Generalized Lerch's formulas

Yoshinori Mizuno (水野 義紀)

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Osaka University (大阪大学理学研究科数学科攻D 3)

1 Generalized Lerch’s formulas

The zeta-regularized product of a countable sequence \( \{\lambda_k\} \subset \mathbb{C} \setminus \{0\} \) is defined by

\[
\prod_{k} \lambda_k = \exp \left( -\frac{\partial}{\partial s} \sum_{k} \lambda_k^{-s} \bigg|_{s=0} \right),
\]

provided that \( \Lambda(s) = \sum_k \lambda_k^{-s} \) is continued holomorphically at \( s = 0 \). Here the branch is chosen so that \(-\pi < \arg(\lambda_k) \leq \pi\).

There are several interesting formulas which can be formulated in terms of zeta-regularized products. Typical examples are Lerch's formula

\[
\prod_{n=0}^{\infty} (n + x) = \frac{\sqrt{2\pi}}{\Gamma(x)}
\] (1)

and Kronecker's limit formula

\[
\prod_{(c,d)=1} \frac{|cz+d|}{\sqrt{y}} = (y^6 |\Delta(z)|)^{-\frac{1}{6}}.
\] (2)

Here \( \Gamma(x) \) is Euler's gamma function and \( \Delta(z) = e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24} \) is Ramanujan's delta function.

In this paper, we generalize Lerch's formula.
Theorem 1 For $z_j \in \mathbb{C} \setminus \{0, -1, -2, ...\}$, we have
\[
\prod_{m=0}^{\infty} \left( \prod_{j=1}^{n} (m + z_j) \right) = \frac{(\sqrt{2\pi})^n}{\prod_{j=1}^{n} \Gamma(z_j)} = \prod_{j=1}^{n} \left( \prod_{m=0}^{\infty} (m + z_j) \right).
\]

As a part of Theorem 1, we can obtain the formula of Lerch, Kurokawa and Wakayama.

Corollary 1 (Lerch)
\[
\prod_{n=0}^{\infty} (n + x)^2 + y^2 = \frac{2\pi}{\Gamma(x + iy)\Gamma(x - iy)}.
\]

Corollary 2 (Kurokawa and Wakayama [5])
\[
\prod_{n=0}^{\infty} (n + x)^m - y^m = \frac{(\sqrt{2\pi})^m}{\prod_{\zeta^{m}=1} \Gamma(x - \zeta y)}.
\]

We would like to mention that our motivation of generalizing Lerch's formula is how $\prod_n (a_n \cdot b_n)$ is connected with $\prod_n a_n \cdot \prod_n b_n$.

Suppose that $a_n$ and $b_n$ depend on some parameters $X$. In many examples, we know
\[
\prod_n (a_n \cdot b_n) = e^{F(X)} \prod_n a_n \cdot \prod_n b_n \tag{3}
\]
with some $F(X)$. An interesting question is to understand $F(X)$.

Theorem 1 is an example of the case where $F(X)$ vanishes in (3). In fact we have

Corollary 3 For monic polynomials $P_j(x)$ such that $P_j(m) \neq 0$ for any $m \in \{0\} \cup \mathbb{N}$, one has
\[
\prod_{m=0}^{\infty} \left( \prod_{j=1}^{n} P_j(m) \right) = \prod_{j=1}^{n} \left( \prod_{m=0}^{\infty} P_j(m) \right).
\]

Corollary 3 is remarkable because it is saying that $F(X) = 0$ in (3), which does not hold in general at all. We can see examples for $F(X) \neq 0$ in Corollary 4 which will be given in Section 2 and Lemma 1 of [8].
2 Two dimensional analogue and $q$-analogue

There are two dimensional analogue and $q$-analogue of Euler's gamma function, so called Barnes' double gamma functions and Jackson's $q$-gamma functions (see [1], [7]). Hence it is natural to seek two dimensional analogue and $q$-analogue of Theorem 1.

Barnes' double gamma function $\Gamma_{2}^{*}(z, (\omega_{1}, \omega_{2}))$ is defined by

$$\log \Gamma_{2}^{*}(z, (\omega_{1}, \omega_{2})) = \frac{\partial}{\partial s} \sum_{m,l=0}^{\infty} (m\omega_{1} + l\omega_{2} + z)^{-s}|_{s=0},$$

$$\Gamma_{2}^{*}(z, (\omega_{1}, \omega_{2}))-1 = \prod_{m,l=0}^{\infty} (m\omega_{1} + l\omega_{2} + z).$$

We get a two dimensional analogue of Theorem 1 by using the following result.

**Theorem 2** Assume that $q_j, \tau_j, z_j \in \mathbb{C}$ satisfy that $\Re(q_j) > 0, \Re(\tau_j) > 0, \Re(z_j) > 0$, and $q_j \neq q_k, \tau_j \neq \tau_k, q_j\tau_k \neq q_k\tau_j$ for $j \neq k$. The function of $s$ defined by

$$H_{2}(s) = \sum_{m,l=0}^{\infty} \prod_{j=1}^{n} (mq_j + l\tau_j + z_j)^{-s}$$

is continued meromorphically to all $s$-plane. $H_{2}(s)$ is holomorphic at $s = 0$ and we have the following formula for $\frac{\partial}{\partial s}H_{2}(s)|_{s=0}$

$$\frac{\partial}{\partial s}H_{2}(s)|_{s=0} = \sum_{j=1}^{n} \log \Gamma_{2}^{*}(z_j, (q_j, \tau_j))$$

$$+ \frac{1}{2n} \sum_{1 \leq j < k \leq n} \left\{ \frac{q_j\tau_k - \tau_j q_k}{q_j q_k} \left( \log q_k - \log q_j \right) B_2 \left( \frac{q_j z_k - q_k z_j}{q_j \tau_k - \tau_j q_k} \right) \right\}$$

$$+ \frac{q_k\tau_j - \tau_k q_j}{\tau_j \tau_k} \left( \log \tau_k - \log \tau_j \right) B_2 \left( \frac{\tau_j z_k - \tau_k z_j}{\tau_j q_k - q_j \tau_k} \right).$$

Here $B_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial. We choose the principal branch for $\log q_i, \log \tau_i$.

This is a generalization of Shintani's result (see [12]). He treated the case $n = 2$ to give a new proof of Kronecker's limit formula (2).
We remark that in order to conclude
\[\exp \left( -\frac{\partial}{\partial s} H_2(s) \bigg|_{s=0} \right) = \prod_{m,l=0}^{\infty} \left( \prod_{j=1}^{n} (mq_j + lr_j + z_j) \right) ,\]
the equation
\[\left\{ \prod_{j=1}^{n} (mq_j + lr_j + z_j) \right\}^{s} = \prod_{j=1}^{n} (mq_j + lr_j + z_j)^s \quad (4)\]
must hold for any \(m, l \in \mathbb{N} \cup \{0\}\). We take this remark into account to give a two dimensional analogue of Theorem 1. As an example of \(q_j, \tau_j, z_j\) which satisfy the equation (4) for any \(m, l \in \mathbb{N} \cup \{0\}\), we can take \(n = 2h, q_j, \tau_j, z_j \in \mathbb{C}, q_{h+j} = \overline{q_j}, \tau_{h+j} = \overline{\tau_j}, z_{h+j} = \overline{z_j}, j = 1, \ldots, h.\)

**Corollary 4** Fix \(q_j, \tau_j, z_j \in \mathbb{C}\) such that \(\Re(q_j) > 0, \Re(\tau_j) > 0, \Re(z_j) > 0,\)
and \(q_j \neq q_k, \tau_j \neq \tau_k, q_j \tau_j \neq q_k \tau_k\) for \(j \neq k\). Suppose that (4) is satisfied for any \(m, l \in \mathbb{N} \cup \{0\}\). Then we have
\[\prod_{m,l=0}^{\infty} \left( \prod_{j=1}^{n} (mq_j + lr_j + z_j) \right) = e^F \prod_{j=1}^{n} \Gamma_2^*(z_j, (q_j, \tau_j))^{-1} = e^F \prod_{j=1}^{n} \left( \prod_{m,l=0}^{\infty} (mq_j + lr_j + z_j) \right),\]
where
\[F = -\frac{1}{2n} \sum_{1 \leq j < k \leq n} \left\{ \frac{q_j \tau_k - \tau_j q_k}{q_j q_k} (\log q_k - \log q_j) B_2 \left( \frac{q_j z_k - q_k z_j}{q_j \tau_k - \tau_j q_k} \right) + \frac{q_k \tau_j - \tau_k q_j}{\tau_j \tau_k} (\log \tau_k - \log \tau_j) B_2 \left( \frac{\tau_j z_k - \tau_k z_j}{\tau_j q_k - q_j \tau_k} \right) \right\} .\]

Next we present \(q\)-analogue of Theorem 1. Usually the zeta-regularized product is defined for a sequence \(\{\lambda_k\} \subset \mathbb{C} \setminus \{0\}\) such that \(\Lambda(s) = \sum_k \lambda_k^{-s}\)
can be continued holomorphically at \(s = 0\). In case \(\Lambda(s)\) is meromorphic at
s = 0, Kurokawa and Wakayama [6] define the generalized zeta regularization by
\[ \prod_k \lambda_k = \exp \left( - \operatorname{Res}_{s=0} \frac{\Lambda(s)}{s^2} \right). \]

They obtained several examples of such product, one of which is the following $q$-analogue of Lerch's formula.

**Theorem 3 (Kurokawa and Wakayama [6])** For $q > 1, x > 0$,
\[ \prod_{n=0}^{\infty} [n + x]_q = \frac{C_q}{\Gamma_q(x)}. \]

Here $[x]_q = \frac{q^x - 1}{q - 1}$ is the $q$-analogue of number $x$,
\[ \Gamma_q(x) = \frac{\prod_{n=1}^{\infty} (1 - q^{-n})}{\prod_{n=0}^{\infty} (1 - q^{-(x+n)})} (q-1)^{1-x} q^{\frac{xe(x-1)}{2}} \]
is Jackson's $q$-gamma function,
\[ C_q = \prod_{n=1}^{\infty} [n]_q = q^{-\frac{1}{12}} (q-1)^{\frac{1}{2} - \frac{\log(q-1)}{2\log q}} \prod_{n=1}^{\infty} (1 - q^{-n}). \]

We obtain the next result which is the $q$-analogue of Theorem 1 including the above Theorem 3.

**Theorem 4** For $q > 1, z_j > 1$, we have
\[
\prod_{m=0}^{\infty} \left( \prod_{j=1}^{n} [m + z_j]_q \right) = \frac{C_q^n}{\prod_{j=1}^{n} \Gamma_q(z_j)} q^{-\frac{1}{2n} (\sum_{j=1}^{n} z_j)^2 + \frac{1}{2} \sum_{j=1}^{n} z_j^2}.
\]

\[
= q^{-\frac{1}{2n} (\sum_{j=1}^{n} z_j)^2 + \frac{1}{2} \sum_{j=1}^{n} z_j^2} \prod_{j=1}^{n} \left( \prod_{m=0}^{\infty} [n + z_j]_q \right).
\]
3 Double Hurwitz zeta

For $\beta > \alpha > 0$, let $H_{\alpha,\beta}(s_1, s_2)$ be Dirichlet series defined by

$$H_{\alpha,\beta}(s_1, s_2) = \sum_{n=0}^{\infty} (n + \alpha)^{-s_1}(n + \beta)^{-s_2}.$$  

This series converges absolutely for $\Re(s_1 + s_2) > 1$.

$H_{\alpha,\beta}(s_1, s_2)$ is an important object in the theory of the zeta-regularized product. For example, as we presented in Section 1, we know generalized Lerch's formula

$$\exp \left( - \frac{\partial}{\partial s} H_{\alpha,\beta}(s, s) \bigg|_{s=0} \right) = \frac{2\pi}{\Gamma(\alpha)\Gamma(\beta)}.$$  

We know also that the spectral zeta function $Z_n(s)$ of the unit $n$-sphere $S^{n-1}$ can be written in terms of $H_{\alpha,\beta}(s_1, s_2)$ as

$$Z_n(s) = \sum_{d=0}^{n-1} T_{n,d} H_{1,n}(s-d, s), \quad (5)$$

where

$$T_{n,d} = \frac{1}{n!} \sum_{r=d+1}^{n} s(n,r) \binom{r}{d} (n^{r-d} - (n-2)^{r-d}),$$

$s(r,d)$ denoting the Stirling numbers of the first kind. See Lemma 2 of [4] p.202. We get the formula for the functional determinant of the Laplacian by evaluating $\frac{\partial}{\partial s} Z_n(s) \bigg|_{s=0}$. See Theorem 1 of [4] p. 200.

In the results mentioned above, the main target is not $H_{\alpha,\beta}(s_1, s_2)$ itself but evaluating derivative of $H_{\alpha,\beta}(s_1, s_2)$. In this section, we analyze $H_{\alpha,\beta}(s_1, s_2)$ itself. First by applying the method described in [2], we can get the following expression for $H_{\alpha,\beta}(s_1, s_2)$.

$$H_{\alpha,\beta}(s_1, s_2) = \frac{\Gamma(s_1 + s_2)}{\Gamma(s_1)\Gamma(s_2)} \int_{0}^{1} u^{s_2-1}(1-u)^{s_1-1}\zeta(s_1 + s_2, \alpha - (\alpha - \beta)u)du, \quad (6)$$

where $\zeta(s, x) = \sum_{n=0}^{\infty} (n+x)^{-s}$ is Hurwitz zeta function. It is very interesting to note that S. Ramanujan already treated the integral of the right hand side on (6) apart from Dirichlet series $H_{\alpha,\beta}(s_1, s_2)$. See (14) of [9] p.166.

Starting from the integral expression (6), we show the following results.

Theorem 5 $H_{\alpha,\beta}(s_1, s_2)$ can be continued meromorphically to all $s_1, s_2 \in \mathbb{C}$. 

Theorem 6 For $\Re(s_1) < 0, \Re(s_2) < 0, 0 < \alpha < \beta < 1$, we have

$$H_{\alpha, \beta}(s_1, s_2) = \frac{\Gamma(1-s_1-s_2)}{(2\pi)^{1-s_1-s_2}} \times \left\{ e^{\frac{\pi i}{2}(1-s_1-s_2)} \sum_{n=1}^{\infty} n^{s_1+s_2-1} e^{-2\pi i n(\beta - \alpha)} 1F_1(s_1, s_1+s_2, 2\pi i n(\beta - \alpha)) + e^{-\frac{\pi i}{2}(1-s_1-s_2)} \sum_{n=1}^{\infty} n^{s_1+s_2-1} e^{2\pi i n \alpha} 1F_1(s_2, s_1+s_2, 2\pi i n(\beta - \alpha)) \right\}. \quad (7)$$

Here $1F_1(a, b, z)$ is the confluent hypergeometric series defined by

$$1F_1(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \quad (8)$$

with $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$.

This is a generalization of well known Hurwitz relation for $\zeta(s, x)$.

Theorem 7 We have

$$H_{\alpha, \beta}(s_1, s_2) = \sum_{n=0}^{\infty} \frac{(s_2)_n}{n!} \zeta(s_1 + s_2 + n, \alpha)(\alpha - \beta)^n.$$

This is a special case of Main Theorem of [3]. However we can prove Main Theorem of [3] by quite different manner using the confluent hypergeometric series $1F_1(a, b, z)$.

Next we give the evaluation formula of $H_{\alpha, \beta}(s_1, s_2)$. We can evaluate the values of $H_{\alpha, \beta}(s_1, s_2)$ at any integers $s_1, s_2$ in terms of the values of Hurwitz zeta function.

Theorem 8 For $p, q \in \mathbb{N}$, we have

$$H_{\alpha, \beta}(q, p) = \frac{\Gamma(p+q)}{(p+q-1)!} \frac{\Gamma(p) \Gamma(q)}{\Gamma(q)} \times \left\{ \sum_{n=0}^{p+q-3} \sum_{m=\max(n-p+1,0)}^{q-1} (-1)^m \binom{q-1}{m} \binom{p+m-1}{n} \right\}$$
\[
\times \frac{n!}{(2 - p - q)_n} \zeta(p + q - n - 1, \beta)(\alpha - \beta)^{-n-1} \\
- \sum_{m=0}^{q-2} (-1)^m \binom{q-1}{m} \frac{(p + m - 1)!}{(2 - p - q)_{p+m-1}} \zeta(q - m, \alpha)(\alpha - \beta)^{-p-m} \\
+ (-1)^{q-1} \frac{(p + q - 2)!}{(2 - p - q)_{p+q-2}} (\alpha - \beta)^{-p-q+1} \left( \frac{\Gamma'(\beta)}{\Gamma(\beta)} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} \right) \}
\]

Here empty sum is considered as zero.

**Theorem 9** For \( p, q \in \mathbb{Z} \) which are not both negative, we have

\[
H_{\alpha, \beta}(-p, -q) = \sum_{k=0}^{q} \binom{q}{k} (\beta - \alpha)^k \zeta(-p - q + k, \alpha) \\
+ \sum_{k=0}^{p} \binom{p}{k} (\alpha - \beta)^k \zeta(-p - q + k, \beta).
\]

Here empty sum is considered as zero.

Finally we mention that we can provide another approach to evaluate the determinant \( \det \Delta_n \) of the Laplacian on the \( n \)-sphere \( S^{n-1} \) starting from the integral expression (6). Here \( \det \Delta_n \) is defined by

\[
\det \Delta_n = \exp \left( -\sum_{d=0}^{n-1} T_{n,d} \frac{\partial}{\partial s} H_{1,n}(s-d, s) \bigg|_{s=0} \right).
\]

Sec (5) for the definition of \( T_{n,d} \).

**Theorem 10**

\[
\frac{\partial}{\partial s} H_{1,n+1}(s-d, s) \bigg|_{s=0} = \zeta'(-d) + \sum_{l=0}^{d} (-n)^{d-l} \binom{d}{l} \zeta'(-l, n+1) \\
- \frac{(-n)^{d+1}}{2(d+1)} \left( \sum_{j=1}^{d} \frac{1}{j} \right).
\]

This is simpler than Kumagai’s formula given in Lemma 3 of [4] p.202. Comparing Theorem 10 and Kumagai’s result, we get the following identity for harmonic numbers.
Corollary 5 The following identity holds:

\[ 2^{1-d} \sum_{l=1, \text{odd}}^{d} \binom{d+1}{l+1} \sum_{j=1, \text{odd}}^{l} \frac{1}{j} = \sum_{j=1}^{d} \frac{1}{j}. \]

References


