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<tr>
<td>引用</td>
<td>数理解析研究所講究録 2005年 1451巻 51-60</td>
</tr>
<tr>
<td>発行年月</td>
<td>2005-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/47750">http://hdl.handle.net/2433/47750</a></td>
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<tr>
<td>タイプ</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>本文バージョン</td>
<td>publisher</td>
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Generalized Lerch formulas

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March 15, 2005

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1 Generalized Lerch's formulas

The zeta-regularized product of a countable sequence \( \{ \lambda_k \} \subset \mathbb{C} \setminus \{ 0 \} \) is defined by

\[
\prod \lambda_k = \exp \left( -\frac{\partial}{\partial s} \sum_k \lambda_k^{-s} \bigg|_{s=0} \right),
\]

provided that \( \Lambda(s) = \sum \lambda_k^{-s} \) is continued holomorphically at \( s = 0 \). Here the branch is chosen so that \( -\pi < \arg(\lambda_k) \leq \pi \).

There are several interesting formulas which can be formulated in terms of zeta-regularized products. Typical examples are Lerch's formula

\[
\prod_{n=0}^{\infty} (n + x) = \frac{\sqrt{2\pi}}{\Gamma(x)}
\]

(1)

and Kronecker's limit formula

\[
\prod_{(c,d)=1} \frac{|cz + d|}{\sqrt{y}} = (y^6 |\Delta(z)|)^{-\frac{1}{6}}.
\]

(2)

Here \( \Gamma(x) \) is Euler's gamma function and \( \Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24} \) is Ramanujan's delta function.

In this paper, we generalize Lerch's formula.
Theorem 1 For \( z_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots \} \), we have
\[
\prod_{m=0}^{\infty} \left( \prod_{j=1}^{n} (m + z_j) \right) = \frac{(\sqrt{2\pi})^n}{\prod_{j=1}^{n} \Gamma(z_j)} = \prod_{j=1}^{n} \left( \prod_{m=0}^{\infty} (m + z_j) \right).
\]

As a part of Theorem 1, we can obtain the formula of Lerch, Kurokawa and Wakayama.

Corollary 1 (Lerch)
\[
\prod_{n=0}^{\infty} \left( (n + x)^2 + y^2 \right) = \frac{2\pi}{\Gamma(x + iy)\Gamma(x - iy)}.
\]

Corollary 2 (Kurokawa and Wakayama [5])
\[
\prod_{n=0}^{\infty} \left( (n + x)^m - y^m \right) = \frac{(\sqrt{2\pi})^m}{\prod_{\zeta^m=1} \Gamma(x - \zeta y)}.
\]

We would like to mention that our motivation of generalizing Lerch's formula is how \( \prod_n (a_n \cdot b_n) \) is connected with \( \prod_n a_n \cdot \prod_n b_n \).

Suppose that \( a_n \) and \( b_n \) depend on some parameters \( X \). In many examples, we know
\[
\prod_n (a_n \cdot b_n) = e^{F(X)} \prod_n a_n \cdot \prod_n b_n \tag{3}
\]
with some \( F(X) \). An interesting question is to understand \( F(X) \).

Theorem 1 is an example of the case where \( F(X) \) vanishes in (3). In fact we have

Corollary 3 For monic polynomials \( P_j(x) \) such that \( P_j(m) \neq 0 \) for any \( m \in \{0\} \cup \mathbb{N} \), one has
\[
\prod_{m=0}^{\infty} \left( \prod_{j=1}^{n} P_j(m) \right) = \prod_{j=1}^{n} \left( \prod_{m=0}^{\infty} P_j(m) \right).
\]

Corollary 3 is remarkable because it is saying that \( F(X) = 0 \) in (3), which does not hold in general at all. We can see examples for \( F(X) \neq 0 \) in Corollary 4 which will be given in Section 2 and Lemma 1 of [8].
2 Two dimensional analogue and $q$-analogue

There are two dimensional analogue and $q$-analogue of Euler's gamma function, so called Barnes' double gamma functions and Jackson's $q$-gamma functions (see [1], [7]). Hence it is natural to seek two dimensional analogue and $q$-analogue of Theorem 1.

Barnes' double gamma function $\Gamma_2^*(z, (\omega_1, \omega_2))$ is defined by

$$\log \Gamma_2^*(z, (\omega_1, \omega_2)) = \frac{\partial}{\partial s} \sum_{m,l=0}^{\infty} (m\omega_1 + l\omega_2 + z)^{-s} \bigg|_{s=0},$$

$$\Gamma_2^*(z, (\omega_1, \omega_2))^{-1} = \prod_{m,l=0}^{\infty} (m\omega_1 + l\omega_2 + z).$$

We get a two dimensional analogue of Theorem 1 by using the following result.

Theorem 2 Assume that $q_j, \tau_j, z_j \in \mathbb{C}$ satisfy that $\Re(q_j) > 0, \Re(\tau_j) > 0, \Re(z_j) > 0$, and $q_j \neq q_k, \tau_j \neq \tau_k, q_j \tau_k \neq q_k \tau_j$ for $j \neq k$. The function of $s$ defined by

$$H_2(s) = \sum_{m,l=0}^{\infty} \prod_{j=1}^{n} (mq_j + l\tau_j + z_j)^{-s}$$

is continued meromorphically to all $s$-plane. $H_2(s)$ is holomorphic at $s = 0$ and we have the following formula for $\frac{\partial}{\partial s} H_2(s) \big|_{s=0}$

$$\frac{\partial}{\partial s} H_2(s) \bigg|_{s=0} = \sum_{j=1}^{n} \log \Gamma_2^*(z_j, (q_j, \tau_j)) + \frac{1}{2n} \sum_{1 \leq j < k \leq n} \left\{ \frac{q_j \tau_k - \tau_j q_k}{q_j q_k} (\log q_k - \log q_j) B_2 \left( \frac{q_j z_k - q_k z_j}{q_j \tau_k - \tau_j q_k} \right) \right\} + \frac{q_k \tau_j - \tau_k q_j}{\tau_j \tau_k} (\log \tau_k - \log \tau_j) B_2 \left( \frac{\tau_j z_k - \tau_k z_j}{\tau_j q_k - q_j \tau_k} \right).$$

Here $B_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial. We choose the principal branch for $\log q_i, \log \tau_i$.

This is a generalization of Shintani's result (see [12]). He treated the case $n = 2$ to give a new proof of Kronecker's limit formula (2).
We remark that in order to conclude
\[
\exp \left( -\frac{\partial}{\partial s} H_2(s) \bigg|_{s=0} \right) = \prod_{m,l=0}^{\infty} \left( \prod_{j=1}^{n} (mq_j + l\tau_j + z_j) \right),
\]
the equation
\[
\left\{ \prod_{j=1}^{n} (mq_j + l\tau_j + z_j) \right\}^s = \prod_{j=1}^{n} (mq_j + l\tau_j + z_j)^s
\]
(4)
must hold for any \( m, l \in \mathbb{N} \cup \{0\} \). We take this remark into account to give a two dimensional analogue of Theorem 1. As an example of \( q_j, \tau_j, z_j \) which satisfy the equation (4) for any \( m, l \in \mathbb{N} \cup \{0\} \), we can take \( n=2h \), \( q_j, \tau_j, z_j \in \mathbb{C}, q_{h+j} = \overline{q_j}, \tau_{h+j} = \overline{\tau_j}, z_{h+j} = \overline{z_j}, j = 1, \ldots, h \).

**Corollary 4** Fix \( q_j, \tau_j, z_j \in \mathbb{C} \) such that \( \Re(q_j) > 0, \Re(\tau_j) > 0, \Re(z_j) > 0 \), and \( q_j \neq q_k, \tau_j \neq \tau_k, q_j \tau_k \neq q_k \tau_j \) for \( j \neq k \). Suppose that (4) is satisfied for any \( m, l \in \mathbb{N} \cup \{0\} \). Then we have
\[
\prod_{m,l=0}^{\infty} \left( \prod_{j=1}^{n} (mq_j + l\tau_j + z_j) \right) = e^{F} \prod_{j=1}^{n} \Gamma_2(z_j, (q_j, \tau_j))^{-1}
\]
\[
= e^{F} \prod_{j=1}^{n} \left( \prod_{m,l=0}^{\infty} (mq_j + l\tau_j + z_j) \right),
\]
where
\[
F = -\frac{1}{2n} \sum_{1 \leq j < k \leq n} \left\{ \frac{q_j \tau_k - \tau_j q_k}{q_j q_k} (\log q_k - \log q_j) B_2 \left( \frac{q_j z_k - q_k z_j}{q_j \tau_k - \tau_j q_k} \right) \right. \\
+ \frac{q_k \tau_j - \tau_k q_j}{\tau_j \tau_k} (\log \tau_k - \log \tau_j) B_2 \left( \frac{\tau_j z_k - \tau_k z_j}{\tau_j q_k - q_j \tau_k} \right) \left. \right\}.
\]

Next we present \( q \)-analogue of Theorem 1. Usually the zeta-regularized product is defined for a sequence \( \{\lambda_k\} \subset \mathbb{C} \setminus \{0\} \) such that \( \Lambda(s) = \sum_k \lambda_k^{-s} \) can be continued holomorphically at \( s = 0 \). In case \( \Lambda(s) \) is meromorphic at
$s = 0$, Kurokawa and Wakayama [6] define the generalized zeta regularisation by
\[ \prod_k \lambda_k = \exp \left( - \text{Res}_{s=0} \frac{\Lambda(s)}{s^2} \right). \]

They obtained several examples of such product, one of which is the following $q$-analogue of Lerch's formula.

Theorem 3 (Kurokawa and Wakayama [6]) For $q > 1, x > 0$,
\[ \prod_{n=0}^{\infty} [n + x]_q = \frac{C_q}{\Gamma_q(x)}. \]

Here $[x]_q = \frac{q^x - 1}{q - 1}$ is the $q$-analogue of number $x$,
\[ \Gamma_q(x) = \frac{\prod_{m=1}^{\infty} (1 - q^{-m})}{\prod_{m=0}^{\infty} (1 - q^{-(x+m)})} (q-1)^{1-x} q^{\frac{x(x-1)}{2}} \]

is Jackson's $q$-gamma function,
\[ C_q = \prod_{n=1}^{\infty} [n]_q = q^{-\frac{1}{12}} (q-1)^{-\frac{1}{2} - \frac{\log(q-1)}{2\log q}} \prod_{n=1}^{\infty} (1 - q^{-n}). \]

We obtain the next result which is the $q$-analogue of Theorem 1 including the above Theorem 3.

Theorem 4 For $q > 1, z_j > 1$, we have
\[ \prod_{m=0}^{\infty} \left( \prod_{j=1}^{n} [m + z_j]_q \right) = \frac{C_q^n}{\prod_{j=1}^{n} \Gamma_q(z_j)} q^{-\frac{1}{2n} (\sum_{j=1}^{n} z_j)^2 + \frac{1}{2} \sum_{j=1}^{n} z_j^2} \]
\[ = q^{-\frac{1}{2n} (\sum_{j=1}^{n} z_j)^2 + \frac{1}{2} \sum_{j=1}^{n} z_j^2} \prod_{j=1}^{n} \left( \prod_{m=0}^{\infty} [n + z_j]_q \right). \]
3 Double Hurwitz zeta

For \( \beta > \alpha > 0 \), let \( H_{\alpha,\beta}(s_1, s_2) \) be Dirichlet series defined by

\[
H_{\alpha,\beta}(s_1, s_2) = \sum_{n=0}^{\infty} (n + \alpha)^{-s_1} (n + \beta)^{-s_2}.
\]

This series converges absolutely for \( \Re(s_1 + s_2) > 1 \).

\( H_{\alpha,\beta}(s_1, s_2) \) is an important object in the theory of the zeta-regularized product. For example, as we presented in Section 1, we know generalized Lerch’s formula

\[
\exp \left( -\frac{\partial}{\partial s} H_{\alpha,\beta}(s, s) \bigg|_{s=0} \right) = \frac{2\pi}{\Gamma(\alpha)\Gamma(\beta)}.
\]

We know also that the spectral zeta function \( Z_n(s) \) of the unit \( n \)-sphere \( S_{n-1} \) can be written in terms of \( H_{\alpha,\beta}(s_1, s_2) \) as

\[
Z_n(s) = \sum_{d=0}^{n-1} T_{n,d} H_{1,n}(s-d, s),
\]

where

\[
T_{n,d} = \frac{1}{n!} \sum_{r=d+1}^{n} s(n, r) \binom{r}{d} (n^{r-d} - (n-2)^{r-d}),
\]

\( s(r,d) \) denoting the Stirling numbers of the first kind. See Lemma 2 of [4] p.202. We get the formula for the functional determinant of the Laplacian by evaluating \( \frac{\partial}{\partial s} Z_n(s) \bigg|_{s=0} \). See Theorem 1 of [4] p. 200.

In the results mentioned above, the main target is not \( H_{\alpha,\beta}(s_1, s_2) \) itself but evaluating derivative of \( H_{\alpha,\beta}(s_1, s_2) \). In this section, we analyze \( H_{\alpha,\beta}(s_1, s_2) \) itself. First by applying the method described in [2], we can get the following expression for \( H_{\alpha,\beta}(s_1, s_2) \).

\[
H_{\alpha,\beta}(s_1, s_2) = \frac{\Gamma(s_1 + s_2)}{\Gamma(s_1)\Gamma(s_2)} \int_0^1 u^{s_2-1}(1-u)^{s_1-1}\zeta(s_1 + s_2, \alpha - (\alpha - \beta)u)du,
\]

where \( \zeta(s, x) = \sum_{n=0}^{\infty} (n+x)^{-s} \) is Hurwitz zeta function. It is very interesting to note that S. Ramanujan already treated the integral of the right hand side on (6) apart from Dirichlet series \( H_{\alpha,\beta}(s_1, s_2) \). See (14) of [9] p.166.

Starting from the integral expression (6), we show the following results.

\[ \textbf{Theorem 5} \] \( H_{\alpha,\beta}(s_1, s_2) \) can be continued meromorphically to all \( s_1, s_2 \in \mathbb{C} \).
Theorem 6 For $\Re(s_1) < 0, \Re(s_2) < 0, 0 < \alpha < \beta < 1$, we have

$$H_{\alpha,\beta}(s_1, s_2) = \frac{\Gamma(1 - s_1 - s_2)}{(2\pi)^{1-s_1-s_2}} \times \left\{ e^{\frac{\pi i}{2}(1-s_1-s_2)} \sum_{n=1}^{\infty} n^{s_1+s_2-1} e^{-2\pi in \beta} \sum_{n=1}^{\infty} n^{s_2-1} e^{2\pi in \alpha} 1F_1(s_1 + s_2, 2\pi in (\beta - \alpha)) 
+ e^{-\frac{\pi i}{2}(1-s_1-s_2)} \sum_{n=1}^{\infty} n^{s_1+s_2-1} e^{2\pi in \alpha} \sum_{n=1}^{\infty} n^{s_1-1} e^{-2\pi in \beta} 1F_1(s_1, s_1+s_2, 2\pi in (\beta - \alpha)) \right\}. \quad (7)$$

Here $1F_1(a, b, z)$ is the confluent hypergeometric series defined by

$$1F_1(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \quad (8)$$

with $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$.

This is a generalization of well known Hurwitz relation for $\zeta(s, x)$.

Theorem 7 We have

$$H_{\alpha,\beta}(s_1, s_2) = \sum_{n=0}^{\infty} \frac{(s_2)_n}{n!} \zeta(s_1 + s_2 + n, \alpha)(\alpha - \beta)^n \zeta(n, \alpha).$$

This is a special case of Main Theorem of [3]. However we can prove Main Theorem of [3] by quite different manner using the confluent hypergeometric series $1F_1(a, b, z)$.

Next we give the evaluation formula of $H_{\alpha,\beta}(s_1, s_2)$. We can evaluate the values of $H_{\alpha,\beta}(s_1, s_2)$ at any integers $s_1, s_2$ in terms of the values of Hurwitz zeta function.

Theorem 8 For $p, q \in \mathbb{N}$, we have

$$H_{\alpha,\beta}(p+q, p) = \frac{\Gamma(p+q)}{(p+q-1)!\Gamma(p)\Gamma(q)} \times \left\{ \sum_{n=0}^{p+q-3} \left( \sum_{m=\max\{n-p+1,0\}}^{q-1} (-1)^m \binom{q-1}{m} \binom{p+m-1}{n} \right) \right\}.$$
\[ \begin{align*}
\times \frac{n!}{(2-p-q)_n} \zeta(p + q - n - 1, \beta) (\alpha - \beta)^{-n-1} & - \sum_{m=0}^{q-2} (-1)^m \binom{q-1}{m} \frac{(p+m-1)!}{(2-p-q)_{p+m-1}} \zeta(q-m, \alpha) (\alpha - \beta)^{-p-m} \\
+ (-1)^{q-1} \frac{(p+q-2)!}{(2-p-q)_{p+q-2}} (\alpha - \beta)^{-p-q+1} \left( \frac{\Gamma'(\beta)}{\Gamma(\beta)} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} \right) \end{align*} \]

Here empty sum is considered as zero.

**Theorem 9** For \( p, q \in \mathbb{Z} \) which are not both negative, we have

\[ H_{\alpha, \beta}(-p, -q) = \sum_{k=0}^{q} \binom{q}{k} (\beta - \alpha)^k \zeta(-p-q+k, \alpha) + \sum_{k=0}^{p} \binom{p}{k} (\alpha - \beta)^k \zeta(-p-q+k, \beta). \]

Here empty sum is considered as zero.

Finally we mention that we can provide another approach to evaluate the determinant \( \det \Delta_n \) of the Laplacian on the \( n \)-sphere \( S^{n-1} \) starting from the integral expression (6). Here \( \det \Delta_n \) is defined by

\[ \det \Delta_n = \exp \left( - \sum_{d=0}^{n-1} T_{n,d} \frac{\partial}{\partial s} H_{1,n}(s-d, s) \bigg|_{s=0} \right). \]

Sec (5) for the definition of \( T_{n,d} \).

**Theorem 10**

\[ \frac{\partial}{\partial s} H_{1,n+1}(s-d, s) \bigg|_{s=0} = \zeta'(-d) + \sum_{l=0}^{d} (-n)^{d-l} \binom{d}{l} \zeta'(-l, n+1) \]

\[ - \frac{(-n)^{d+1}}{2(d+1)} \left( \sum_{j=1}^{d} \frac{1}{j^2} \right). \]

This is simpler than Kumagai's formula given in Lemma 3 of [4] p.202. Comparing Theorem 10 and Kumagai's result, we get the following identity for harmonic numbers.
Corollary 5 The following identity holds:

$$2^{1-d} \sum_{l=1, \text{odd}}^{d} \left( \frac{d+1}{l+1} \right) \sum_{j=1, \text{odd}}^{l} \frac{1}{j} = \sum_{j=1}^{d} \frac{1}{j}.$$ 

References


