\sigma \in I'} \notin W_{\{n,v\}}^\text{cl} \). We define the space of \( p \)-adic automorphic forms of weight \( (s,t) \) and level \( \Gamma_1(N) \) on \( G \) (defined over \( K_{(s,t)} \)) as
\[
S_{(s,t)}(G; \Gamma_1(N)) := \{ f : G(\mathbb{Q}) \backslash G(A_f) \to S_{(s,t)} : \text{function} | f(xu) = [u^{-1}]_{(s,t)} \cdot f(x), u \in \Gamma_1(N), x \in G(A_f) \}.
\]

Then we have an isomorphism
\[
(11) \quad S_{(s,t)}(G; \Gamma_1(N)) \sim \bigoplus_{i=1}^{h} S_{\Gamma_i}^{(s,t)}, \quad f \mapsto (f(t_1), \ldots, f(t_h))
\]
of $K_{(s,t)}$-Banach spaces satisfying the property $(Pr)$ of [3, Section 2], where we define a norm in $S_{(s,t)}(G; \Gamma_1(N))$ as

$$|f| := \sup_{1 \leq i \leq h} |f(t_i)|.$$ 

Putting $x_\sigma = \frac{X_\sigma}{Y_\sigma}$ for each $\sigma \in I'$, we then see easily the following

**Lemma 1.1.** For any $(n'_\sigma)_{\sigma \in I'} \in W_{(n,v)}^{cl}$, we have a natural $K$-inclusion

$$L(n',v';K) \hookrightarrow S_{(n',v')}^c,$$

and $P((X_\tau,Y_\tau)_{\tau \in I};(x_\sigma,1)_{\sigma \in I'})$ is compatible with $\gamma(n',v')$ for all $\gamma$ in $\Gamma_1(N)$ and the double coset $\Gamma_1(N) \left( \begin{array}{ll} 1 & 0 \\ 0 & \pi \end{array} \right) \Gamma_1(N)$ on these spaces. Thus we have an inclusion

$$S_{(n',v')}^{cl}(G; \Gamma_1(N); K) \hookrightarrow S_{(n',v')}^c(G; \Gamma_1(N))$$

of $K$-Banach spaces satisfying the property $(Pr)$ of [3, Section 2].

2. $p$-Adic automorphic forms of small $T(\pi)$-slope

Let the notation be as in Section 1.2. In this section, we shall introduce the Hecke operator $T(\pi)$ on the spaces of $p$-adic automorphic forms. Then we shall investigate some properties of $p$-adic automorphic forms having small $T(\pi)$-slope.

2.1. The Hecke operator $T(\pi)$

In this subsection, we assume the condition (ram), i.e., $e(p_i) < p - 1$ for all $i = 1, \ldots, s$, unless we deal with the $I'$-parts of classical weights in $W_{(n,v)}^{cl}$. In order to define the Hecke operator $T(\pi)$, we decompose the double coset $\Gamma_1(N) \left( \begin{array}{ll} 1 & 0 \\ 0 & \pi \end{array} \right) \Gamma_1(N)$ in a disjoint union of right cosets as

$$\Gamma_1(N) \left( \begin{array}{ll} 1 & 0 \\ 0 & \pi \end{array} \right) \Gamma_1(N) = \bigsqcup_{i=1}^{l} \zeta_i \Gamma_1(N).$$

For $f \in S(G; \Gamma_1(N))$ (resp. $S_{(s,t)}(G; \Gamma_1(N))$ for $(s_\sigma)_{\sigma \in I'} \in W_{(n,v)}(\mathbb{C}_p)$), we put

$$(12) \quad (f | T(\pi))(x) := \sum_{i=1}^{l} \zeta_i \cdot f (x \zeta_i) \quad (\text{resp.} \sum_{i=1}^{l} \zeta_i | (s,t) \cdot f (x \zeta_i))$$

for $x \in G(\mathbb{Q}) \backslash G(\mathbb{A}_f)$. Note that this definition is independent of choices of representatives $\{\zeta_i\}$ and $f | T(\pi)$ is also an element of $S(G; \Gamma_1(N))$ (resp. $S_{(s,t)}(G; \Gamma_1(N))$) (cf. [10, Section 2]).
Proposition 2.1. Assume the condition $(\text{ram})$ unless $(s_{\sigma})_{\sigma \in I'} \in W_{(n,v)}^\text{c1}$. The Hecke operator $T(\pi)$ is completely continuous on $S(G; \Gamma_1(N))$ and $S_{(s,t)}(G; \Gamma_1(N))$ for any $(s_{\sigma})_{\sigma \in I'} \in \mathcal{W}_{(n,w)}(\mathbb{Q})$ with operator norm $\leq 1$.

Proof. We shall prove the proposition for $S(G; \Gamma_1(N))$, because we can prove in the case of $S_{(s,t)}(G; \Gamma_1(N))$ as well. To see the complete continuity of $T(\pi)$, we calculate the action of $T(\pi)$ on $\bigoplus_{j=1}^{h}S^\Gamma_{j}$ via the isomorphism (10) by means of the decomposition

\[ \Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \Gamma_1(N) = \bigsqcup_{i=1}^{l} \xi_i \Gamma_1(N). \]

For $f \in S(G; \Gamma_1(N))$, the image of $f|T(\pi)$ under the isomorphism (10) is

\[ ((f|T(\pi))(t_1), \ldots, (f|T(\pi))(t_h)) \]

\[ = \sum_{i=1}^{l} ([\xi_i] \cdot f(t_1 \xi_i), \ldots, [\xi_i] \cdot f(t_h \xi_i)). \]

We fix $1 \leq i \leq l$. For each $j = 1, \ldots, h$, there exist $1 \leq \sigma_i(j) \leq h$ and $u_i(j) \in \Gamma_1(N)$ such that

\[ t_j \xi_i = t_{\sigma_i(j)} u_i(j) \]

in $G(\mathbb{Q}) \backslash G(A_f)$. Then we see that

\[ f(t_j \xi_i) = f(t_{\sigma_i(j)} u_i(j)) = [u_i(j)^{-1}] \cdot f(t_{\sigma_i(j)}). \]

by the definition of automorphic forms of level $\Gamma_1(N)$. Therefore we see that

\[ ((f|T(\pi))(t_1), \ldots, (f|T(\pi))(t_h)) \]

\[ = \sum_{i=1}^{l} ([\xi_i u_i(1)^{-1}] \cdot f(t_{\sigma_i(1)}), \ldots, [\xi_i u_i(h)^{-1}] \cdot f(t_{\sigma_i(h)})). \]

Thus the proposition is proven, because the endomorphisms $[\cdot]$ given by the double coset $\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \Gamma_1(N)$ on $S$ are completely continuous with operator norm $\leq 1$. \qed

We denote by $K\langle \xi_{\sigma} | \sigma \in I' \rangle\{\{X\}\}$ the subring of the formal power series ring $K\langle \xi_{\sigma} | \sigma \in I' \rangle[X]$ consisting of power series $\sum_{i \geq 0} c_i X^i$ such that

\[ |c_i| M^i \to 0 \quad \text{as} \quad i \to \infty. \]
for all $M \in \mathbb{R}$. By Proposition 2.1 and the arguments in [3, Section 2] dealing with Banach modules satisfying the property $(Pr)$, we have the following

Proposition 2.2. Assume the condition $(\text{ram})$. We have the characteristic power series

$$P((\xi_{\sigma})_{\sigma \in I'}, X) := \det(1 - XT(\pi)|_{S(G; \Gamma_{1}(N))}) = 1 + \sum_{i \geq 1} c_{i}X^{i} \in K(\xi_{\sigma}|\sigma \in I')\{\{X\}\}$$

of $T(\pi)$ on $S(G; \Gamma_{1}(N))$ with $|c_{i}| \leq 1$. Furthermore, for any $(s_{\sigma})_{\sigma \in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_{p})$, we see that

$$P((s_{\sigma})_{\sigma \in I'}, X) = 1 + \sum_{i \geq 1} c_{i}(s_{\sigma})_{\sigma \in I'}X^{i} \in K_{(s,t)}\{\{X\}\}$$

is the characteristic power series of $T(\pi)$ on $S_{(s,t)}(G; \Gamma_{1}(N))$. 

Let $\alpha$ be a non-negative rational number. For $(s_{\sigma})_{\sigma \in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_{p})$, let $S_{(s,t)}(G; \Gamma_{1}(N))_{\mathbb{C}_{p}}$ be the $\mathbb{C}_{p}$-subspace of $S_{(s,t)}(G; \Gamma_{1}(N)) \otimes_{K_{(s,t)}} \mathbb{C}_{p}$ generated by all generalized $T(\pi)$-eigenspaces for all eigenvalues $\lambda$ such that $\text{ord}_{p}(\lambda) = \alpha$. In the following subsections, we shall investigate $p$-adic automorphic forms which have small $T(\pi)$-slope.

2.2. Classicality of $p$-adic automorphic forms

In Lemma 1.1 without the condition $(\text{ram})$, we have seen that the spaces of classical automorphic forms are included in the ones of $p$-adic automorphic forms. Now we shall see that $p$-adic automorphic forms of small $T(\pi)$-slope are classical. Namely,

Theorem 2.3. Let $\alpha \in \mathbb{Q}_{\geq 0}$ and $(n_{\sigma}')_{\sigma \in I'} \in W_{(n,v)}^{cl}$. If the condition

$$\alpha < \nu_{n'} := \min \left\{ \frac{1}{\prod_{1 \leq i \leq s} e(p_{i})} \left( \min_{\sigma \in I_{i}} \{n_{\sigma}'\} + 1 \right) \right\}$$

is satisfied, then we have (without the condition $(\text{ram})$)

$$S_{(n',v')}(G; \Gamma_{1}(N))_{\mathbb{C}_{p}}^{\alpha} = S_{(n',v')}(G; \Gamma_{1}(N); \mathbb{C}_{p})^{\alpha}.$$

Proof. By the isomorphism (11) in Section 1, we see that the $\mathbb{C}_{p}$-Banach quotient space $(S_{(n',v')}(G; \Gamma_{1}(N)) \otimes_{K} \mathbb{C}_{p})/S_{(n',v')}(G; \Gamma_{1}(N); \mathbb{C}_{p})$ is isomorphic to a direct summand of the direct sum of $h$-copies of the orthonormalizable $\mathbb{C}_{p}$-Banach quotient space $S_{(n',v')} \otimes_{K} \mathbb{C}_{p}/L(n',v'; \mathbb{C}_{p})$.
whose orthonormal basis is
\[
\{(\prod_{\tau \in I \setminus I'} X_{\tau}^{a_{\tau}}Y_{\tau}^{b_{\tau}}) \prod_{\sigma \in I'} x_{\sigma}^{m_{\sigma}} | a_{\tau}, b_{\tau} \geq 0, \ m_{\sigma} \geq 0
\]
and \(m_{\sigma} > n'_{\sigma}\) for some \(\sigma\).

By the actions (3), (4) and (6) on the variables \(X_{\tau}, Y_{\tau}\) and \(x_{\sigma}\) in Section 1.2, we then see easily that
\[
|T(\pi)| \leq p^{-\nu_{n'}}
\]
on \((S_{(n', v')} \otimes_{K} \mathbb{C}_{p}/L(n', v'; \mathbb{C}_{p}))^{h}\). Hence we see that if \(\alpha < \nu_{n'}\), then the image of any generalized \(T(\pi)\)-eigenvector of slope \(\alpha\) is 0 in the quotient space \((S_{(n', v')}(G; \Gamma_{1}(N)) \otimes_{K} \mathbb{C}_{p})/S_{(n', v')}^{cl}(G; \Gamma_{1}(N); \mathbb{C}_{p})\). So we have
\[
S_{(n', v')}(G; \Gamma_{1}(N))_{\mathbb{C}_{p}}^{\alpha} = S_{(n', v')}^{cl}(G; \Gamma_{1}(N); \mathbb{C}_{p})^{\alpha}.
\]

\[\square\]

**Remark 2.1.** It is known that the spaces of definite quaternionic automorphic forms over \(\mathbb{Q}\) defined by means of homogeneous polynomials of degree \(n\) are isomorphic to the spaces of elliptic cusp forms of weight \(k = n + 2\) by Jacquet-Langlands' theorem (cf. [2, Theorem 2]). Coleman [5, Theorem 6.1 and Theorem 8.1] showed that \(p\)-adic over-convergent modular forms of weight \(k\) and \(U_{p}\)-slope \(\alpha\) are classical if \(\alpha < k - 1(=n + 1)\). Since \(s = 1\) and \(e(p) = 1\) in the case of \(F = \mathbb{Q}\), Theorem 2.3 is a generalization of the result of Coleman to the case over totally real fields.

2.3. **The local constancy of \(\dim_{\mathbb{C}_{p}} S_{(s, t)}(G; \Gamma_{1}(N))^\alpha_{\mathbb{C}_{p}}\)**

We assume the condition (ram), i.e., \(e(p_i) < p - 1\) for all \(i = 1, \ldots, s\). Let \(\alpha \in \mathbb{Q}_{>0}\). In this subsection, we shall give an explicit description of \(m(\alpha)\) such that if \((s_{\sigma})_{\sigma \in I'}, (s'_{\sigma})_{\sigma \in I'} \in W_{(n, v)}(\mathbb{C}_{p})\) satisfy that \(|s_{\sigma} - s'_{\sigma}| \leq p^{-m(\alpha)}\) for all \(\sigma \in I'\), then we have
\[
\dim_{\mathbb{C}_{p}} S_{(s, t)}(G; \Gamma_{1}(N))^\alpha_{\mathbb{C}_{p}} = \dim_{\mathbb{C}_{p}} S_{(s', t')}(G; \Gamma_{1}(N))_{\mathbb{C}_{p}}^{\alpha}
\]

by applying Chenevier's argument in [4, Section 5] to our case.

By Definition 1.3 (2), we regard \(S_{(s, t)}(G; \Gamma_{1}(N))\) as a direct summand of the orthonormalizable \(K_{(s, t)}\)-Banach module \(S_{(s, t)}^{h}\) for which we can also have the characteristic power series
\[
P'((s_{\sigma})_{\sigma \in I'}, X) = 1 + \sum_{i \geq 1} c_i'((s_{\sigma})_{\sigma \in I'})X^i \in K_{(s, t)}\{\{X\}\}.
with $|c'_i((s_{\sigma})_{\sigma\in I'})| \leq 1$. To obtain $m(\alpha)$ as above; we shall investigate the Newton polygon $N'_{(s,t)}$ of $P'((s_{\sigma})_{\sigma\in I'}, X)$. We can take the set

$$\{e_{M,a} := (0, \ldots, M, \ldots, 0) \}_{M \in \mathfrak{M}, 1 \leq a \leq h}$$

as an orthonormal basis of $S^h_{(s,t)}$, where we put the set of monomials

$$\mathfrak{M} := \{(\prod_{\tau \in I\setminus I'} X_{\tau}^{a_{\tau}} Y_{\tau}^{b_{\tau}}) \prod_{\sigma \in I'} x_{\sigma}^{m_{\sigma}} | a_{\tau} + b_{\tau} = n_{\tau} \text{ with } a_{\tau}, b_{\tau} \geq 0, m_{\sigma} \geq 0\}$$

and $M$ sits in the $a$-th component in $e_{M,a}$. We shall calculate the $p$-adic valuations of coefficients $c'_i((s_{\sigma})_{\sigma\in I'})$ of $P'((s_{\sigma})_{\sigma\in I'}, X)$ by means of this basis. For $\gamma = \gamma_1 \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \gamma_2 \in \Gamma_1(N) \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \Gamma_1(N)$ with $\gamma_1, \gamma_2 \in \Gamma_1(N)$ and a monomial $M = (\prod_{\tau \in I\setminus I'} X_{\tau}^{a_{\tau}} Y_{\tau}^{b_{\tau}}) \prod_{\sigma \in I'} x_{\sigma}^{m_{\sigma}} \in \mathfrak{M}$, we have

$$(13) \ [\gamma]_{(s,t)} \cdot M = \prod_{\tau \in I\setminus I'} \det(\gamma_{\tau}) \prod_{\sigma \in I'} j_{\sigma}(\gamma_{\sigma})^{s_{\sigma}} \det(\gamma_{1\sigma}\gamma_{2\sigma})^{t_{\sigma}}$$

$$\times (\prod_{\tau \in I\setminus I'} X_{\tau}^{a_{\tau}} Y_{\tau}^{b_{\tau}})^{t_{\tau}} \prod_{\sigma \in I'} (\gamma_{\sigma} \cdot x_{\sigma})^{m_{\sigma}}.$$ 

By the definition of $j_{\sigma}(\gamma_{\sigma})^{s_{\sigma}}$ and the action (6) on the variable $x_{\sigma}$ in Section 1.2 for each $\sigma \in I'$, we see that the $p$-adic valuations of all coefficients of monomials of the form $(\prod_{\tau \in I\setminus I'} X_{\tau}^{a_{\tau}} Y_{\tau}^{b_{\tau}}) \prod_{\sigma \in I'} x_{\sigma}^{k_{\sigma}}$ in the expansion of (13) in $S_{(s,t)}$ are at least $\lambda \sum_{\sigma \in I'} k_{\sigma}$, where we put the positive rational number $\lambda := \min_{1 \leq i \leq s} \{\frac{1}{e(p_i)}\} - \frac{1}{p-1}$. Now we order the basis $\{e_{M,a}\}_{M,a}$ as follows: For $k \geq 0$, we define the subset

$$A_k := \{e_{M,a} | 1 \leq a \leq h, M \text{ is of the form}$$

$$(\prod_{\tau \in I\setminus I'} X_{\tau}^{a_{\tau}} Y_{\tau}^{b_{\tau}}) \prod_{\sigma \in I'} x_{\sigma}^{k_{\sigma}} \text{ with } \sum_{\sigma \in I'} k_{\sigma} = k\}$$

of $\{e_{M,a}\}_{M,a}$. Then we see that the cardinality $\#A_k = h_n(k+g'-1)$ for $k \geq 0$, where $h_n := h \prod_{\tau \in I\setminus I'} (n_{\tau} + 1)$, and that for $k \geq 1$,

$$(14) \ \sum_{q=0}^{k} q \cdot \#A_q = h_n g'(k + g') \frac{k + g'}{g' + 1}.$$ 

We then exhibit elements of $A_0$ as $e_1^{(0)}, \ldots, e_{h_n}^{(0)}$ arbitrarily. Next we exhibit elements of $A_1$ as $e_1^{(1)}, \ldots, e_{h_n(g'+1)}^{(1)}$ arbitrarily. We then repeat this operation for all $k \geq 2$ as

$e_1^{(k)}, \ldots, e_{h_n(k+g'-1)}^{(k)}.$
We are going to obtain the representation matrix of infinite degree of $T(\pi)$ with respect to the basis $\{e_j^{(l)}\}_{j,l}$ ordered as above. For each $e_j^{(l)}$, we write

$$e_j^{(l)}|T(\pi) = \sum_{i_0=1}^{h_n} \alpha_{i_0}^{(0)}(j, l)e_{i_0}^{(0)} + \sum_{k \geq 1} \sum_{i_k = h_n\left(k+g'-1\right)+1}^{h_n\left(k+g'-1\right)+1} \alpha_{i_k}^{(k)}(j, l)e_{i_k}^{(k)}$$

with $\alpha_{i_k}^{(k)}(j, l) \in \mathcal{O}_{(s,t)}$ for all $k \geq 0$, where $\mathcal{O}_{(s,t)}$ is the ring of integers in $K_{(s,t)}$. As mentioned above, we then see that

(15) \hspace{1cm} \text{ord}_p(\alpha_{i_k}^{(k)}(j, l)) \geq k\lambda

for all $k \geq 0$, $j \geq 1$ and $l \geq 0$. The representation matrix of $T(\pi)$ with respect to the ordered basis $\{e_1^{(0)}, \ldots, e_{h_n}^{(0)}, \ldots\}$ is of the form

$$\begin{pmatrix}
\alpha_1^{(0)}(1, 0) & \cdots & \alpha_1^{(0)}(h_n, 0) & \cdots \\
\vdots & & \vdots & \\
\alpha_{h_n}^{(0)}(1, 0) & \cdots & \alpha_{h_n}^{(0)}(h_n, 0) & \cdots \\
\vdots & & \vdots & \\
\vdots & & \vdots & \\
\end{pmatrix}$$

It is known that the coefficient $c_i'((s_{\sigma})_{\sigma \in I'})$ of $\sum_{\sigma \in \mathbb{I}'} X$ is given by $(-1)^i \times \text{(the convergent sum of } i\text{-th minors of the above matrix)}$ for each $i \geq 1$ (cf. [13, Proposition 7 (a)]). So we see easily that

$$\text{ord}_p(c_i'((s_{\sigma})_{\sigma \in I'})) > i^{1+\frac{1}{g'}} \frac{2\lambda g'}{(g'+1)(g'+2)^2} \left(\frac{g!}{h_n}\right)^{\frac{1}{g'}}$$

by (14) and (15) in the case where

$$h_n\left(\begin{array}{c} k+g'-1 \\ g' \end{array}\right) + 1 \leq i \leq h_n\left(\begin{array}{c} k+g' \\ g' \end{array}\right)$$

with some $k \geq 2$. On the other hand, in the case where $1 \leq i \leq h_n(g'+1)$, we see that $\text{ord}_p(c_i'((s_{\sigma})_{\sigma \in I'})) \geq 0$ by Proposition 2.2. Therefore we have

(16) \hspace{1cm} \text{ord}_p(c_i'((s_{\sigma})_{\sigma \in I'})) \geq \frac{2\lambda g'}{(g'+1)(g'+2)^2} \left(\frac{g!}{h_n}\right)^{\frac{1}{g'}} i (i^{\frac{1}{g'}} - (h_n(g'+1))^{\frac{1}{g'}})

for all $i \geq 1$. We put the function

$$\mu(x) := \frac{2\lambda g'}{(g'+1)(g'+2)^2} \left(\frac{g!}{h_n}\right)^{\frac{1}{g'}} x(x^{\frac{1}{g'}} - (h_n(g'+1))^{\frac{1}{g'}})$$
on $\mathbb{R}_{\geq 0}$, which is a monotone increasing function. Since the Newton polygon $N_{(s,t)}$ of the characteristic power series $P((s_{\sigma})_{\sigma \in I'}, X)$ of $T(\pi)$ acting on $S(\sigma, t)(G; \Gamma_{1}(N))$ is bounded by $N'_{(s,t)}$ from the bottom, we then obtain the following

**Proposition 2.4.** Assume the condition (ram). Then we have

$$N_{(s,t)}(x) \geq \mu(x)$$

for all $(s_{\sigma})_{\sigma \in I'} \in \mathcal{W}_{(n,v)}(\mathbb{R})$ and $x \in \mathbb{R}_{\geq 0}$.

Secondly, the characteristic power series $P((\xi_{\sigma})_{\sigma \in I'}, X)$ for $T(\pi)$ on $S(G; \Gamma_{1}(N))$ shall be investigated. The coefficients $c_{i} \in K\langle \xi_{\sigma} | \sigma \in I' \rangle$ of $P((\xi_{\sigma})_{\sigma \in I'}, X)$ can be regarded as analytic functions on $\mathcal{W}(n,v)$. We then have the following

**Proposition 2.5.** Assume the condition (ram). We take two elements $(s_{\sigma})_{\sigma \in I'}, (s'_{\sigma})_{\sigma \in I'} \in \mathcal{W}_{(n,v)}(\mathbb{Q})$. We assume that there exists an integer $m \geq 0$ such that

$$|s_{\sigma} - s'_{\sigma}| \leq p^{-m \cdot \max_{1 \leq i \leq s} \{\frac{1}{e(\mathfrak{p}_{i})}\}}$$

for all $\sigma \in I'$. Then we have

$$|c_{i}(s_{\sigma})_{\sigma \in I'} - c_{i}(s'_{\sigma})_{\sigma \in I'}| \leq p^{-(m+\lambda') \min_{1 \leq i \leq s} \{\frac{1}{e(\mathfrak{p}_{i})}\}}$$

for all $i \geq 1$, where we put $\lambda' := \min_{1 \leq i \leq s} \{1 - \frac{e(\mathfrak{p}_{i})}{p-1}\}$.

**Proof.** Since $S(G; \Gamma_{1}(N))$ can be regarded as a direct summand of $S^{h}$ via the isomorphism (10) in Definition 1.3 (1), it is enough to show the statement for the coefficients $c'_{i}$ of the characteristic power series $P'(((\xi_{\sigma})_{\sigma \in I'}, X)$ of $T(\pi)$ on $S^{h}$.

Note that both $S(s,t)$ and $S(s',t')$ can be generated by the same orthonormal basis $\alpha n$ over $K(s,t)$ and $K(s',t')$, respectively. For $M = \left(\prod_{\tau \in I \setminus I'} X_{\tau}^{a_{\tau}}Y_{\tau}^{b_{\tau}}\right) \prod_{\sigma \in I'} x_{\sigma}^{m_{\sigma}} \in M$ and $\gamma = \gamma_{1} \left(\begin{array}{ll}1 & 0 \\ 0 & \pi \end{array}\right) \gamma_{2} \in \Gamma_{1}(N)$ with $\gamma_{1}, \gamma_{2} \in \Gamma_{1}(N)$, we see that

\begin{align}
\gamma(s,t) \cdot M &= \prod_{\tau \in I \setminus I'} \det(\gamma_{\tau})^{v_{r}} \left(\prod_{\sigma \in I'} j_{\sigma}(\gamma_{\sigma})^{s_{\sigma}} \det(\gamma_{1\sigma} \gamma_{2\sigma})^{t_{\sigma}}\right) \\
&\times \left(\prod_{\tau \in I \setminus I'} X_{\tau}^{a_{\tau}}Y_{\tau}^{b_{\tau}}\right) \prod_{\sigma \in I'} (\gamma_{\sigma} \cdot x_{\sigma})^{m_{\sigma}} \quad \text{and} \\
\gamma(s',t') \cdot M &= \prod_{\tau \in I \setminus I'} \det(\gamma_{\tau})^{v_{r}} \left(\prod_{\sigma \in I'} j_{\sigma}(\gamma_{\sigma})^{s'_{\sigma}} \det(\gamma_{1\sigma} \gamma_{2\sigma})^{t'_{\sigma}}\right) \\
&\times \left(\prod_{\tau \in I \setminus I'} X_{\tau}^{a_{\tau}}Y_{\tau}^{b_{\tau}}\right) \prod_{\sigma \in I'} (\gamma_{\sigma} \cdot x_{\sigma})^{m_{\sigma}}.
\end{align}
By the assumption that \(|s_{\sigma} - s'_{\sigma}| \leq p^{-\frac{m}{\epsilon(p_{i})}}\) for each \(\sigma \in I'\) with \(i\) such that \(\sigma \in I_{i}\), we can write in \(\mathbb{C}_{p}\)

\[s'_{\sigma} = s_{\sigma} + (\pi_{i}^{\sigma})^{m}u_{\sigma}\quad \text{and} \quad t'_{\sigma} = t_{\sigma} - \frac{u_{\sigma}}{2}(\pi_{i}^{\sigma})^{m}\]

with some \(u_{\sigma} \in \mathcal{O}_{\mathbb{C}_{p}}\) by Definition 1.2. Then we have

\[
\begin{align*}
\det(\gamma_{1\sigma}\gamma_{2\sigma})^{t'_{\sigma}} &= \det(\gamma_{1\sigma}\gamma_{2\sigma})^{t_{\sigma}}(\det(\gamma_{1\sigma}\gamma_{2\sigma})^{(\pi_{i}^{\sigma})^{m}}1^{u_{\vec{2}}} \text{ because we can see easily that} \\
\left| \frac{(\pi_{i}^{\sigma})^{km}(\pi_{i}^{\sigma})^{k}}{k!}u_{\sigma}'(u_{\sigma}' - 1)\cdots(u_{\sigma}' - k + 1) \right| &\leq |\pi_{i}^{\sigma}|^{m+\lambda'} (k \geq 1, \ m \geq 0)
\end{align*}
\]

under the condition (ram). Here the symbol \(u_{\sigma}'\) stands for both \(u_{\sigma}\) and \(\frac{u_{\sigma}}{2}\). By Proposition 2.2 and the isomorphism (11) in Definition 1.3, this implies that the absolute values of all components of the difference of the representation matrices of \(T(\pi)\) on \(S_{(s,t)}^{h}\) and the one on \(S_{(s',t')}^{h}\), calculated before are at most \(p^{-(m+\lambda')\min_{1\leq i \leq s}\left\{ \frac{1}{e(p_{i})} \right\}}\). This implies that

\[
|c_{i}((s_{\sigma})_{\sigma \in I'}) - c_{i}((s'_{\sigma})_{\sigma \in I'})| \leq p^{-(m+\lambda')\min_{1\leq i \leq s}\left\{ \frac{1}{e(p_{i})} \right\}}
\]

for all \(i \geq 1\).

Let \((s_{\sigma})_{\sigma \in I'}, (s'_{\sigma})_{\sigma \in I'} \in \mathcal{W}_{(n,v)}(\emptyset)\). By Proposition 2.4, we see that \(N_{(s,t)}(x) \geq \mu(x)\).

We put

\[
\nu(x) := \frac{2\lambda g'}{(g' + 1)(g' + 2)^{2}}(g'!\frac{1}{h_{n}})^{\frac{1}{2}}(x^{\frac{1}{2}}(g' + 1))^{\frac{1}{2}}
\]

for \(x \in \mathbb{R}_{\geq 0}\). Then \(\nu\) is a strictly monotone increasing function, and we have

\[
\nu(0) < 0 \quad \text{and} \quad \lim_{x \to \infty} \nu(x) = \infty.
\]

Moreover, the inverse function

\[
\nu^{-1}(x) = h_{n}(\frac{(g' + 1)(g' + 2)^{2}}{2\lambda g'(g'!)}\frac{1}{2}x + (g' + 1)^{\frac{1}{2}})g'
\]
of $\nu$ is also a monotone increasing function on $\mathbb{R}_{\geq 0}$ and $\nu^{-1}(x) \geq 0$ for $x \geq 0$. For $\alpha \in \mathbb{Q}_{\geq 0}$, we put

$$m(\alpha) := \left( \frac{\max_{1 \leq i \leq s} \{ e(p_i) \} } { \min_{1 \leq i \leq s} \{ e(p_i) \} } \right) [\alpha \nu^{-1}(\alpha)].$$

By Proposition 2.5, we then see that if $|s_{\sigma} - s'_{\sigma}| \leq p^{-m(\alpha)}$ for all $\sigma \in I'$, then

$$|c_i((s_{\sigma})_{\sigma \in I'}) - c_i((s'_{\sigma})_{\sigma \in I'})| \leq p^{-\min_{1 \leq i \leq s} \{ \frac{1}{e(p_i)} \} \min\{ \frac{\max_{1 \leq i \leq s} \{ e(p_i) \} } { \min_{1 \leq i \leq s} \{ e(p_i) \} } \} [\alpha \nu^{-1}(\alpha)] + \lambda')$$

for all $i \geq 1$. Since we can replace $\mathbb{Z}_p$ (resp. $m_{\nu}(\alpha) + 1$) by $\mathcal{O}_{\mathbb{C}_p}$ (resp. $\min_{1 \leq i \leq s} \{ \frac{1}{e(p_i)} \} \max\{ \frac{\max_{1 \leq i \leq s} \{ e(p_i) \} } { \min_{1 \leq i \leq s} \{ e(p_i) \} } \} [\alpha \nu^{-1}(\alpha)] + \lambda'$) in the statement of [14, Lemma 4.1], we have the following

**Proposition 2.6.** Assume the condition (ram). For any $\alpha \in \mathbb{Q}_{\geq 0}$, we put

$$m(\alpha) := \left( \frac{\max_{1 \leq i \leq s} \{ e(p_i) \} } { \min_{1 \leq i \leq s} \{ e(p_i) \} } \right) [\alpha h_n((g' + 1)(g' + 2)^2) + (g' + 1)^{\frac{1}{3}}]^{\frac{1}{3}}.$$

If $(s_{\sigma})_{\sigma \in I'}$, $(s'_{\sigma})_{\sigma \in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_p)$ satisfy $|s_{\sigma} - s'_{\sigma}| \leq p^{-m(\alpha)}$ for all $\sigma \in I'$, then the slope-$\alpha$-part of the Newton polygons of $P((s_{\sigma})_{\sigma \in I'}, X)$ and $P((s'_{\sigma})_{\sigma \in I'}, X)$ are equal.

By combining this proposition with [12, Corollary of Section IV.4], we obtain the following

**Theorem 2.7.** Assume the condition (ram). Let $\alpha \in \mathbb{Q}_{\geq 0}$ and $(s_{\sigma})_{\sigma \in I'}$, $(s'_{\sigma})_{\sigma \in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_p)$. If $|s_{\sigma} - s'_{\sigma}| \leq p^{-m(\alpha)}$ for all $\sigma \in I'$, then we have

$$\dim_{\mathcal{O}_p} S_{(s,t)}(G; \Gamma(1)(N))_{\mathbb{Q}}^{\alpha} = \dim_{\mathcal{O}_p} S_{(s',t')}((s_{\sigma})_{\sigma \in I'}, X)_{\mathbb{Q}}^{\alpha}.$$

Further, by Theorem 2.3, we then have immediately the following

**Corollary 2.8.** Assume the condition (ram). If $(n'_{\sigma})_{\sigma \in I'}$, $(n''_{\sigma})_{\sigma \in I'} \in W_{(n,v)}(\mathbb{C}_p)$ satisfy the conditions that $|n'_{\sigma} - n''_{\sigma}| \leq p^{-m(\alpha)}$ for all $\sigma \in I'$ and $\nu_{n'}, \nu_{n''} > \alpha$, then we have

$$\dim_{\mathcal{O}_p} S_{(n',t')}((s_{\sigma})_{\sigma \in I'}, X)_{\mathbb{Q}}^{\alpha} = \dim_{\mathcal{O}_p} S_{(n'',t'')}((s_{\sigma})_{\sigma \in I'}, X)_{\mathbb{Q}}^{\alpha}.$$

**Remark 2.2.** In Corollary 2.8, we need to assume the condition (ram) to apply the modified Wan's lemma with the positive rational number $\lambda'$. This corollary is a generalization of Coleman's result [5, Theorem B3.4] which gives a solution to a conjecture of Gouvêa and Mazur [7, Conjecture 1 in Section 5].

**Remark 2.3.** Kassaei [11] has constructed overconvergent $\mathcal{P}$-adic modular forms on quaternion algebras defined over any totally real field $F$ which are unramified at $\mathcal{P}$ and exactly one infinite place, where $\mathcal{P}$ is a
prime ideal of $F$ above $p$ whose residue field has cardinality $> 3$. Then he has also showed the local constancy of dimensions of the spaces of overconvergent forms ([11, Theorem 1.1]).

References


DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KYOTO, 606-8502, JAPAN

E-mail address: yamagami@math.kyoto-u.ac.jp