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Kyoto University
Generalized Lerch formulas

Yoshinori Mizuno (水野 義紀)

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Osaka University （大阪大学理学研究科学数学専攻 D3）

1 Generalized Lerch’s formulas

The zeta-regularized product of a countable sequence \(\{\lambda_k\} \subset \mathbb{C} \setminus \{0\}\) is defined by

\[
\prod_n \lambda_k = \exp \left( \frac{-\partial}{\partial s} \sum_k \lambda_k^{-s} \bigg|_{s=0} \right),
\]

provided that \(A(s) = \sum \lambda_k^{-s}\) is continued holomorphically at \(s = 0\). Here the branch is chosen so that \(-\pi < \arg(\lambda_k) \leq \pi\).

There are several interesting formulas which can be formulated in terms of zeta-regularized products. Typical examples are Lerch’s formula

\[
\prod_{n=0}^{\infty} (n + x) = \frac{\sqrt{2\pi}}{\Gamma(x)}
\] (1)

and Kronecker’s limit formula

\[
\prod_{(c,d)=1} \frac{|cz+d|}{\sqrt{y}} = (y^6 |\Delta(z)|)^{-\frac{1}{6}}.
\] (2)

Here \(\Gamma(x)\) is Euler’s gamma function and \(\Delta(z) = e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^2\) is Ramanujan’s delta function.

In this paper, we generalize Lerch’s formula.
**Theorem 1** For \( z_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots \} \), we have
\[
\prod_{m=0}^{\infty} \left( \prod_{j=1}^{n} (m + z_j) \right) = \frac{(\sqrt{2\pi})^n}{\prod_{j=1}^{n} \Gamma(z_j)} = \prod_{j=1}^{n} \left( \prod_{m=0}^{\infty} (m + z_j) \right).
\]

As a part of Theorem 1, we can obtain the formula of Lerch, Kurokawa and Wakayama.

**Corollary 1 (Lerch)**
\[
\prod_{n=0}^{\infty} ((n+x)^{2} + y^{2}) = \frac{2\pi}{\Gamma(x+iy)\Gamma(x-iy)}.
\]

**Corollary 2 (Kurokawa and Wakayama [5])**
\[
\prod_{n=0}^{\infty} ((n+x)^{m} - y^{m}) = \frac{(\sqrt{2\pi})^m}{\prod_{\zeta=1}^{m} \Gamma(x-\zeta y)}.
\]

We would like to mention that our motivation of generalizing Lerch's formula is how \( \prod_{n}(a_n \cdot b_n) \) is connected with \( \prod_{n}a_n \cdot \prod_{n}b_n \).

Suppose that \( a_n \) and \( b_n \) depend on some parameters \( X \). In many examples, we know
\[
\prod_{n}(a_n \cdot b_n) = e^{F(X)} \prod_{n}a_n \cdot \prod_{n}b_n \tag{3}
\]
with some \( F(X) \). An interesting question is to understand \( F(X) \).

Theorem 1 is an example of the case where \( F(X) \) vanishes in (3). In fact we have

**Corollary 3** For monic polynomials \( P_j(x) \) such that \( P_j(m) \neq 0 \) for any \( m \in \{0\} \cup \mathbb{N} \), one has
\[
\prod_{m=0}^{\infty} \left( \prod_{j=1}^{n} P_j(m) \right) = \prod_{j=1}^{n} \left( \prod_{m=0}^{\infty} P_j(m) \right).
\]

Corollary 3 is remarkable because it is saying that \( F(X) = 0 \) in (3), which does not hold in general at all. We can see examples for \( F(X) \neq 0 \) in Corollary 4 which will be given in Section 2 and Lemma 1 of [8].
2 Two dimensional analogue and $q$-anologue

There are two dimensional analogue and $q$-analogue of Euler's gamma function, so called Barnes' double gamma functions and Jackson's $q$-gamma functions (see [1], [7]). Hence it is natural to seek two dimensional analogue and $q$-analogue of Theorem 1.

Barnes' double gamma function $\Gamma_{2}(z, (\omega_{1}, \omega_{2}))$ is defined by

$$\log \Gamma_{2}(z, (\omega_{1}, \omega_{2})) = \frac{\partial}{\partial s} \sum_{m,l=0}^{\infty} (m\omega_{1} + l\omega_{2} + z)^{-s} \bigg|_{s=0},$$

$$\Gamma_{2}^{-1}(z, (\omega_{1}, \omega_{2})) = \prod_{m,l=0}^{\infty} (m\omega_{1} + l\omega_{2} + z).$$

We get a two dimensional analogue of Theorem 1 by using the following result.

**Theorem 2** Assume that $q_{j}, \tau_{j}, z_{j} \in \mathbb{C}$ satisfy that $\Re(q_{j}) > 0, \Re(\tau_{j}) > 0, \Re(z_{j}) > 0$, and $q_{j} \neq q_{k}, \tau_{j} \neq \tau_{k}, q_{j}\tau_{k} \neq q_{k}\tau_{j}$ for $j \neq k$. The function of $s$ defined by

$$H_{2}(s) = \sum_{m,l=0}^{\infty} \prod_{j=1}^{n} (mq_{j} + l\tau_{j} + z_{j})^{-s}$$

is continued meromorphically to all $s$-plane. $H_{2}(s)$ is holomorphic at $s = 0$ and we have the following formula for $\frac{\partial}{\partial s} H_{2}(s) |_{s=0}$,

$$\frac{\partial}{\partial s} H_{2}(s) |_{s=0} = \sum_{j=1}^{n} \log \Gamma_{2}(z_{j}, (q_{j}, \tau_{j}))$$

$$+ \frac{1}{2n} \sum_{1 \leq j < k \leq n} \left\{ \frac{q_{j}\tau_{k} - \tau_{j}q_{k}}{q_{j}q_{k}} (\log q_{k} - \log q_{j}) B_{2} \left( \frac{q_{j}z_{k} - q_{k}z_{j}}{q_{j}\tau_{k} - \tau_{j}q_{k}} \right) \right\}.$$ 

Here $B_{2}(x) = x^{2} - x + 1/6$ is the second Bernoulli polynomial. We choose the principal branch for $\log q_{i}, \log \tau_{i}$.

This is a generalization of Shintani's result (see [12]). He treated the case $n = 2$ to give a new proof of Kronecker's limit formula (2).
We remark that in order to conclude

\[
\exp \left( -\frac{\partial}{\partial s} H_2(s) \bigg|_{s=0} \right) = \prod_{m,l=0}^{\infty} \left( \prod_{j=1}^{n} (mq_j + lr_j + z_j) \right),
\]

the equation

\[
\left\{ \prod_{j=1}^{n} (mq_j + lr_j + z_j) \right\}^s = \prod_{j=1}^{n} (mq_j + lr_j + z_j)^s
\]

must hold for any \( m, l \in \mathbb{N} \cup \{0\} \). We take this remark into account to give a two dimensional analogue of Theorem 1. As an example of \( q_j, \tau_j, z_j \) which satisfy the equation (4) for any \( m, l \in \mathbb{N} \cup \{0\} \), we can take \( n = 2h, q_j, \tau_j, z_j \in \mathbb{C}, q_{h+j} = \overline{q_j}, \tau_{h+j} = \overline{\tau_j}, z_{h+j} = \overline{z_j}, j = 1, \ldots, h \).

**Corollary 4** Fix \( q_j, \tau_j, z_j \in \mathbb{C} \) such that \( \Re(q_j) > 0, \Re(\tau_j) > 0, \Re(z_j) > 0 \), and \( q_j \neq q_k, \tau_j \neq \tau_k, q_j \tau_k \neq q_k \tau_j \) for \( j \neq k \). Suppose that (4) is satisfied for any \( m, l \in \mathbb{N} \cup \{0\} \). Then we have

\[
\prod_{m,l=0}^{\infty} \left( \prod_{j=1}^{n} (mq_j + lr_j + z_j) \right) = e^F \prod_{j=1}^{n} \Gamma_2^*(z_j, (q_j, \tau_j))^{-1} = e^F \prod_{j=1}^{n} \left( \prod_{m,l=0}^{\infty} (mq_j + lr_j + z_j) \right),
\]

where

\[
F = -\frac{1}{2n} \sum_{1 \leq j < k \leq n} \left\{ \frac{q_j \tau_k - \tau_j q_k}{q_j q_k} (\log q_k - \log q_j) B_2 \left( \frac{q_j z_k - q_k z_j}{q_j \tau_k - \tau_j q_k} \right) + \frac{q_k \tau_j - \tau_k q_j}{\tau_j \tau_k} (\log \tau_k - \log \tau_j) B_2 \left( \frac{\tau_j z_k - \tau_k z_j}{\tau_j q_k - q_j \tau_k} \right) \right\}.
\]

Next we present \( q \)-analogue of Theorem 1. Usually the zeta-regularized product is defined for a sequence \( \{\lambda_k\} \subset \mathbb{C} \setminus \{0\} \) such that \( \Lambda(s) = \sum_k \lambda_k^{-s} \) can be continued holomorphically at \( s = 0 \). In case \( \Lambda(s) \) is meromorphic at
s = 0, Kurokawa and Wakayama [6] define the generalized zeta regularization by

$$\prod_{k} \lambda_{k} = \exp \left( - \text{Res}_{s=0} \frac{\Lambda(s)}{s^{2}} \right).$$

They obtained several examples of such product, one of which is the following $q$-analogue of Lerch's formula.

**Theorem 3 (Kurokawa and Wakayama [6])** For $q > 1$, $x > 0$,

$$\prod_{n=0}^{\infty} [n + x]_{q} = \frac{C_{q}}{\Gamma_{q}(x)}.$$  

Here $[x]_{q} = \frac{q^{x} - 1}{q - 1}$ is the $q$-analogue of number $x$,

$$\Gamma_{q}(x) = \frac{\prod_{n=1}^{\infty} (1 - q^{-n})}{\prod_{n=0}^{\infty} (1 - q^{-(x+n)})} (q - 1)^{1-x} q^{\frac{x(x-1)}{2}},$$

is Jackson's $q$-gamma function,

$$C_{q} = \prod_{n=1}^{\infty} [n]_{q} = q^{-\frac{1}{12}} (q - 1)^{\frac{1}{2} - \frac{\log(q-1)}{2\log q}} \prod_{n=1}^{\infty} (1 - q^{-n}).$$

We obtain the next result which is the $q$-analogue of Theorem 1 including the above Theorem 3.

**Theorem 4** For $q > 1$, $z_{j} > 1$, we have

$$\prod_{m=0}^{\infty} \left( \prod_{j=1}^{n} [m + z_{j}]_{q} \right) = \frac{C_{q}^{n}}{\prod_{j=1}^{n} \Gamma_{q}(z_{j})} q^{-\frac{1}{2n} (\sum_{j=1}^{n} z_{j})^{2} + \frac{1}{2} \sum_{j=1}^{n} z_{j}^{2}} \prod_{m=0}^{\infty} [n + z_{j}]_{q}. $$
3 Double Hurwitz zeta

For $\beta > \alpha > 0$, let $H_{\alpha, \beta}(s_1, s_2)$ be Dirichlet series defined by

$$H_{\alpha, \beta}(s_1, s_2) = \sum_{n=0}^{\infty} (n + \alpha)^{-s_1} (n + \beta)^{-s_2}.$$  

This series converges absolutely for $\Re(s_1 + s_2) > 1$.

$H_{\alpha, \beta}(s_1, s_2)$ is an important object in the theory of the zeta-regularized product. For example, as we presented in Section 1, we know generalized Lerch’s formula

$$\exp \left( -\frac{\partial}{\partial s} H_{\alpha, \beta}(s_1, s_2) \right) \bigg|_{s=0} = \frac{2\pi}{\Gamma(\alpha)\Gamma(\beta)}.$$  

We know also that the spectral zeta function $Z_n(s)$ of the unit $n$-sphere $S^{n-1}$ can be written in terms of $H_{\alpha, \beta}(s_1, s_2)$ as

$$Z_n(s) = \sum_{d=0}^{n-1} T_{n, d} H_{1, n}(s - d, s), \quad (5)$$

where

$$T_{n, d} = \frac{1}{n!} \sum_{r=d+1}^{n} s(n, r) \binom{r}{d} (n^{r-d} - (n-2)^{r-d}),$$

$s(r, d)$ denoting the Stirling numbers of the first kind. See Lemma 2 of [4] p.202. We get the formula for the functional determinant of the Laplacian by evaluating $\frac{\partial}{\partial s} Z_n(s) \bigg|_{s=0}$. See Theorem 1 of [4] p. 200.

In the results mentioned above, the main target is not $H_{\alpha, \beta}(s_1, s_2)$ itself but evaluating derivative of $H_{\alpha, \beta}(s_1, s_2)$. In this section, we analyze $H_{\alpha, \beta}(s_1, s_2)$ itself. First by applying the method described in [2], we can get the following expression for $H_{\alpha, \beta}(s_1, s_2)$.

$$H_{\alpha, \beta}(s_1, s_2) = \frac{\Gamma(s_1 + s_2)}{\Gamma(s_1)\Gamma(s_2)} \int_0^1 u^{s_2-1}(1-u)^{s_1-1}\zeta(s_1 + s_2, \alpha - (\alpha - \beta)u)du, \quad (6)$$

where $\zeta(s, x) = \sum_{n=0}^{\infty} (n + x)^{-s}$ is Hurwitz zeta function. It is very interesting to note that S. Ramanujan already treated the integral of the right hand side on (6) apart from Dirichlet series $H_{\alpha, \beta}(s_1, s_2)$. See (14) of [9] p.166.

Starting from the integral expression (6), we show the following results.

**Theorem 5** $H_{\alpha, \beta}(s_1, s_2)$ can be continued meromorphically to all $s_1, s_2 \in \mathbb{C}$.  

Theorem 6 For $\Re(s_1) < 0, \Re(s_2) < 0, 0 < \alpha < \beta < 1$, we have

$$H_{\alpha,\beta}(s_1, s_2) = \frac{\Gamma(1 - s_1 - s_2)}{(2\pi)^{1-s_1-s_2}} \times \left\{ e^{\frac{\pi i}{2} (1-s_1-s_2)} \sum_{n=1}^{\infty} n^{s_1+s_2-1} e^{-2\pi i n\beta} 1F_1(s_1, s_1+s_2, 2\pi in(\beta-\alpha)) + e^{-\frac{\pi i}{2} (1-s_1-s_2)} \sum_{n=1}^{\infty} n^{s_1+s_2-1} e^{2\pi i n\alpha} 1F_1(s_2, s_1+s_2, 2\pi in(\beta-\alpha)) \right\}.$$ \hspace{1cm} (7)

Here $1F_1(a, b, z)$ is the confluent hypergeometric series defined by

$$1F_1(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \hspace{1cm} (8)$$

with $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$.

This is a generalization of well known Hurwitz relation for $\zeta(s, x)$.

Theorem 7 We have

$$H_{\alpha,\beta}(s_1, s_2) = \sum_{n=0}^{\infty} \frac{(s_2)_n}{n!} \zeta(s_1 + s_2 + n, \alpha)(\alpha - \beta)^n.$$ 

This is a special case of Main Theorem of [3]. However we can prove Main Theorem of [3] by quite different manner using the confluent hypergeometric series $1F_1(a, b, z)$.

Next we give the evaluation formula of $H_{\alpha,\beta}(s_1, s_2)$. We can evaluate the values of $H_{\alpha,\beta}(s_1, s_2)$ at any integers $s_1, s_2$ in terms of the values of Hurwitz zeta function.

Theorem 8 For $p, q \in \mathbb{N}$, we have

$$H_{\alpha,\beta}(q, p) = \frac{\Gamma(p+q)}{(p+q-1)\Gamma(p)\Gamma(q)} \times \left\{ \sum_{n=0}^{p+q-3} \left\{ \sum_{m=\max\{n-p+1,0\}}^{q-1} (-1)^m \binom{q-1}{m} \binom{p+m-1}{n} \right\} \right\}.$$
\[ \times \frac{n!}{(2-p-q)_{n}} \zeta(p+q-n-1, \beta)(\alpha-\beta)^{-n-1} \]
\[ - \sum_{m=0}^{q-2} (-1)^{m} \binom{q-1}{m} \frac{(p+m-1)!}{(2-p-q)_{p+m-1}} \zeta(q-m, \alpha)(\alpha-\beta)^{-p-m} \]
\[ + \frac{(-1)^{q-1}}{(2-p-q)_{p+q-2}} \frac{(p+q-2)!}{(2-p-q)_{p+q-2}} (\alpha-\beta)^{p+q+1} \left\{ \left( \frac{\Gamma'}{\Gamma}(\beta) - \frac{\Gamma'}{\Gamma}(\alpha) \right) \right\} . \]

Here empty sum is considered as zero.

**Theorem 9** For \( p, q \in \mathbb{Z} \) which are not both negative, we have

\[ H_{\alpha,\beta}(-p, -q) = \sum_{k=0}^{q} \binom{q}{k} (\beta-\alpha)^{k} \zeta(-p-q+k, \alpha) \]
\[ + \sum_{k=0}^{p} \binom{p}{k} (\alpha-\beta)^{k} \zeta(-p-q+k, \beta) . \]

Here empty sum is considered as zero.

Finally we mention that we can provide another approach to evaluate the determinant \( \det \Delta_{n} \) of the Laplacian on the \( n \)-sphere \( S^{n-1} \) starting from the integral expression (6). Here \( \det \Delta_{n} \) is defined by

\[ \det \Delta_{n} = \exp \left( - \sum_{d=0}^{n-1} T_{n,d} \frac{\partial}{\partial s} H_{1,n}(s-d, s) \bigg|_{s=0} \right) . \]

See (5) for the definition of \( T_{n,d} \).

**Theorem 10**

\[ \frac{\partial}{\partial s} H_{1,n+1}(s-d, s) \bigg|_{s=0} = \zeta'(-d) + \sum_{l=0}^{d} (-n)^{d-l} \binom{d}{l} \zeta'(-l, n+1) \]
\[ - \frac{(-n)^{d+1}}{2(d+1)} \sum_{j=1}^{d} \frac{1}{j} . \]

This is simpler than Kumagai's formula given in Lemma 3 of [4] p.202. Comparing Theorem 10 and Kumagai's result, we get the following identity for harmonic numbers.
Corollary 5 The following identity holds:

\[ 2^{1-d} \sum_{l=1, odd}^{d} \left( \frac{d+1}{l+1} \right) \sum_{j=1, odd}^{l} \frac{1}{j} = \sum_{j=1}^{d} \frac{1}{j}. \]

References


