数理解析研究所講究録 1452 巻 2005 年 152-159

GENERALIZED MEASUREMENTS, CP-MEASURE AND CP-CHOQUET THEOREM

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ABSTRACT. In C*-physics, quantum interactions are described by contractive CP-maps, and *CP-convexity* is essential to describe the decomposition of CP-maps. We introduce the notion of *CP-measure* as a new mathematical formulation of generalized measurements, and develop its integration theory. We then prove a generalized Choquet's theorem for completely positive maps, i.e., every contractive CP-map from a C*-algebra to $B(H)$ can be represented by a CP-measure supported by CP-extreme elements, which solves the decomposition problem for completely positive maps.

1. Introduction.

In operational quantum physics, it is now standard that quantum measurement process is described by a completely positive map, i.e., assuming that observables are represented by (self-adjoint) bounded linear operators on a Hilbert space K, and those of apparatus on a Hilbert space H, then after an interaction between the two physical systems, an observable $a \in B(K)$ is measured by the apparatus as an operator $\psi(a) \in B(H)$, where ψ is a normal contractive completely positive map

$$\psi(a) = \sum_i V_i^* a V_i \quad \text{with} \quad V_i \in B(H,K) \quad \text{such that} \quad \sum_i V_i^* V_i \leq I_H.$$

Note that ψ is called an *operation* if $K = H$ (cf. K. Kraus [14]).

In CP-convexity theory [5-11], this can be interpreted as a CP-convex combination of CP-extreme elements, i.e., using the polar decomposition $V_i = u_i|V_i|$, ψ can be rewritten as

$$\psi(a) = \sum_i |V_i^*| \varphi_i(a) |V_i| \text{ with } |V_i| \in B(H)^+ \text{ such that } \sum_i |V_i|^2 = \sum_i V_i^* V_i \leq I_H,$$

where $\varphi_i(\cdot) = u_i^* \cdot u_i$ is a *conditional transform* with a partial isometry u_i from H to K. We can also show that it can be decomposed as

$$\psi(a) = \sum_i W_i^* \phi_i(a) W_i \quad \text{with } W_i \in B(H) \quad \text{such that } \sum_i W_i^* W_i \leq I_H,$$

where $\phi_i(\cdot) = U_i^* \cdot U_i$ is a *unitary transform* with a unitary U_i from H to K, but this time W_i is not positive in general (cf. [11]). Thus ψ can be considered as the "barycenter" of an operation-valued atomic measure $\{|V_i| \cdot |V_i|\}$ [or $\{W_i^* \cdot W_i\}$] supported by CP-extreme states $\psi_i(\cdot) = u_i^* \cdot u_i$ (*conditional transform*) [or $\phi_i(\cdot) = U_i^* \cdot U_i$] (*unitary transform*)].

Since a physical system is described by a C*-algebra in general, we shall generalize the setting of $B(K)$ to a C*-algebra A. Then, a contractive CP-map ψ from A to $B(H)$ cannot be decomposed into a CP-convex combination of extreme CP-maps as above unless ψ is atomic (cf. §3). Therefore we are naturally led to develop the concept of the measure generalization of CP-convex combination, that is *CP-measure,* which is mathematically defined as an operation-valued measure, and physically it represents a distribution of operational effects over the extreme interactions. Our main result in [9] states that, if A and H are separable, then every contractive CP-map from A to $B(H)$ (generalized measurement process) can be represented by a CP-measure supported by CP-extreme elements (extreme measurements) which are characterized in § 2, in the sense of our integration theory discussed in § 4.

The formulation of generalized measurements above is different from that of conventional quantum physics, even when the observables are described by $B(K)$, where the generalized measurement ψ was used to be described by a POV-measure $\{V_i^* V_i\}$ on the discrete index set (e.g. [4]). In this case however the information about the extreme interactions $\{\varphi_i\}$ is eliminated, so that it is impossible to recover the original measurement process as a CP-map. On the other hand, the notion of CP-measure provides the perspective of the decomposition of the measurement into the extreme ones with operational weights recovering the original measurement.

2. CP-convexity and notations.

Recall that every CP-map $\psi \in CP(A, B(H))$ can be represented as $\psi = V^*\pi V$ where π is a representation of A, and V is a bounded linear operator from H to H_π (cf. [15]). A CP-map $\psi \in CP(A, B(H))$ is called a *CP-state* if it is contractive, and we denote the set of all CP-states by $Q_H(A)$ and call the *CP-state space* of A;

$$Q_H(A) = \{\psi = V^*\pi V \in CP(A; B(H)); \|V\| \leq 1\}.$$

A CP-map $\psi = V^*\pi V \in CP(A, B(H))$ is unital iff $V^*V = I_H$, and pure iff π is irreducible (cf. [2]). We denote by $S_H(A)$ [resp. $P_H(A)$, $PS_H(A)$] the set of all unital [resp. pure, unital pure] CP-states. We also denote by $Rep(A : H)$ [resp.

$Rep_c(A:H), Irr(A:H)$] the set of all [resp. cyclic, irreducible] representations of A on H (i.e., whose representation spaces are subspaces of H).

A CP-state $\psi \in Q_H(A)$ is said to be a *CP-convex combination* of CP-states $\psi_i \in Q_H(A)$ if it can be decomposed as

$$\psi = \sum_i S_i^* \psi_i S_i \quad \text{with } S_i \in B(H) \text{ such that } \sum_i S_i^* S_i \leq I_H,$$

which will be abbreviated by $\psi = CP\text{-}\sum_i S_i^* \psi_i S_i$.

A CP-state is defined to be *CP-extreme* if $\psi = CP\text{-}\sum_i S_i^* \psi_i S_i$ implies that ψ_i is unitarily equivalent to ψ. We denote by $D_H(A)$ the set of all CP-extreme states. It is shown in [11] that $D_H(A) = Irr(A:H)$ if $\dim H = \infty$, $D_H(A) = Irr(A:H) \cup PS_H(A)$ if $dim H < \infty$, and in particular $D_{\mathbb{C}}(A) = P(A)$.

The definition of CP-extreme state is not unique, which is a distinctive feature compared with the scalar convexity. For example, a CP-state is defined to be *conditionally CP-extreme* if $\psi = CP-\sum_i S_i \psi_i S_i$ with $S_i \geq 0$ implies that $s(S_i)\psi_i s(S_i) = \psi$, where $s(S_i)$ is the support projection of S_i. Then, $\psi = u^* \pi u \in Q_H(A)$ is conditionally CP-extreme iff ψ is pure and u is a partial isometry from H to H_π (cf. [11]).

If $A = B(K)$, then the CP-extreme states are the unitary transforms, and the conditionally CP-extreme states are the conditional transforms as we observed in the introduction. For example, annihilations and creations in Fock Hilbert space are conditionally CP-extreme, so extreme interactions as expected, where CP-coefficients include the information about the correlation and probability in quantum information theory (cf. [12]).

3. Special decompositions.

Besides the physical motivation outlined in the introduction, we had the mathematical motivation to answer the so-called decomposition problem of CP-maps, i.e., "How can a CP-state be decomposed into the extreme elements?" The answer should generalize the Choquet theory on the state space $S(A)$ as we shall find out later.

As illustrated in [9], the preliminary attempt to apply scalar convexity theory would fail here, since the extreme points of the CP-state space $Q_H(A)$ are not pure CP-maps in general, so not extreme CP-maps (cf.[2]). For example, note that any representation $\pi \in Rep(A:H)$ is an extreme point of the CP-state space $Q_H(A)$, but it is not pure in general. Also the attempt to apply the disintegration theory of representations confronts the difficulty caused by the unboundedness arguments (cf. [9]). As we shall see later, this is possible only if it has a strong continuity.

After all, we realize that CP-convex combination would be essential to describe the decomposition of CP-maps. It is straightforward to see that, if $\psi = V^*\pi V \in Q_H(A)$ is atomic (i.e., π can be embedded into a subrepresentation of a direct sum of irreducible representations), then ψ can be decomposed into a CP-convex combination of CP-extreme states. In fact, let $\pi = (\oplus_i \pi_i)|_{H_\pi}$ with $\pi_i \in Irr(A)$, then for each $a \in A$,

$$\begin{aligned}\psi(a) = V^*\pi(a)V = V^*(\oplus_i p_i)(\oplus \pi_i(a))(\oplus_i p_i)V = \sum_i V^* p_i \pi_i(a) p_i V \\ = \sum_i V^* p_i u_i (u_i^* \pi_i(a) u_i) u_i^* p_i V = \sum_i V_i^* \varphi_i(a) V_i\end{aligned}$$

where p_i is the projection of $H_{\oplus_i \pi_i}$ onto H_{π_i}, u_i is a partial isometry from H to H_{π_i}, $V_i = u_i^* p_i V \in B(H)$ and $\varphi_i = u_i^* \pi_i u_i \in D_H(A)$. For example, as we mentioned in the introduction, every normal CP-map from $B(K)$ to $B(H)$ is atomic, i.e., it is represented by a CP-convex combination of CP-extreme states.

More generally, as the generalization of the Krein-Milman theorem (e.g. [1]), every CP-state can be approximated by CP-convex combinations of CP-extreme elements, i.e.,

$$Q_H(A) = \text{BW-cl. CP-conv } D_H(A),$$

which is a preliminary version of the CP-Choquet theorem. As we shall show later, to eliminate "BW-cl.", we have to replace the CP-convex combination by "CP-measure."

The next fundamental decomposition would be the decomposition by a scalar measure. We call $\psi = V^*\pi V \in Q_H(A)$ to be *nuclear* if V is a nuclear mapping from H to a nuclear space $\Omega \subset H_\pi$, with $\Omega \subset H_\pi \subset \Omega'$ being a rigged Hilbert space. Then we have the following result (cf. [9]).

Theorem 1. *Let A be a separable C^*-algebra and H be a separable Hilbert space, and assume that $\psi \in CP(A, B(H))$ is nuclear. Then there exists a standard measure space (Z, μ) and measurable families of irreducible representations $\pi(\zeta) \in Irr(A)$ and trace class operators $V(\zeta) \in T(H, H_\pi(\zeta))$ such that*

$$\psi = \int_Z V(\zeta)^* \pi(\zeta) V(\zeta) d\mu(\zeta).$$

Actually, the above result can be applied to slightly more general CP-maps, called *pre-nuclear* CP-maps, for which the readers are referred to [9]. In particular, finite rank CP-maps are nuclear, and this fact will be used in the proof of the CP-Choquet theorem.

4. CP-measure and integration.

We shall develop the measure and integration theory especially for operator algebras, and we present it in a more general setting than we actually need in § 5.

Let A be an order unit space with an order unit e, and let B be a dual order unit space with a predual base norm space B_* (cf. [1]). We denote by $P(A, B)$ the set of all positive linear maps from A to B. Let $(X, \mathcal{B})$ be a measurable space, and λ be a $P(A, B)$-valued BW-countably additive measure, i.e., if $E = \sum_{i=1}^{\infty} E_i$ with $E_i \in \mathcal{B}$ (disjoint), then $\lambda(E) = \sum_{i=1}^{\infty} \lambda(E_i)$ converging in the BW-topology. We denote by $BS(X, A)$ the set of all bounded (countably valued) simple functions from X to A, i.e., $f \in BS(X, A)$ is a function of the form

$$f = \sum_{i=1}^{\infty} \chi_{E_i} a_i \quad \text{with} \quad (E_i) \subset \mathcal{B} \ \text{(disjoint) and} \ (a_i) \subset A \ \text{(bounded)},$$

where χ_E denotes the characteristic function of $E \in \mathcal{B}$. We then define the integral of f with respect to λ by

$$\int_X f \, d\lambda := \sum_{i=1}^{\infty} \lambda(E_i) a_i \ \in B,$$

which converges in the $\sigma(B, B_*)$-topology. We note that the integration map λ : $BS(X, A) \to B$ is bounded with $\|\lambda\| = \|\lambda(X)e\|$.

Let τ be a topology in A, where, in the subsequent arguments, τ will denotes the *norm* topology ($\tau = s$), or the *weak** topology ($\tau = w$) if A is a dual order unit space. By $BM_\tau(X, A)$ we denote the set of all bounded τ-measurable functions from X to A, and call the *strongly* [*weakly*] measurable functions if $\tau = s$ [$\tau = w$].

Lemma 2. *Assume that any bounded part of A is τ-metrizable. Then $BS(X, A)$ is dense in $BM_\tau(X, A)$ in the τ-uniformly convergence topology.*

Definition. By the boundedness of the integration map $\lambda : BS(X, A) \to B$ and the above lemma with the norm topology ($\tau = s$), λ has the unique norm continuous extention $\lambda : BM_s(X, A) \to B$, which we shall call the *strong integration* of the A-valued strongly measurable functions on X with respect to λ.

We next assume that A is a dual order unit space, whose predual consists of all monotone continuous linear functionals on A (which is satisfied for example for JBW-algebras and of course for W*-algebras). Let $\lambda : \mathcal{B} \to P(A,B)_n$ be a BW-countably additive measure, where $P(A,B)_n$ denotes the set of all normal positive linear maps from A to B. For each $a \in A$, $\rho \in B_*$ and $E \in \mathcal{B}$, we shall define a scalar measure $\lambda_{a,\rho}(E) := (\lambda(E)a, \rho)$. Then, if there exists a scalar measure ν such that $\lambda_{a,\rho} \ll \nu$ uniformly for $a \in A$ and $\rho \in B_*$, we say that λ is *BW-absolutely continuous with respect to ν*. This constraint would suffice for our purpose.

Lemma 3. *Assume that λ satisfies the above conditions, and let $(f_n) \subset BS(X,A)$ be a Cauchy sequence in the uniform w^*-convergence topology. Then, $(b_n) = (\int_X f_n d\lambda)$ is a Cauchy sequence in the $\sigma(B,B_*)$-topology.*

Definition. Let A and B be dual order unit spaces, and assume that any bounded part of A is w*-metrizable. Then by Lemmas 2 and 3, the integration map $\lambda : BS(X,A) \to B$ has the unique w*-continuous extention $\lambda : BM_w(X,A) \to B$, which we shall call the *weak integration* of A-valued weakly measurable functions on X with respect to λ.

Remark. We note that the strong integral can be extended for unbounded measurable functions, and this integration has some advantages compared with the usual method in the vector measure theory in Banach spaces (cf. [3], [5]). On the other hand, it would seem hardly possible to establish the weak integral without any restrictions.

5. CP-Choquet theorem.

We shall now apply our measure and integration theory to the measurable space $(X,\mathcal{B})$ where X is the CP-state space $Q_H(A)$ for a C*-algebra A, and $\mathcal{B}$ is the Borel sets $\mathcal{B}_{BW}$ in $Q_H(A)$ induced from the BW-topology. Let $\lambda : \mathcal{B}_{BW} \to Q_H(B(H))_n$ be an operation valued countably additive measure, which we shall call *CP-measure* in the following. By the CP-duality theorem (cf. [7]), every element $a \in A$ defines a BW-w continuous function $\hat{a} : \psi \in Q_H(A) \to \psi(a) \in B(H)$ by the natural evaluation map, so a w-measurable function, so that we can consider the weak integral $\int_{Q_H(A)} \hat{a}\, d\lambda$ as we defined in the previous section.

Lemma 4. *If A and H are separable, then $D_H(A) \in \mathcal{B}_{BW}$.*

We can now state our main theorem.

Theorem 5. *Let A be a separable C^*-algebra, and H be a separable Hilbert space. Then for any CP-state $\psi \in Q_H(A)$, there exists a CP-measure λ_ψ supported by the CP-extreme states $D_H(A)$ such that*

$$\psi(a) = \int_{D_H(A)} \hat{a}\, d\lambda.$$

Sketch of proof. Let $\psi = V^*\pi V$ be the Stinespring representation of ψ, and we take an irreducible decomposition of π, i.e.,

$$\pi = \int_Z^{\oplus} \pi(\zeta) d\mu(\zeta) \quad \text{and} \quad H_\pi = \int_Z^{\oplus} H_\pi(\zeta) d\mu(\zeta).$$

We first claim that there exists a partial isometry

$$U : H_\pi \to L^2(Z, \mu, H) \quad \text{with} \quad U = \int_Z^{\oplus} u(\zeta) d\mu(\zeta)$$

where $u(\zeta) : H_\pi(\zeta) \to H$ is a measurable field of isometries or co-isometries. To prove this, if H is infinite dimensional, then we can take an embedding $u(\zeta)$: $H_\pi(\zeta) \to H$, and if H is finite dimensional, then we can use the decomposition of nuclear CP-map ψ (Theorem 1) (see [9] for details).

We define $\tilde{\pi}(\zeta) := u(\zeta)\pi(\zeta)u(\zeta)^* \in D_H(A)$. We also denote by κ the natural representation of $B(H)$ on $L^2(Z, \mu, H)$, and by P_E the projection in $L^2(Z, \mu, H)$ which is defined by

$$P_E = \int_Z^{\oplus} P_E(\zeta) d\mu(\zeta) \text{ where } P_E(\zeta) = \begin{cases} I_H & \text{for } \zeta \in \tilde{\pi}^{-1}(E). \\ 0 & \text{for } \zeta \notin \tilde{\pi}^{-1}(E). \end{cases}$$

We define a CP-measure λ_ψ by

$$\lambda_\psi(E) := (P_E U V)^* \kappa (P_E U V) \in Q_H(B(H))_n,$$

then we can observe that it is supported by $D_H(A)$, and BW-absolutely continuous with respect to a scalar measure $\nu := \mu \circ \tilde{\pi}^{-1}$.

We now apply our weak integral discussed in the previous section, and the left of the proof is devoted to prove the equality

$$\left(\int_{D_H(A)} \hat{a}\, d\lambda_\psi, \rho\right) = (\psi(a), \rho).$$

for all $\rho \in B(H)_* = T(H)$, which depends on the Radon-Nikodym theorem for the weak integral and the definition of $u(\zeta)$ above (cf. [9] for details).

As an application of the CP-Choquet theorem, we mention the non-commutative spectral theory in [10]. We expect that it will find more applications in mathematics and quantum physics where the decomposition of CP-maps play an important role.

Remarks. 1. In particular, if H is infinite dimensional and $\psi = \pi \in Rep(A : H)$, then $D_H(A) = Irr(A : H)$ and the above theorem gives an analytic expression of the algebraic decomposition of π. Also, if $dimH = 1$, then $D_H(A) = P(A)$, and this reduces to the classical Choquet's theorem. Thus the CP-Choquet theorem interpolates between the classical Choquet' theorem and the algebraic decomposition theory for representations.

2. By modifying the proof, we can show that there exists a representing CP-measure supported by the conditionally CP-extreme states, where if $A = B(K)$ for example, then thc CP-measure is the atomic measure $\{|V_i| \cdot |V_i|\}$ as we saw in the introduction.

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